Taming the dualizing complex

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Philosophy. We have proven Grothendieck-Serre duality in the setting of derived categories. The aim of this talk is to give an impression how classical Serre duality arises from this machinery. In particular, we want to show that in the case of smooth proper schemes, the dualizing complex "is" a sheaf and that this sheaf coincides with the canonical sheaf. The goal is to prove the following:

Theorem 1. *Let $f : X \to S$ be a proper smooth morphism of noetherian schemes of relative dimension n with associated dualizing complex $f^!\mathcal{O}_S$. Then $H^i(f^!\mathcal{O}_S) = 0$ whenever $i \neq -n$ and $H^{-n}(f^!\mathcal{O}_S)$ is isomorphic to the relative canonical sheaf $\omega_{X/S}$.*

1 Dualizing sheaf and canonical sheaf

Proposition 2. *Let $f : X \to S$ be a proper smooth morphism of noetherian schemes of relative dimension n with associated dualizing complex $f^!\mathcal{O}_S$. Suppose that $H^i(f^!\mathcal{O}_S) = 0$ whenever $i \neq -n$ and assume further that $H^{-n}(f^!\mathcal{O}_S)$ is a flat $\mathcal{O}_X$-module. Then the sheaf $H^{-n}(f^!\mathcal{O}_S)$ is isomorphic to the relative canonical sheaf $\omega_{X/S}$.*

Application of Grothendieck-Serre duality. Consider the diagonal morphism $\Delta : X \to X \times_S X$ and the projection onto the first factor $\pi : X \times_S X \to X$. Then Δ is a closed immersion and $\pi \circ \Delta = \mathrm{id}_X$. Furthermore, the morphism π satisfies all necessary conditions for Grothendieck-Serre duality. Hence:

If $\mathscr{F} \in \mathcal{D}^+(\mathrm{QCoh}(X \times_S X))$ and $\mathscr{G} \in \mathcal{D}^+(\mathrm{QCoh}(X))$, then the natural map

$$\mathrm{R}\pi_*\mathrm{R}\mathscr{H}om_{\mathcal{O}_{X\times_S X}}(\mathscr{F}, \pi^!\mathscr{G}) \to \mathrm{R}\mathscr{H}om_{\mathcal{O}_X}(\mathrm{R}\pi_*\mathscr{F}, \mathscr{G}).$$

is an isomorphism.

Applying the duality theorem for $\mathscr{F} = \Delta_*\mathcal{O}_X[0]$ and $\mathscr{G} = \mathcal{O}_X[0]$, we obtain a natural isomorphism

$$\mathrm{R}\pi_*\mathrm{R}\mathscr{H}om_{\mathcal{O}_{X\times_S X}}(\Delta_*\mathcal{O}_X[0], \pi^!\mathcal{O}_X) \cong \mathrm{R}\mathscr{H}om_{\mathcal{O}_X}(\mathrm{R}\pi_*\Delta_*\mathcal{O}_X[0], \mathcal{O}_X[0]). \qquad (*)$$

Computing the right-hand side of $(*)$**.** Let $\mathcal{O}_X \to \mathscr{J}$ be an injective resolution. Since Δ_* maps flabby sheaves to flabby sheaves, $\Delta_*\mathcal{O}_X \to \Delta_*\mathscr{J}$ is a π_*-acyclic resolution. Hence

$$(\mathrm{R}^i\pi_*)\Delta_*\mathcal{O}_X = H^i(\pi_*\Delta_*\mathscr{J}) = H^i(\mathscr{J}) = \begin{cases} \mathcal{O}_X & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Therefore $\mathrm{R}\pi_*\Delta_*\mathcal{O}_X[0]$ is supported in degree zero only, and the degree zero part is simply given by $\mathcal{O}_X$. The right-hand side of $(*)$ reads now

$$\mathrm{R}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X[0],\mathcal{O}_X[0]) = \mathcal{O}_X[0].$$

Left-hand side of $(*)$. We need a few lemmas.

Lemma 3. *Let Y be an S-scheme, $j : X \to Y$ a regular closed immersion, and $\mathscr{F} \in \mathrm{QCoh}(Y)$ a flat $\mathcal{O}_Y$-module. If $Y \to S$ is of relative dimension m and $X \to S$ is of relative dimension n, then*

$$\mathscr{E}xt^i_{\mathcal{O}_Y}(j_*\mathcal{O}_X,\mathscr{F}) \cong \begin{cases} j_*j^*\mathscr{F}\otimes_{\mathcal{O}_Y}\bigwedge^k(\mathscr{I}_X/\mathscr{I}_X^2)^\vee & \text{if } i = m-n, \\ 0 & \text{if } i \neq m-n. \end{cases}$$

Proof: Since j is a regular immersion, we can cover Y by affine opens where the ideal sheaf $\mathscr{I}_X$ is regular. Hence if $U = \operatorname{Spec} A$ is one of those, then the ideal $I = H^0(U,\mathscr{I}_X) \subseteq A$ is generated by an A-regular sequence $f_1,\dots,f_k$, where $k := m-n$.

Let $\mathcal{K}$ denote the Koszul complex

$$0 \to \bigwedge^k A^k \to \bigwedge^{k-1} A^k \to \cdots \to \bigwedge^0 A^k \to 0,$$

where the maps are given by

$$\bigwedge^i A^k \to \bigwedge^{i-1} A^k, \quad e_{j_1}\wedge\cdots\wedge e_{j_i} \mapsto \sum_{\ell=1}^{i}(-1)^{\ell+1} f_{j_\ell}\cdot e_{j_1}\wedge\cdots\wedge\widehat{e_{j_\ell}}\wedge\cdots\wedge e_{j_i}$$

In particular, the map $A^k \cong \bigwedge^1 A^k \to \bigwedge^0 A^k \cong A$ is given by $(a_1,\dots,a_k) \mapsto \sum_{\ell=1}^k a_\ell f_\ell$. As the sequence $f_1,\dots,f_k$ was regular, the Koszul complex is a free resolution. So all cohomology modules vanish, except for the last one, which equals A/I.

Furthermore, $\mathrm{Ext}^i(A/I,-)$ is computed by $H^i(\mathrm{Hom}(\mathcal{K},-))$. But for an A-module M,

$$\mathrm{Hom}_A\left(\bigwedge^i A^k, M\right) \cong \left(\bigwedge^i A^k\right)^\vee \otimes_A M \cong \bigwedge^{k-i} A^k \otimes_A M,$$

thus $\mathrm{Ext}^i(A/I,M)$ is isomorphic to the cohomology of the complex $\mathcal{K}\otimes_A M$ at the $(k-i)$-th wedge product. Hence we obtain isomorphisms

$$\psi^i_{f_1,\dots,f_k} : \mathrm{Ext}^i(A/I,M) = H^i(\mathrm{Hom}(\mathcal{K},M)) \cong H^{k-i}(\mathcal{K}\otimes_A M),$$

and we can view $\psi^k_{f_1,\dots,f_k}$ as isomorphism $\mathrm{Ext}^i(A/I,M) \cong M/IM$.

The residue classes of $f_1,\dots,f_k$ generate I/I^2 as a free A/I-module of rank k. Thus the k-th exterior power of I/I^2 is free of rank one with generator $f_1\wedge\cdots\wedge f_k$. Consider the map

$$\varphi : \mathrm{Ext}^k_A(A/I,M) \to \mathrm{Hom}_{A/I}\left(\bigwedge^k I/I^2, M/IM\right), \quad \varphi(x)(f_1\wedge\cdots\wedge f_k) := \psi^k_{f_1,\dots,f_k}(x).$$

This is well-defined: If $g_1, \dots, g_k$ form another regular sequence, then inspecting the Koszul maps implies that $\psi^k_{f_1,\dots,f_k} = \det B \cdot \psi^k_{g_1,\dots,g_k}$, where B is a base change matrix. Consequently, φ is an isomorphism.

We take now $M := H^0(U, \mathscr{F})$, which is a flat A-module. Hence $\mathcal{K} \otimes_A M$ remains exact except for the last spot, in particular, $\operatorname{Ext}^i(A/I, M) = 0$ for $i \neq k$. Since I/I^2 is a free A-module, we have an isomorphism

$$\operatorname{Hom}_{A/I}\left(\bigwedge^k I/I^2, M/IM\right) \cong \bigwedge^k (I/I^2)^\vee \otimes_A M/IM.$$

Now gluing yields the sheafified version of φ :

$$\begin{aligned}\mathscr{E}xt^k_{\mathcal{O}_Y}(j_*\mathcal{O}_X, \mathscr{F}) &\cong \begin{cases} j_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathscr{F} \otimes_{\mathcal{O}_Y} \bigwedge^k (\mathscr{I}_X/\mathscr{I}_X^2)^\vee & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \\ &\cong \begin{cases} j_*j^*\mathscr{F} \otimes_{\mathcal{O}_Y} \bigwedge^k (\mathscr{I}_X/\mathscr{I}_X^2)^\vee & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}\end{aligned}$$

where the last isomorphism is an application of the projection formula for the closed immersion j. □

Lemma 4 (Flat base change). *Consider a Cartesian square of noetherian schemes*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{g}} & X \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \xrightarrow{g} & Y \end{array},$$

where $f, \widetilde{f}$ are proper and $g, \widetilde{g}$ are flat. Then there is a natural isomorphism

$$\widetilde{g}^* f^! \cong \widetilde{f}^! g^*.$$

Proof: This was proved for $g, \widetilde{g}$ open immersions in the last talk. The more general statement when $g, \widetilde{g}$ are flat is proven in [4, Theorem 4.4.1]. □

Finishing the proof of Proposition 2. Using the flat base change for dualizing complexes,

$$\pi^* f^! \mathcal{O}_S = \pi^! f^* \mathcal{O}_S = \pi^! \mathcal{O}_X.$$

Since π is flat, π^* is exact and we obtain a flat $\mathcal{O}_{X \times_S X}$-module $\omega := H^{-n}(\pi^! \mathcal{O}_X) = \pi^* H^{-n}(f^! \mathcal{O}_S)$. In particular, since

$$H^i(\mathrm{R}\mathscr{H}om_{\mathcal{O}_{X \times_S X}}(\Delta_* \mathcal{O}_X[0], \omega[0])) = \mathscr{E}xt^i_{\mathcal{O}_{X \times_S X}}(\Delta_* \mathcal{O}_X, \omega) \quad \text{for all } i,$$

applying Lemma 3 (note that Δ is a regular closed immersion) yields

$$\mathrm{R}\mathscr{H}om_{\mathcal{O}_{X \times_S X}}(\Delta_* \mathcal{O}_X[0], \omega[0]) \cong \left(\Delta_* \Delta^* \omega \otimes_{\mathcal{O}_{X \times_S X}} \bigwedge^n (\mathscr{I}_X/\mathscr{I}_X^2)^\vee\right)[n].$$

Using the canonical isomorphism
$$\mathscr{I}_X/\mathscr{I}_X^2 \cong \Delta_*\Omega_{X/S},$$
the left-hand side of (∗) transforms to

$$\begin{aligned}
\mathrm{R}\pi_*\mathrm{R}\mathscr{H}om_{\mathcal{O}_{X\times_S X}}(\Delta_*\mathcal{O}_X[0],\omega[-n]) &= \mathrm{R}\pi_*\mathrm{R}\mathscr{H}om_{\mathcal{O}_{X\times_S X}}(\Delta_*\mathcal{O}_X[0],\omega[0])[-n] \\
&\cong \mathrm{R}\pi_*\left(\Delta_*\Delta^*\omega \otimes_{\mathcal{O}_{X\times_S X}} \bigwedge^n (\mathscr{I}_X/\mathscr{I}_X^2)^\vee\right)[0] \\
&\cong (\Delta^*\omega \otimes_{\mathcal{O}_{X\times_S X}} \omega_{X/S}^\vee)[0].
\end{aligned}$$

Note that $\Delta^*\omega = \Delta^*\pi^*H^{-n}(f^!\mathcal{O}_S) = H^{-n}(f^!\mathcal{O}_S)$. Since $\omega_{X/S}$ is an invertible sheaf due to smoothness of f, we finally arrive at an isomorphism
$$H^{-n}(f^!\mathcal{O}_S) \cong \omega_{X/S}.$$

2 The dualizing complex becomes a sheaf

Projective space over a field. We need a little bit of classical theory:

Proposition 5. *Let $f : \mathbb{P}^n_S \to S$ be the structure morphism of projective space over a noetherian scheme S. Then $H^i(f^!\mathcal{O}_S) = 0$ whenever $i \neq -n$ and $H^{-n}(f^!\mathcal{O}_S)$ is isomorphic to the canonical sheaf $\omega_{\mathbb{P}^n_S/S}$.*

Proof: [2, Theorem III.5.1] shows that the functor Rf_*- has the right adjoint $\omega_{\mathbb{P}^n_S/S} \otimes_{\mathcal{O}_{\mathbb{P}^n_S}} f^*-$. □

The Cohen-Macaulay case.

Definition (Cohen-Macaulay [5, 045Q])**.**

- A module M over a local ring $(A,\mathfrak{m})$ is Cohen-Macaulay if the projective dimension $\mathrm{pd}_A\, M$ of M over A equals the codimension $\mathrm{codim}_A\, M$ of M in A.
- A morphism $f : X \to S$ of schemes is Cohen-Macaulay at $x \in X$ if f is flat at x and the local ring $\mathcal{O}_{X_{f(x)},x}$ is Cohen-Macaulay.
- A morphism $f : X \to S$ of schemes is Cohen-Macaulay if f is Cohen-Macaulay at all points of X.

Proposition 6. *Let $f : X \to S$ be a flat morphism of noetherian schemes of relative dimension n with associated dualizing complex $f^!\mathcal{O}_S$. Assume that f factors as $X \xrightarrow{j} Y \xrightarrow{g} S$, where j is a closed immersion and g is separated and smooth. Then the following are equivalent:*

(a) f is Cohen-Macaulay,

(b) $H^i(f^!\mathcal{O}_S) = 0$ whenever $i \neq -n$ and $H^{-n}(f^!\mathcal{O}_S)$ is flat over S.

Proof: We sketch the proof in the case that $Y = \mathbb{P}^m_S$. For the general statement and the flatness condition, see [1, Theorem 3.5.1].

By Grothendieck-Serre duality applied to $j : X \hookrightarrow \mathbb{P}^m_S$, we have
$$\mathrm{R}j_*\mathrm{R}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, f^!\mathcal{O}_S) = \mathrm{R}j_*\mathrm{R}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, j^!g^!\mathcal{O}_S) \cong \mathrm{R}\mathscr{H}om_{\mathcal{O}_{\mathbb{P}^m_S}}(Rj_*\mathcal{O}_X, g^!\mathcal{O}_S).$$

Since j is a closed immersion, j_* is exact. Taking the i-th cohomology and using duality for projective space (Proposition 5),

$$\begin{aligned} H^i(j_*\mathrm{R}\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, f^!\mathcal{O}_S)) &\cong H^i(\mathrm{R}\mathscr{H}om_{\mathcal{O}_{\mathbb{P}^n_S}}(j_*\mathcal{O}_X, g^!\mathcal{O}_S)) \\ &= H^{i+m}(\mathrm{R}\mathscr{H}om_{\mathcal{O}_{\mathbb{P}^m_S}}(j_*\mathcal{O}_X, \omega_{\mathbb{P}^m_S/S}) \\ &= \mathscr{E}xt^{i+m}_{\mathcal{O}_{\mathbb{P}^m_S}}(j_*\mathcal{O}_X, \omega_{\mathbb{P}^m_S/S}). \end{aligned}$$

Pick a point $x \in j(X)$, corresponding to a maximal ideal $\mathfrak{m} \subseteq A$, where A is the coordinate ring of a suitable affine open subset in $\mathbb{P}^m_S$. Since $\omega_{\mathbb{P}^m_K}$ is an invertible sheaf, we have thus an isomorphism

$$H^i(j_*f^!\mathcal{O}_S)_x \cong \mathrm{Ext}^{i+m}_{A_\mathfrak{m}}((A/I)_\mathfrak{m}, A_\mathfrak{m}),$$

In the following, we will write A instead of $A_\mathfrak{m}$. Note that if $\mathcal{P}$ is a projective resolution of length ℓ of A/I over A, then

$$\mathrm{Ext}^{i+m}_A(A/I, A) = H^{i+m}(\mathrm{Hom}_A(\mathcal{P}, A))$$

vanishes for $i + m > \ell$. Thus

$$\mathrm{Ext}^{i+m}_A(A/I, A) = 0 \quad \text{whenever } i + m > \mathrm{pd}_A\, A/I.$$

On the other hand, since A is a regular local ring, there is a local duality isomorphism [3, Theorem 4.4]

$$\mathrm{Ext}^{i+m}_A(A/I, A) \cong H^{-i}_\mathfrak{m}(A/I)^\vee,$$

where the right-hand sied is the $(-i)$-th local cohomology of A/I at $\mathfrak{m}$, and $^\vee$ denotes the so-called Matlis dual. By the properties of local cohomology, $H^{-i}_\mathfrak{m}(A/I) = 0$ for $-i > \dim A/I$. Phrased differently,

$$\mathrm{Ext}^{i+m}_A(A/I, A) = 0 \quad \text{whenever } i + m < m - \dim A = \mathrm{codim}_A\, A/I.$$

As a consequence, $\mathrm{Ext}^{i+m}_A(A/I, A) \neq 0$ can occur only for $m - n = \mathrm{codim}_A\, A/I \leq i + m \leq \mathrm{pd}_A\, A/I$. Hence the fact that only $\mathrm{Ext}^{m-n}_A(A/I, A)$ is possibly non-zero is equivalent to A/I being a Cohen-Macaulay module of codimension n over A. ☐

The Gorenstein case. For completeness, we mention the case of Gorenstein morphisms.

Definition (Gorenstein).

- A local ring $(A, \mathfrak{m})$ of dimension n is Gorenstein if $\mathrm{Ext}^i_A(A/\mathfrak{m}, A) = 0$ whenever $i \neq n$ and $\mathrm{Ext}^n_A(A/\mathfrak{m}, A) \cong A/\mathfrak{m}$.
- A morphism $f : X \to S$ of schemes is Gorenstein at $x \in X$ if f is flat at x and the local ring $\mathcal{O}_{X_{f(x)},x}$ is Gorenstein.
- A morphism $f : X \to S$ of schemes is Gorenstein if f is Gorenstein at all points of X.

Proposition 7 ([1, Theorem 3.5.1])**.** *With the same hypotheses as in Proposition 6, the following are equivalent:*

(a) f is Gorenstein,

(b) $H^i(f^!\mathcal{O}_S) = 0$ whenever $i \neq -n$ and $H^{-n}(f^!\mathcal{O}_S)$ is an invertible $\mathcal{O}_X$-module.

Remark. If A is a local ring, then

$$A \text{ regular local ring } \Rightarrow A \text{ Gorenstein } \Rightarrow A \text{ Cohen-Macaulay}.$$

Thus Proposition 6 and Proposition 2 imply Theorem 1.

The dualizing complex for general projective varieties. As a final application, we show that the dualizing complex has at most $\dim X$ non-vanishing cohomology sheaves when X is projective over a field.

Proposition 8. *Let $f : X \to \operatorname{Spec} K$ be projective of dimension n, K a field. Then $H^i(f^!\mathcal{O}_{\operatorname{Spec} K}) = 0$ for $i < -n$ and $i > 0$.*

Proof: We apply Grothendieck-Serre to a closed immersion $j : X \hookrightarrow \mathbb{P}^n_K$ and proceed as in the proof of Proposition 6 to obtain

$$H^i(j_*f^!\mathcal{O}_{\operatorname{Spec} K}) \cong \mathscr{E}xt^{i+m}_{\mathcal{O}_{\mathbb{P}^m_K}}(j_*\mathcal{O}_X, \omega_{\mathbb{P}^m_K}).$$

Going local to some open affine $\operatorname{Spec} A$, we have to consider $\operatorname{Ext}^{i+m}_A(A/I, A)$, since $\omega_{\mathbb{P}^m_K}$ is locally free of rank one. $\operatorname{Ext}^{i+m}_A(A/I, A) \neq 0$ means that the projective dimension of A/I over A is at least $i + m$. But the projective dimension is always bounded from above by $\dim A$, which equals m. Thus $\operatorname{Ext}^{i+m}_A(A/I, A) \neq 0$ implies $i + m \leq m$ and therefore $i \leq 0$.

Since $\mathscr{E}xt^{i+m}_{\mathcal{O}_{\mathbb{P}^m_K}}(j_*\mathcal{O}_X, \omega_{\mathbb{P}^m_K})$ is coherent on $\mathbb{P}^m_K$, the sheaves $\mathscr{E}xt^{i+m}_{\mathcal{O}_{\mathbb{P}^m_K}}(j_*\mathcal{O}_X, \omega_{\mathbb{P}^m_K})(d)$ are globally generated for sufficiently large $d \gg 0$. To show that $\mathscr{E}xt^{i+m}_{\mathcal{O}_{\mathbb{P}^m_K}}(j_*\mathcal{O}_X, \omega_{\mathbb{P}^m_K})$ vanishes, it is enough to show that

$$H^0(\mathbb{P}^m_K, \mathscr{E}xt^{i+m}_{\mathcal{O}_{\mathbb{P}^m_K}}(j_*\mathcal{O}_X, \omega_{\mathbb{P}^m_K})(d)) = 0 \quad \text{for } d \gg 0.$$

But now we know Serre duality for $\mathbb{P}^m_K$:

$$\begin{aligned} H^0(\mathbb{P}^m_K, \mathscr{E}xt^{i+m}_{\mathcal{O}_{\mathbb{P}^m_K}}(j_*\mathcal{O}_X, \omega_{\mathbb{P}^m_K})(d)) &\cong H^0(\mathbb{P}^m_K, \mathscr{E}xt^{i+m}_{\mathcal{O}_{\mathbb{P}^m_K}}(j_*\mathcal{O}_X, \omega_{\mathbb{P}^m_K}(d))) \\ &= \operatorname{Ext}^{i+m}(j_*\mathcal{O}_X, \omega_{\mathbb{P}^m_K}(d)) \\ &\cong H^{-i}(\mathbb{P}^m_K, j_*\mathcal{O}_X(-d))^\vee \\ &= H^{-i}(X, \mathcal{O}_X(-d))^\vee \\ &= 0 \quad \text{for } -i > n. \end{aligned}$$

So $H^i(j_*f^!\mathcal{O}_{\operatorname{Spec} K}) \neq 0$ can occur only for $-n \leq i \leq 0$. □

References

[1] Brian Conrad, *Grothendieck duality and base change*, Springer, 2000.

[2] Robin Hartshorne, *Residues and duality*, Springer, 1966.

[3] Craig Huneke, *Lectures on local cohomology*, available at `http://homepages.math.uic.edu/~bshipley/huneke.pdf`.

[4] J. Lipman, *Notes on derived functors and Grothendieck duality*, Foundations of Grothendieck Duality for Diagrams of Schemes, Lecture Notes in Math. **1960** (2009), 1–259, available at `http://http://www.math.purdue.edu/~lipman/Duality.pdf`.

[5] The Stacks Project Authors, *Stacks Project* (2014), available at `http://stacks.math.columbia.edu`.