Assignment-set 4 Applied Analysis

Due on 3 December 2007, 13.45u

1.) Find and classify the bifurcation points (x_c, μ_c) for the following equations

(a)

$$\dot{x} = \mu x - \frac{x}{1+x}$$

(b)

$$\dot{x} = 1 - \mu e^{-x^2}$$

2.) Consider the system

$$\dot{x} = rx - \sin(x).$$

(a) For the case $r = 0$, find and classify all the fixed points, and sketch the vector field.

(b) Show that when $r > 1$, there is only one fixed point. What kind of fixed point is it?

(c) As r decreases from ∞ to zero, classify all the bifurcations that occur.

(d) For $0 < r \ll 1$, find an approximate expression for values of r at which bifurcations occur.

(e) Sketch the bifurcation diagram for $r > 0$ (including indicating the stability).

3.) One of the most important developments in the study of firing nerve cells or neurons was the development of a model for this phenomenon in giant squid in the 1950s by Hodgkin and Huxley. They developed a four-dimensional system of differential equations that described the electrochemical transmission of neuronal signals along the cell membrane, a work for which they later received the Nobel prize. Roughly speaking, this system is similar to systems that arise in electrical circuits. The neuron consists of a cell body which receives electrical stimuli. This stimulus is then conducted along the axon, which can be thought of as a electrical cable that connects to other neurons via a collection of synapses. Of course, the motion is not really electrical, because the current is not really made up of electrons, but rather ions.

The four-dimensional Hodgkin-Huxley system is difficult to deal with, primarily because of the highly nonlinear nature of the equations. An important breakthrough from mathematical point of view came from Fitzhugh and Nagumo, who produced a simpler model of the Hodgkin-Huxley model. Although this system is not as biologically accurate as the original system, it nevertheless does capture the essential behaviour of nerve impulses.

The Fitzhugh-Nagumo system of equations is given by

$$\begin{aligned} \dot{x} &= y + x - \frac{x^3}{3} - I \\ \dot{y} &= -x + a - by \end{aligned}$$

where a and b are constants satisfying

$$0 < \frac{3}{2}(1-a) < b < 1$$

and I is a parameter. In these equations x is similar to the voltage and represents the *excitability* of the system; the variable y represents a combination of other forces that tend to return the system to rest. The parameter I is a stimulus parameter that leads to excitation of the system; I is like an applied current.

(a) First assume that $I = 0$. Prove that this system has a unique equilibrium point (x_0, y_0).
Hint: Use the geometry of the nullclines for this.

(b) Prove that this equilibrium point is always a sink.

(c) Now assume that $I > 0$. Prove that there is still a unique equilibrium point (x_I, y_I) and that x_I varies monotonically with I.

(d) Determine the values of x_I for which the equilibrium point is a source and show that there must be a stable limit cycle in this case.

(e) When $I > 0$, the point (x_0, y_0) is no longer an equilibrium point. Nonetheless we can still look at the solution through this point. Describe the qualitative nature of the solution as I moves away from 0. Make a sketch of x versus t of such a solution. Explain why biologists consider this phenomenon the 'excitement' of the neuron.