Assignment-set 3 Applied Analysis

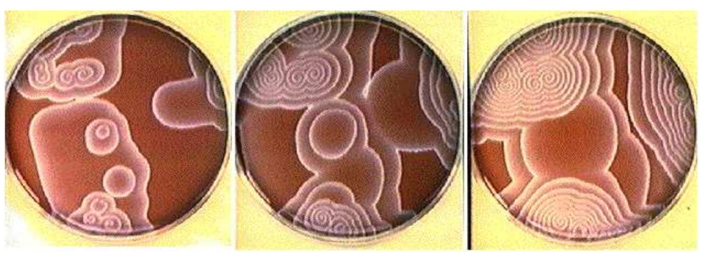
Due on 12 November 2007, 13.45u

1.) Show that

$$\begin{aligned} \dot{x} &= \frac{y}{1+x^2} \\ \dot{y} &= \frac{-x+(1+x^2+x^4)y}{1+x^2} \end{aligned}$$

has no limit cycle in $\mathbf{R}^2$.

2.) For much of the 20th century, chemists believed that all chemical reactions tended monotonically to equilibrium. This belief was shattered in the 1950s when the Russian Belousov discovered that a certain reaction involving acid, bromate ions, and sulfric acid, when combined with a cerium catalyst, could oscillate for long periods of time before settling to equilibrium. The concoction would turn yellow for a while, then fade, then turn yellow again, then fade, and on and on like this for over an hour. In the literature, this behaviour is called a chemical clock, and is often studied as a prototype of a simple biological system. The reaction, now called the Belousov-Zhabotinski reaction (the BZ reaction, for short), was a major turning point in the history of chemical reactions. Nowadays, many systems are known to oscillate. Some have even been shown to behave chaotically. Also, behaviour as shown below is found.



From left to right: Propagating oxidation waves in an unstirred layer of the ferroin-malonic acid BZ reaction. When the wave is broken at a certain point (for example by a gentle airflow through a pipette) a pair of spiral waves develop at this point.

One particular chemical reaction is given by a chlorine dioxide-iodine-malonic acid interaction. The exact differential equations modelling this reaction are extremely complicated. However, there is a planar nonlinear system that closely approximates the concentrations of two of the reactants. The system is

$$\begin{aligned} \dot{x} &= a - x - \frac{4xy}{1+x^2} \\ \dot{y} &= bx\left(1 - \frac{y}{1+x^2}\right) \end{aligned}$$

where x and y represent concentrations of I^- and ClO_2^-, respectively, and a and b are positive parameters.

(a) Find all the equilibrium points for this system and characterise these equilibria.

(b) In the (a, b)-plane, sketch the regions where you find asymptotically stable or unstable equilibria.

(c) Identify the a, b-values where the system undergoes bifurcations.

(d) Find the a, b-values for which a stable limit cycle exists. Why do these values correspond to oscillating chemical reactions?
Hint: Use the nullclines and the Poincaré-Benixson theorem.

3.) In this exercise we will study the travelling wave solutions to the sine-Gordon equation:

$$u_{tt} - u_{xx} + \sin(u) = 0.$$

Travelling wave solutions depend only on the travelling coordinate $\xi = x - ct + x_0$ with speed c. That is, we look for solutions of the form $u(x, t) = \phi(\xi)$.

(a) Insert this assumption into the equation and give the equation that ϕ has to satisfy. Then, write this second order ODE into a first order system.

(b) Determine the fixed points and characterise these.

(c) The system is Hamiltonian, determine the Hamiltonian H.

(d) Combining the results from b. and c. sketch the phase-plane. Note that the sketch is different for $|c| < 1$ and for $|c| > 1$.
Sketch all the heteroclinic and homoclinic orbits in the phase-plane in a different colour.

(e) Make a sketch of the functions $\phi(\xi)$ and $\phi_\xi(\xi)$ corresponding to these heteroclininc and homoclinic orbits.

(f) Now, set $|c| < 1$. Show that

$$\phi(\xi) = 4 \arctan(e^{a\xi})$$

is a travelling wave solution and determine a.