SEMICONDUCTOR LASER WITH FILTERED OPTICAL FEEDBACK: BRIDGE BETWEEN CONVENTIONAL FEEDBACK AND OPTICAL INJECTION

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Abstract

We study a semiconductor laser subject to filtered optical feedback, where the filter is characterised by a mean frequency Ω_m and a filter width λ. In the limit of a narrow filter ($\lambda \rightarrow 0$) the laser equations reduce under some conditions to the equations for a laser with optical injection, whereas they become the Lang-Kobayashi equations in the limit of an unbounded filter width ($\lambda \rightarrow \infty$). We vary the parameter λ and study bifurcations of steady state solutions, to get insight in the relation between the filter width and the number of possible steady states. In particular we obtain the unexpected result, that there exist parameter regions in which the number of possible steady states decreases when λ is increased.

Key words

semiconductor laser, filter, bifurcation analysis

1 Introduction

The laser with filtered external optical feedback is in its simplest version modelled mathematically by three equations. The first two, for the (complex) electric field E and the (real) inversion n with respect to the threshold inversion, are given by a rescaled version of the so-called Lang-Kobayashi equations (Lang and Kobayashi, 1980; Krauskopf and Lenstra, 2000):

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2}(1+i\alpha)\xi n(t)E(t) + \kappa E(t-\tau)e^{-i\omega_0\tau}, \\ \frac{dn}{dt} &= J - J_{thr} - \frac{n(t)}{T_1} - (\Gamma_0 + \xi n(t))|E(t)|^2. \end{aligned} \tag{1.1}$$

Here time t is scaled with ω_r^{-1}, where ω_r is the frequency of the relaxation oscillation, an intrinsic resonance of the laser. The parameter α is the linewidth enhancement factor, ξ is the differential gain, ω_0 is the solitary laser angular frequency, Γ_0 is the photon decay rate, T_1 is the carrier lifetime and J and J_{thr} are the pump current and its value at solitary laser threshold. The parameter κ measures the injected field strength.

A straightforward way to let a laser operate in a single longitudinal mode, is by filtering the feedback light. Light with certain frequencies can pass the filter, while other frequencies are filtered. In the Lang-Kobayashi equations (1.1) a delay term $\kappa E(t-\tau)e^{-i\omega_0\tau}$ appears, modelling feedback from an external flat mirror. To incorporate the filter, this term is replaced by a term γF, where the filtered electric field F satisfies

$$\frac{dF}{dt} - (i\omega_m - \Lambda)F = \Lambda E(t-\tau)e^{-i\omega_0\tau} \tag{1.2}$$

and Λ is related to the filter width. Hence the full model we analyse is

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2}(1+i\alpha)\xi nE + \gamma F, \\ \frac{dn}{dt} &= J - J_{thr} - \frac{n(t)}{T_1} - (\Gamma_0 + \xi n(t))|E(t)|^2, \\ \frac{dF}{dt} &= \Lambda E(t-\tau)e^{-i\omega_0\tau} + (i\omega_m - \Lambda)F. \end{aligned} \tag{1.3}$$

See (Yousefi and Lenstra, 1999) and references therein for an explanation, a derivation and numerical studies of the resulting system.

These first studies were followed by a series of numerical and experimental studies (Fischer *et al.*, 2000; Yousefi *et al.*, 2001*a*; Yousefi *et al.*, 2002; Yousefi *et al.*, 2003) in which various types of behaviour were found and studied. However, analytically, even the simplest behaviour of the laser with filtered feedback is still hardly understood. A first attempt towards an analytical approach is to study limit cases of the problem. One case is the limit $\Lambda \rightarrow \infty$, in which the problem reduces to a laser with conventional optical feedback, that is, a laser without filtering. In this limit the equation for F simply becomes the algebraic relation

$F = E(t-\tau)e^{-i\omega_0\tau}$, which means that the equations reduce to the Lang-Kobayashi equations (1.1). The second limit case is the limit $\Lambda \to 0$ of the small filter width. We show that this small filter width limit of the laser with filtered feedback is in fact a single optically injected semiconductor laser. However, since the injection is still obtained through feedback, the feedback delay appears to play a role in the selection of frequency and phase shift of the electric field E of the laser.

Although in experiments filters will satisfy neither of these limits, analysis of the parameter regions $0 < \Lambda \ll 1$ and $0 < \frac{1}{\Lambda} \ll 1$ may help to understand the dynamics that are observed in other regions. The aim of this article is to try and understand how the filter equations form a bridge between these two limits, and thus between the standard single injection laser equations and Lang-Kobayashi (LK-)equations. To do so, we rescale the system (1.3) to a set of dimensionless equations as is done more often in the literature. Defining new variables and dimensionless parameters by

$$\begin{gathered} N = \frac{\xi n}{2\Gamma_0}, \quad E \to \sqrt{\frac{\xi T_1}{2}}E, \quad F \to \sqrt{\frac{\xi T_1}{2}}F, \\ s = \Gamma_0 t, \quad T \equiv \Gamma_0 T_1, \quad \theta \equiv \Gamma_0 \tau, \\ P \equiv \frac{\xi T_1}{2\Gamma_0}(J - J_{thr}), \quad \Gamma \equiv \frac{\gamma}{\Gamma_0}, \quad \lambda \equiv \frac{\Lambda}{\Gamma_0}, \\ \Omega_0 \equiv \frac{\omega_0}{\Gamma_0} \quad \text{and} \quad \Omega_m \equiv \frac{\omega_m}{\Gamma_0}, \end{gathered} \tag{1.4}$$

equations (1.3) can be rewritten in the following form

$$\begin{aligned} E' &= (1+i\alpha)NE + \Gamma F, \\ TN' &= P - N - (1+2N)|E|^2, \\ F' &= \lambda E(s-\theta)e^{-i\Omega_0\theta} + (i\Omega_m - \lambda)F. \end{aligned} \tag{1.5}$$

Here prime means differentiation with respect to s and the filter width is represented by λ. Note that ω_m is in fact the detuning of the mean filter frequency with respect to the solitary laser frequency ω_0; hence the center frequency of the filter is denoted by $\omega_f \equiv \omega_m + \omega_0$ or $\Omega_f = \Omega_m + \Omega_0$.

We restrict ourselves to the study of 'fixed points' or so-called external filtered modes (EFMs) and the bifurcations associated to these states that occur when the filter parameter λ (Λ) is varied. Although the equations for the fixed points have been written down many times before, they haven't been analysed in a concise way yet. In this article, we assemble various pieces of the EFM puzzle.

Firstly, we carefully analyse the EFMs in the limit $\lambda \to 0$ and find that solutions either have a frequency Ω with $|\Omega - \Omega_m| = \mathcal{O}(\lambda)$, or a frequency $\Omega = \mathcal{O}(\lambda)$. In the first case, the laser synchronises with the filter, as one wishes in this set-up. In the second case, only a small portion of the electric field E passes the filter, *i.e.*, $|F| = \mathcal{O}(\lambda)$ while $|E| = \mathcal{O}(1)$, and the laser operates close to its solitary laser frequency.

We also find that for $0 < \lambda \ll 1$ the laser can only operate $\mathcal{O}(\lambda)$ close to mean filter frequency Ω_m if Ω_m is chosen such that it lies in one of a number of bounded intervals $I^j_{\Omega_m}$. The number of such intervals varies depending on the parameters α, Γ, θ and Ω_0. One, $I^0_{\Omega_m}$, is centered around $\Omega_m = 0$ and exists for all parameter choices.

Secondly, by comparing the limits $\lambda \to 0$ and $\lambda \to \infty$, we find that there exist parameter regions in which the injection limit $\lambda \to 0$ has three EFM solutions (steady states, named locked modes if they are stable), whereas the LK-limit $\lambda \to \infty$ has only one such solution (named ECM – external cavity mode – for a laser with conventional feedback). This is somewhat surprising, since numerically, so far, only increments of the number of EFMs to (1.5) have been observed for increasing λ.

The last part of this work is a bifurcation analysis, in which we analyse saddle-node (SN-)bifurcations of EFMs that occur as λ is varied. We determine conditions under which the bifurcation is creating of nature (two EFMs appear as λ is increased) or annihilating of nature (two EFMs disappear as λ is increased). We conclude that all bifurcations of EFMs with frequency Ω between 0 (solitary laser frequency) and Ω_m (mean frequency of the filter) are creating of nature, which corresponds to observations in numerical studies (Yousefi and Lenstra, 1999; Yousefi *et al.*, 2001*b*; Yousefi *et al.*, 2001*a*): the two 'islands' of EFMs that have been observed for low values of λ get connected in a series of bifurcations, in which additional EFMs appear for increasing λ and no annihilations occur in between.

2 Existence of EFMs

The basic solution of laser equations is the trivial solution $(E, N, F) = (0, P, 0)$ of (1.5). When this basic solution is stable there is no lasing activity; the laser is off. Increasing the value of the pump P it becomes unstable at the laser threshold $P = 0$: the laser starts lasing and so-called external filtered modes or EFMs are formed. Such solutions have constant intensities and inversion, and a phase that depends linearly on time:

$$E(s) = Re^{i(\Omega s + \phi_0)}, \quad F(s) = Se^{i\Omega s} \text{ and } N(s) = N, \tag{2.1}$$

where R, S, N, Ω and ϕ_0 are constants. They are often referred to as 'fixed points'. Since the electrical fields E and F are optically related, both have the same frequency, possibly with a phase shift ϕ_0. These solutions are easiest studied in a polar coordinate setting, with Ωs defined as the linear part of the phase of F:

$$\begin{aligned} E(s) &= R(s)e^{i\tilde{\phi}(s)} = R(s)e^{i(\Omega s + \phi(s))} \\ F(s) &= S(s)e^{i\tilde{\psi}(s)} = S(s)e^{i(\Omega s + \psi(s))}. \end{aligned} \tag{2.2}$$

Note here, that Ω is the detuning of the frequencies of E and F with respect to the solitary laser frequency

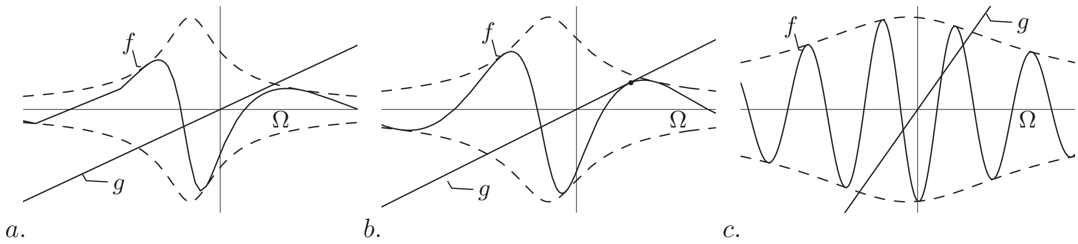

Figure 1. Plot of f and g as functions of Ω as the parameter λ is increased. a. Only one EFM exists b. Two EFMs are formed in a saddle-node bifurcation. c. There are three EFMs.

Ω_0, *i.e.*, the laser operates at $\Omega_L = \Omega + \Omega_0$.

Remark 2.1. In standard modelling of lasers with injection, Ω is defined as the detuning of the *injected field* from the solitary laser frequency. Here F is in fact the injected field, and it is thus natural to put Ωs as the full linear part of F rather than E. This choice means that $\psi(s)$ is purely nonlinear, and that $\phi(s)$ may contain constant terms. Note however, that in (Yousefi and Lenstra, 1999) the opposite choice has been made in the definition of the 'fixed points'.

Using (2.2) for $R, S \neq 0$, we write (1.5) as

$$\begin{aligned} R' &= NR + \Gamma S \cos(\psi - \phi) \\ \phi' &= \alpha N + \Gamma \tfrac{S}{R} \sin(\psi - \phi) - \Omega \\ S' &= \lambda R(s-\theta)\cos(\phi(s-\theta) - \psi(s) - \Omega_0\theta - \Omega\theta) \\ &\quad -\lambda S \\ \psi' &= \lambda \tfrac{R(s-\theta)}{S(s)} \sin(\phi(s-\theta) - \psi(s) - \Omega_0\theta - \Omega\theta) \\ &\quad + \Omega_m - \Omega \\ TN' &= P - N - (1+2N)R^2. \end{aligned} \tag{2.3}$$

EFMs (2.1) satisfy $R' = \phi' = S' = \psi' = N' = 0$. The resulting conditions can, for $\lambda \neq 0$, be written as

$$\Gamma^2 \tfrac{S^2}{R^2} = N^2 + (\Omega - \alpha N)^2 \tag{2.4}$$
$$\tan(\phi_0) = \tfrac{\Omega - \alpha N}{N} \tag{2.5}$$
$$\tfrac{S^2}{R^2} = \tfrac{\lambda^2}{\lambda^2 + (\Omega - \Omega_m)^2} \tag{2.6}$$
$$\tan(\phi_0 - \Omega_0\theta - \Omega\theta) = \tfrac{\Omega - \Omega_m}{\lambda} \tag{2.7}$$
$$R^2 = \tfrac{P-N}{1+2N}. \tag{2.8}$$

In the LK-limit, $\lambda \to \infty$, equation (2.6) reduces to $S^2 = R^2$ and hence (2.4) is an ellipse in the (Ω, N)-plane: it resembles the fixed point ellipse that is well-known in the analysis of the LK-equations. The equation (2.8) is the relation between R and N as also found in the Lang-Kobayashi and injection laser equations. Note that (2.8) implies that solutions with real amplitude R, the physically relevant solutions, should satisfy $-\frac{1}{2} < N < P$, for $P > -\frac{1}{2}$. See (Rottschäfer and Krauskopf, 2004).

From (2.3) one can also deduced that EFMs satisfy $N = -\Gamma \frac{S}{R}\cos(\phi_0) = -\Gamma \cos(\phi_0 - \Omega_0\theta - \Omega\theta)\cos(\phi_0)$ for $\lambda \neq 0$, so (2.5) and (2.7) can be rewritten as

$$\begin{aligned} \Omega &= -\Gamma \cos(\phi_0 - \Omega_0\theta - \Omega\theta)\cos(\phi_0)(\tan(\phi_0) + \alpha) \\ \Omega &= \lambda \tan(\phi_0 - \Omega_0\theta - \Omega\theta) + \Omega_m. \end{aligned} \tag{2.9}$$

Although the equality $\frac{S}{R} = \cos(\phi_0 - \Omega_0\theta - \Omega\theta)$ is not necessarily satisfied in an EFM in the limit $\lambda \to 0$, it is useful to analyse the resulting equations (2.9) in this limit. By elimination of ϕ_0, we derive from (2.9)

$$\begin{aligned} \Omega &= -\Gamma \cos(\phi_0 - \Omega_0\theta - \Omega\theta)(\sin\phi_0 + \alpha\cos\phi_0) \\ &= -\Gamma \cos(\phi_0 - \Omega_0\theta - \Omega\theta) \times \\ &\qquad \sqrt{1+\alpha^2}\sin(\phi_0 + \arctan(\alpha)) \\ &= -\Gamma \sqrt{\tfrac{\lambda^2(1+\alpha^2)}{\lambda^2 + (\Omega-\Omega_m)^2}} \times \\ &\qquad \sin(\Omega_0\theta + \Omega\theta + \arctan\left(\tfrac{\Omega-\Omega_m}{\lambda}\right) + \arctan(\alpha)), \end{aligned} \tag{2.10}$$

which corresponds to the equations in (Yousefi and Lenstra, 1999). The ϕ_0 is only uniquely defined up to π-periodicity, but Ω is equal for all $\phi_0 + k\pi$, $k \in \mathbf{Z}$.

Every solution Ω of (2.10) corresponds to exactly one EFM. Once such a value for Ω is determined, the unknowns R, S, N and ϕ_0 can be obtained. However, equation (2.10) is transcendental in Ω and it can therefore not be solved explicitly.

The main question adressed in this article is, how the number of EFMs varies with λ, *i.e.*, when the set-up is varied from an injection laser to a laser with external feedback. In order to study the EFMs, we define the functions

$$\begin{aligned} f(\Omega;\lambda) &= -\Gamma \sqrt{\tfrac{\lambda^2(1+\alpha^2)}{\lambda^2 + (\Omega-\Omega_m)^2}} \times \\ &\qquad \sin(\Omega_0\theta + \Omega\theta + \arctan\left(\tfrac{\Omega-\Omega_m}{\lambda}\right) + \arctan(\alpha)), \end{aligned} \tag{2.11}$$
$$g(\Omega;\lambda) = \Omega. \tag{2.12}$$

Thus the solutions of (2.10) correspond to intersections of f and g, and SN-bifurcations of EFMs occur when both $f = g$ and $f_\Omega = g_\Omega$. A combination of these two requirements will lead to our bifurcation results. In figure 1, the functions f and g are plotted as λ is increased.

3 Analysis in the limit $\lambda \to 0$

In this section, we study system (1.5) in the limit $\lambda \to 0$ and especially focus on the number of EFMs. If $\Omega_m = \mathcal{O}(1)$ and if the solution satisfies $|F| = \mathcal{O}(1)$ with respect to λ, the third equation in (1.5) resembles, for $0 < \lambda \ll 1$, standard circular motion with frequency Ω_m in first order. Indeed, for $\lambda = 0$, the equations in polar coordinates (2.3) yield $S' = 0$ and $\psi' = \Omega_m - \Omega$, which, by the requirement that ψ contains only nonlinear terms in s, results in $\Omega = \Omega_m$ and $S = \text{const.}$, so $F(s) = Se^{i\Omega_m s}$. Substituting this into the remaining equations of (1.5) yields

$$\begin{aligned} E' &= (1 + i\alpha)NE + \Gamma S e^{i\Omega_m s}, \\ TN' &= P - N - (1 + 2N)|E|^2, \end{aligned} \tag{3.1}$$

which are exactly the equations for a single injection laser. Hence when $\lambda = 0$, the injected field strength $|F|$ is fixed and the detuning frequency Ω equals Ω_m. The 5-dimensional system (1.5) for the filtered laser reduces to the 3-dimensional system describing the single injection laser. The consequence is, that the phase space of a single injection laser serves as a skeleton for the behaviour of the semiconductor laser with filtered feedback, as long as the filter width is sufficiently small ($0 < \lambda \ll 1$).

However, one should be careful here. Equations (2.6), (2.7), (2.10) suggest a singular perturbation analysis and introduction of a new independent variable $y = (\Omega - \Omega_m)/\lambda$. Considering y as an $\mathcal{O}(1)$ independent variable corresponds well with the intuitive expectation that the laser will operate at a frequency close to the filter mean: $\Omega - \Omega_m = \mathcal{O}(\lambda)$. In the sequel, we analyse the fixed points, studying equation (2.10) for $\Omega = \Omega_m + \lambda y$, with $y = \mathcal{O}(1)$ and $\lambda \ll 1$. When $\Omega - \Omega_m = \mathcal{O}(1)$ however, we introduce $x = \Omega - \Omega_m$ and study (2.10) for $\Omega = \Omega_m + x$, with $x = \mathcal{O}(1)$.

Remark 3.1. We assume that $\Omega_m = \mathcal{O}(1)$ throughout the analysis. The more particular case $\Omega_m = \mathcal{O}(\lambda)$ should be considered separately. We refer to (Hek and Rottschäfer, 2005) for this analysis.

3.1 The case $\Omega - \Omega_m = \mathcal{O}(\lambda)$

For $\Omega - \Omega_m = \mathcal{O}(\lambda)$ we set $\Omega = \Omega_m + \lambda y$ so that equation (2.10) becomes

$$\Omega_m + \lambda y = -\Gamma\sqrt{\tfrac{1+\alpha^2}{1+y^2}} \times \sin[(\Omega_0 + \Omega_m + \lambda y)\theta + \arctan(y) + \arctan(\alpha)], \tag{3.2}$$

Although (2.10) is not necessarily satisfied in the EFMs for $\lambda = 0$, see section 2, the $\lambda \downarrow 0$ limit of (3.2)

$$\Omega_m = -\Gamma\sqrt{\tfrac{1+\alpha^2}{1+y^2}} \sin[(\Omega_0 + \Omega_m)\theta + \arctan(y) + \arctan(\alpha)]. \tag{3.3}$$

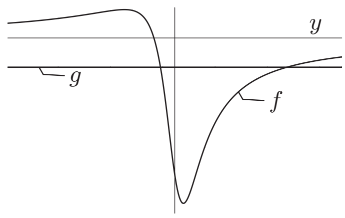

Figure 2. Plot of $f(y;0)$ and $g(y;0)$ as functions of y for a general (negative) value of Ω_0 with $\cos D \neq 0, \sin D \neq 0$. When Ω_m is varied g moves up or down and f changes form, which leads to bifurcation of intersection points of f and g, the EFMs.

does play a role as limiting equation. Equation (3.2) can again be written as $f(y;\lambda) = g(y;\lambda)$, with $\Omega = \Omega_m + \lambda y$ substituted into (2.11) and (2.12). Hence, EFMs for zero or small λ can be studied as intersections of f and g. With $C := \Gamma\theta\sqrt{1+\alpha^2}$, the classical feedback strength, $D := (\Omega_0 + \Omega_m)\theta + \arctan(\alpha)$ and $\lambda = 0$ we write

$$\begin{aligned} f(y;0) &= -\frac{C}{\theta\sqrt{1+y^2}} \sin[\arctan(y) + D] \\ &= \frac{-C(y\cos D + \sin D)}{\theta(1+y^2)}, \end{aligned} \tag{3.4}$$

$$g(y;0) = \Omega_m. \tag{3.5}$$

The function $f(y;0)$ has, unless $\cos D = 0$, a single zero at $y = -\tan D$ and two extreme values, see figure 2. Furthermore, f satisfies $\lim_{|y|\to\infty} f(y;0) = 0$.

We only treat the case $\cos D \neq 0$ here; for the more degenerate case $\cos D = 0$ we refer to (Hek and Rottschäfer, 2005). When $\cos D \neq 0$ the extrema are

$$f(y_+) = \frac{-C\cos^2 D}{\theta(2 - 2\sin D)} = -\frac{C}{2\theta}(1 + \sin D) < 0 \quad \text{with } y_+ = \frac{-\sin D + 1}{\cos D}, \tag{3.6}$$

$$f(y_-) = \frac{C\cos^2 D}{\theta(2 + 2\sin D)} = \frac{C}{2\theta}(1 - \sin D) > 0 \quad \text{with } y_- = \frac{-\sin D - 1}{\cos D}. \tag{3.7}$$

See figure 2. Note that $|f(y_+)| \neq |f(y_-)|$ unless $\sin D = 0$, in which case also $f(0;0) = 0$ and f is an odd function. Since $g(y;0) = \Omega_m$ for $\lambda = 0$, it is clear that $f(y;0) = g(y;0)$ has two solutions when Ω_m lies between the two extrema of f:

$$-\frac{C}{2\theta}(1 + \sin D) < \Omega_m < \frac{C}{2\theta}(1 - \sin D). \tag{3.8}$$

An equality on either side corresponds to a bifurcation value: SN-bifurcations where two EFMs are formed or disappear take place at $\Omega_m = \pm\frac{C}{2\theta}(1 \mp \sin D)$. Substituting the expressions for C and D, we see that the bifurcations occur for those Ω_m satisfying

$$-\tfrac{\Gamma}{2}\sqrt{1+\alpha^2}(\pm 1+\sin((\Omega_0+\Omega_m)\theta+\arctan\alpha)))=\Omega_m. \tag{3.9}$$

So depending on the parameters, there exist a number of bounded intervals in Ω_m-space in which $f = g$ has, for $\lambda = 0$, two solutions. The parameter Ω_m can be varied to enter or leave these intervals, *i.e.*, to gain or loose two EFMs via a SN-bifurcation. See figure 2.

Furthermore, the laser operates at a frequency $\Omega = \Omega_m + \lambda y$. Hence the fact that there are only a number of bounded intervals in Ω_m-space in which $f(y;0) = g(y;0)$ has two solutions, while $f(y;0) = g(y;0)$ has no solutions for parameter values Ω_m outside these intervals, says the following: given fixed parameters α, Γ, θ and Ω_0, the laser can in the small-λ regime only operate at frequencies Ω in a number of bounded intervals. If Ω_m is chosen in such interval, the laser will (probably) operate at a frequency $\mathcal{O}(\lambda)$ close to this Ω_m. If Ω_m is more than $\mathcal{O}(\lambda)$ away from these intervals, there are no EFMs with $\Omega = \Omega_m + \lambda y$.

We now study the Ω_m-intervals where two EFMs exist. Their number changes when both equation (3.9) holds and the derivatives with respect to Ω_m at both sides of this equation are equal. Differentiating (3.9) with respect to Ω_m yields $\cos(\Omega_m\theta + D_1) = -\frac{2}{C}$, with C as above and $D_1 := \Omega_0\theta + \arctan(\alpha)$. Therefore,

$$\Omega_m\theta + D_1 = \pm\arccos\left(-\tfrac{2}{C}\right) + 2k\pi, \tag{3.10}$$

$$\sin(\Omega_m\theta + D_1) = \pm\sqrt{1-\tfrac{4}{C^2}}. \tag{3.11}$$

These two equations combined with (3.8) yield that an Ω_m-interval is created or annihilated precisely when

$$\begin{aligned} -\tfrac{1}{2}C\left[1 \pm \sqrt{1-\tfrac{4}{C^2}}\right] + D_1 &= \Omega_m\theta + D_1 \\ &= \pm\arccos\left(-\tfrac{2}{C}\right) + 2k\pi \end{aligned} \tag{3.12}$$

$$\text{or}\quad \begin{aligned} -\tfrac{1}{2}C\left[1 \pm \sqrt{1-\tfrac{4}{C^2}}\right] + D_1 + C &= \Omega_m\theta + D_1 \\ &= \pm\arccos\left(-\tfrac{2}{C}\right) + 2k\pi, \end{aligned} \tag{3.13}$$

where the plus and minus signs correspond to each other. Rewriting these equations and substituting D_1, we see that creation/annihilation of an interval is in $(\alpha, \Gamma, \theta, \Omega_0)$-parameter space determined by

$$\begin{aligned} \Omega_0\theta &= \pm\arccos(-\tfrac{2}{C}) + \tfrac{1}{2}C \\ &\quad \pm\sqrt{\tfrac{C^2}{4}-1} - \arctan\alpha + 2k\pi, \\ \Omega_0\theta &= \pm\arccos(-\tfrac{2}{C}) - \tfrac{1}{2}C \\ &\quad \pm\sqrt{\tfrac{C^2}{4}-1} - \arctan\alpha + 2k\pi. \end{aligned} \tag{3.14}$$

Note that these expressions do not depend on the parameter P, the pump, and that these equalities can only occur when $\frac{C^2}{4} - 1 \geq 0$, hence when $C > 2$. When $C < 2$, the number of Ω_m-intervals remains fixed to one that always exists: an interval around $\Omega_m = 0$ of which the size decreases to zero as $\Gamma \to 0$. The curves (3.14) are plotted in figure 3 in the $(\Omega_0\theta, \Gamma)$-parameter space where the number of Ω_m-intervals is given in the different regions.

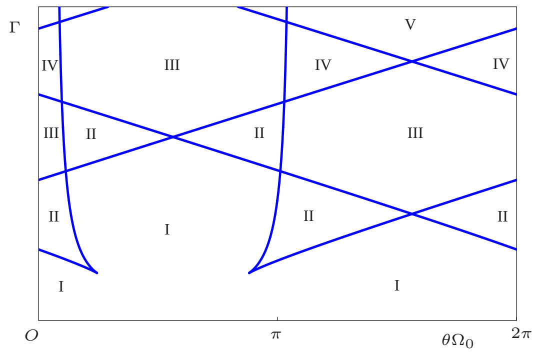

Figure 3. The curves (3.14) and the number of intervals $I^j_{\Omega_m}$ for Ω_m in $(\Omega_0\theta, \Gamma)$-parameter space for $\lambda \to 0$. When Ω_m in chosen in these intervals, there exist 3 EFMs.

From (3.9) and (3.14) we conclude the following

Lemma 3.2. *Let $\lambda \to 0$ and fix the parameters α, Ω_0, Γ and θ such that $\cos D := \cos(\Omega_0 + \Omega_m)\theta + \arctan(\alpha) \neq 0$. If $C = \Gamma\theta\sqrt{1+\alpha^2} < 2$, there is a single open interval $I^0_{\Omega_m}$ around $\Omega_m = 0$ such that there are two solutions Ω to (2.10), hence two EFMs, with $y = \frac{\Omega-\Omega_m}{\lambda} = \mathcal{O}(1)$ when $\Omega_m \in I^0_{\Omega_m}$ and none if $\Omega_m \notin \overline{I^0_{\Omega_m}}$.*
For $C = \Gamma\theta\sqrt{1+\alpha^2} > 2$, there is a number of intervals $I^j_{\Omega_m}$ such there are two solutions Ω to (2.10) with $y = \frac{\Omega-\Omega_m}{\lambda} = \mathcal{O}(1)$ if $\Omega_m \in I^j_{\Omega_m}$ for some j and none if $\Omega_m \notin \overline{I^j_{\Omega_m}}$. The locus in parameter space of the creations of these regions is given by (3.14).

Hence, the number of EFMs with $\frac{\Omega-\Omega_m}{\lambda} = \mathcal{O}(1)$ changes between zero and two as Ω_m is varied. Note that, since θ is large in practice, the condition $C > 2$ will be satisfied unless there is a very small $\mathcal{O}(\frac{1}{\theta})$ coupling strength Γ.

The boundaries of the intervals $I^j_{\Omega_m}$ are given by the (transcendental) equation (3.9).

3.2 The case $\Omega - \Omega_m = \mathcal{O}(1)$

To analyse the case $\Omega - \Omega_m = \mathcal{O}(1)$, we introduce $\Omega = \Omega_m + x$ with $\Omega_m = \mathcal{O}(1)$ and $x = \mathcal{O}(1)$, and write the functions f and g as functions of x:

$$\begin{aligned} f(x;\lambda) &= -\lambda\tfrac{C}{\theta\sqrt{\lambda^2+x^2}}\sin\left(x\theta + \arctan\left(\tfrac{x}{\lambda}\right) + D\right), \\ g(x;\lambda) &= x + \Omega_m, \end{aligned} \tag{3.15}$$

where again $C = \Gamma\theta\sqrt{1+\alpha^2}$ and $D = (\Omega_0 + \Omega_m)\theta + \arctan\alpha$. The limit of $f(x;\lambda)$ as $\lambda \to 0$ is well-defined for all $|x| > \mathcal{O}(\lambda)$ and satisfies $f(x;0) = 0$ for every $|x| > \mathcal{O}(\lambda)$. For $x = \mathcal{O}(\lambda)$ one should consider $x = \lambda y$, and (3.2) would be obtained. The function f in (3.15) has its zeroes where $\arctan(\frac{x}{\lambda}) + x\theta + D = k\pi$. As $\lambda \to 0$, $\arctan(\frac{x}{\lambda}) \to \frac{\pi}{2}$ for $x > 0$ and $\arctan(\frac{x}{\lambda}) \to -\frac{\pi}{2}$ for $x < 0$, hence the zeroes of $f(x;\lambda)$ at either side of $x = 0$ are π/θ apart as $\lambda \to 0$.

For $\lambda = 0$, the functions $f(x;0)$ and $g(x;0)$ intersect in $x = -\Omega_m$ since f is identical zero. By the implicit function theorem it follows that there exists a unique intersection point of $f(x;\lambda)$ and $g(x;\lambda)$ with $(x;\lambda) \in (\mathbf{R}\backslash[-X,X]) \times [0,\lambda_0)$, for some $X = \mathcal{O}(\lambda)$, as long as λ is small enough. We approximate this intersection point using that its x-coordinate must lie close to $-\Omega_m$: we set $x = -\Omega_m + \lambda z$. Then $f(z;\lambda) = g(z;\lambda)$ gives in the limit $\lambda \to 0$, with C and D substituted

$$z = -\tfrac{\Gamma\sqrt{1+\alpha^2}}{\Omega_m} \cos(\Omega_0\theta + \arctan\alpha). \tag{3.16}$$

An expression for Ω where f and g intersect, hence for the Ω-value of an EFM, can now be obtained by using $\Omega = \Omega_m + x = \lambda z$. We conclude

Lemma 3.3. *For $\lambda \to 0$ and α, Ω_0, $\Omega_m = \mathcal{O}(1)$, Γ and θ fixed, there exists a solution Ω to (2.10) with $x = \Omega - \Omega_m = \mathcal{O}(1)$. This solution is constant up to and including order $\mathcal{O}(\lambda)$, and is approximated by* $\Omega = -\lambda\frac{\Gamma\sqrt{1+\alpha^2}}{\Omega_m} \cos(\Omega_0\theta + \arctan\alpha)$.

Remark 3.4. This solution with $\Omega - \Omega_m = \mathcal{O}(1)$, corresponds to a solution with $|\Omega_L - \Omega_0| = \mathcal{O}(\lambda)$ in (Erneux *et al.*, 2004), whereas the solutions with $\Omega - \Omega_m = \mathcal{O}(\lambda) = \lambda y$ correspond to the limit $|\Omega_L - \Omega_f| = \mathcal{O}(\lambda)$.

Remark 3.5. The solution with $\Omega - \Omega_m = \mathcal{O}(1)$ must by (2.6) necessarily satisfy $\frac{S}{R} = \mathcal{O}(\lambda)$. This corresponds to the intuition that if the injected field strength S is (too) small, the laser prefers to operate at its own solitary laser frequency Ω_0 instead of the frequency of the injected light (with mean $\Omega_f = \Omega_m + \Omega_0$).

To conclude we combine Lemmas 3.2 and 3.3 that together give a complete overview of the number of EFMs (number of solutions of (2.10)) that exist in various parameter regions. From Lemma 3.3 it follows that for $\lambda \to 0$, there always exists one EFM. Combining this with Lemma 3.2, where depending on Ω_m zero or two EFMs exist, we find that as Ω_m is varied, the total number of EFMs for (1.5) in the limit $\lambda \to 0$ changes between one and three, see figure 3.

4 Analysis in the limit $\lambda \to \infty$

In this section, we study the system (1.5) as $\lambda \to \infty$. Again, there are different cases to study for the two orders of magnitude of Ω_m, of which we only treat $\Omega_m = \mathcal{O}(1)$ here.

In the limit $\lambda \to \infty$, with $\Omega_m = \mathcal{O}(1)$ the equation for F in (1.5) simply reduces to the algebraic relation $F(s) = E(s-\theta)e^{-i\Omega_0\theta}$ as long as $F = \mathcal{O}(1)$. This means that the equations (1.3) reduce to the LK-equations (1.1), and system (1.5) becomes the rescaled form of the LK-equations as studied in (Rottschäfer and Krauskopf, 2004). With $\mu \equiv \frac{1}{\lambda}$ it can be written as

$$\begin{aligned} E' &= (1+i\alpha)NE + \Gamma F, \\ TN' &= P - N - (1+2N)|E|^2, \\ \mu F' &= E(s-\theta)e^{-i\Omega_0\theta} + (i\mu\Omega_m - 1)F. \end{aligned} \tag{4.1}$$

By introducing rescaled time $s = \mu\tilde{s}$ and $\theta = \mu\tilde{\theta}$ we obtain the equivalent system

$$\begin{aligned} \dot{E} &= \mu(1+i\alpha)NE + \mu\Gamma F, \\ T\dot{N} &= \mu P - \mu N - \mu(1+2N)|E|^2, \\ \dot{F} &= E(\tilde{s}-\tilde{\theta})e^{-i\Omega_0\theta} + (i\mu\Omega_m - 1)F, \end{aligned} \tag{4.2}$$

where the dot denotes differentiation with respect to $\tilde{s}$. Setting $\mu = 0$ ($\mu \to 0$), equations (4.2) reduce to $\dot{E} = \dot{N} = 0$ and $\dot{F} = E(\tilde{s}-\tilde{\theta})e^{-i\Omega_0\theta} - F = Ee^{-i\Omega_0\theta} - F$, as long as $\Omega_m = \mathcal{O}(1)$ and $|F| = \mathcal{O}(1)$ with respect to μ. So, E and N are constant on the $\tilde{s}$-time-scale. For $\mu = 0$ the last equation of system (4.1) yields $F(s) = E(s-\theta)e^{-i\Omega_0\theta}$, meaning that the dynamics of (4.1) is only defined when this holds. The flow is then prescribed by the first two equations of system (4.1), the (rescaled) LK-equations,

$$\begin{aligned} E' &= (1+i\alpha)NE + \Gamma E(s-\theta)e^{-i\Omega_0\theta}, \\ TN' &= P - N - (1+2N)|E|^2. \end{aligned} \tag{4.3}$$

Hence the EFMs of system (4.1) are the ECMs (external cavity modes) of the LK-equation, with the additional restriction that they lie in $\{F = Ee^{-i\Omega_0\theta}\}$. In the coordinates (2.1), these fixed points therefore satisfy $R = S$ and $\phi_0 = \Omega_0\theta$.

Indeed, the relation for the Ω-values of the EFMs (2.10) reduces in the limit $\lambda \to \infty$ to the equation $f(\Omega,\infty) = g(\Omega,\infty)$, or

$$\Omega = -\Gamma\sqrt{1+\alpha^2}\sin(\Omega_0\theta + \Omega\theta + \arctan(\alpha)), \tag{4.4}$$

which equals, after a rescaling of parameters, the relation for ECMs in the LK-equations given in (Rottschäfer and Krauskopf, 2004). Hence the results from (Rottschäfer and Krauskopf, 2004) concerning ECMs and their formation in SN-bifurcations immediately apply. As concluded there, SN-bifurcations should also satisfy $f_\Omega = g_\Omega$, which together with (4.4) yields the locus of SN-bifurcations in $(\Omega_0\theta, \Gamma)$-parameter space:

$$\Omega_0\theta^{\pm} = \pm[\sqrt{C^2-1} + \arccos(-\frac{1}{C})] - \arctan\alpha + 2k\pi. \tag{4.5}$$

Note that this condition does not depend on the parameter P, the pump.

5 Comparison between limits $\lambda \to 0$ and $\lambda \to \infty$

In this section, we compare the results we so far obtained in the limits $\lambda \to 0$ (injection limit) and $\lambda \to \infty$ (LK-limit). By comparing the equations (4.4) and (3.3) we conclude:

Lemma 5.1. *The equation (3.3), $\lambda \to 0$ limit, has a solution $y = 0$ if and only if Ω_m is a solution for the equation (4.4), $\lambda \to \infty$ limit.*

This means, that when the parameters α, Γ and θ are chosen such that there is in the limit $\lambda \to 0$ an EFM with $\Omega = \Omega_m$ (the laser with filtered feedback acts exactly as a laser with injected optical field $\Gamma S e^{i\Omega_m \theta}$), then the laser with conventional feedback (the limit $\lambda \to \infty$) has an EFM with $\Omega = \Omega_m$ for the same choices of the parameters α, Γ and θ.

For a laser operating at a fixed point determined by (3.2), we have by (2.3) $\lambda y = \Omega - \Omega_m = \lambda \frac{R}{S} \sin(\phi_0 - (\Omega_0 + \Omega_m + \lambda y)\theta)$, from which the angle ϕ_0 can be solved. It follows that if it exists, so if the parameters are chosen as in Lemma 5.1, an EFM with $y = 0$ satisfies $\phi_0 = (\Omega_m + \Omega_0)\theta + k\pi$, $k \in \mathbf{Z}$.

A second comparison between the LK-limit and injection limit can be made by looking at the curves where EFMs are created/annihilated in both limits, hence, by comparing (3.14) to (4.5). Recall that the curves (3.14) for $\lambda \to 0$ separate regions in the $(\Omega_0\theta, \Gamma)$-plane where the number of Ω_m-intervals $I^j_{\Omega_m}$ changes, see figure 3. When Ω_m is chosen in an interval $I^j_{\Omega_m}$ there are three EFMs, of which two satisfy $|\Omega - \Omega_m| = \mathcal{O}(\lambda)$ and one satisfies $\Omega = \mathcal{O}(\lambda)$. When Ω_m is chosen outside any interval $I^j_{\Omega_m}$, only the one EFM with $\Omega = \mathcal{O}(\lambda)$ exists. The curves (4.5) in the LK-limit on the other hand separate regions in the $(\Omega_0\theta, \Gamma)$-plane with different number of EFMs.

In Figure 4, we plotted the curves (3.14) and (4.5) in the $(\Omega_0\theta, \Gamma)$-parameter space for an $\Omega_0\theta$ window of length 2π. When a black (thinner) line is crossed, a pair of EFMs is created or annihilated in the LK-limit $\lambda \to \infty$. When a blue (thicker) line is crossed, a region $I^j_{\Omega_m}$, where three EFMs exist, appears or disappears in the injection limit $\lambda \to 0$. In the figure, the number of EFMs for the LK-limit is given in the various regions separated by the curves. For the $\lambda \to 0$ limit, the number of Ω_m-intervals where three EFMs exist are given (roman numerals).

We define the various regions in $(\Omega_0\theta, \Gamma)$-parameter space separated by the curves as follows:

$$\mathcal{V}_k := \{(\Omega_0\theta, \Gamma) |\ \text{eq. (4.4) has } k \text{ solutions}\}, \quad k = 1, 3, 5, \ldots$$
$$\mathcal{W}_l := \{(\Omega_0\theta, \Gamma) |\ \text{ineq. (3.8) is satisfied in } l \text{ regions}\}, \quad l = \text{I, II, III}, \ldots \tag{5.1}$$

The bifurcation curves in figure 4 do not depend on the pump parameter P, so the figure is the same for any value of P. Moreover, the cusp point of the LK-limit is

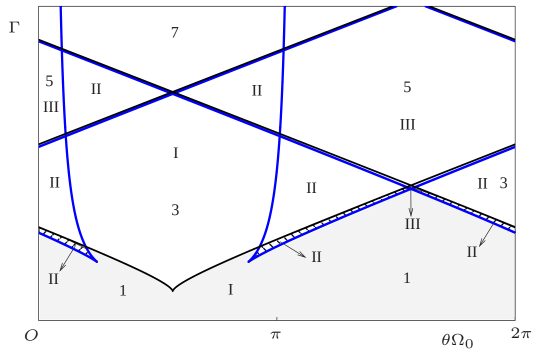

Figure 4. The SN-bifurcation curves (3.14) (blue thicker lines) and the number of intervals $I^j_{\Omega_m}$ (roman numerals) in $(\Omega_0\theta, \Gamma)$-parameter space for $\lambda \to 0$. For $\lambda \to \infty$, the SN-bifurcation curves (4.5) (black thinner curves) and the number of EFMs are given. The tiny region $\mathcal{V}_1 \cap \mathcal{W}_{\text{III}}$ is denoted by the arrow pointing down to the III. See the text for further explanations.

found at $(\theta\Omega_0, \Gamma) = (\pi - \arctan\alpha, \frac{1}{\theta\sqrt{\alpha^2+1}})$ and the cusp points of the $\lambda \to 0$ limit lies at $(\theta\Omega_0, \Gamma) = (\pm 1 + \pi - \arctan\alpha, \frac{2}{\theta\sqrt{\alpha^2+1}})$; hence the arrangement of the curves is exactly as given in figure 4. Furthermore, it can be shown that the first LK-curve lies 'above' the curve for $\lambda \to 0$ for any choice of the parameters.

We thus conclude

Theorem 5.2. *Let $\alpha, P > 0$ be given. Then the sets $\mathcal{V}_1 \cap \mathcal{W}_{\text{I}}$, $\mathcal{V}_1 \cap \mathcal{W}_{\text{II}}$ and $\mathcal{V}_1 \cap \mathcal{W}_{\text{III}}$ in $(\Omega_0\theta, \Gamma)$-parameter space are all nonempty.*

Theorem 5.2 and figure 4 can be explained as follows. Regardless of $\Omega_0\theta$, the LK-limit, $\lambda \to \infty$, has one EFM in the limit $\Gamma \downarrow 0$, and a second and third solution appear when Γ is increased above the first bifurcation curve. Also regardless of $\Omega_0\theta$, the injection limit, $\lambda \to 0$, has one window $I^0_{\Omega_m}$ such that in the limit $\Gamma \downarrow 0$ there is a single EFM when $\Omega_m \notin I^0_{\Omega_m}$ and there are three EFMs if $\Omega_m \in I^0_{\Omega_m}$, see section 3. The same is true for both limits in the whole (large) region $\mathcal{V}_1 \cap \mathcal{W}_{\text{I}}$, the shaded part of figure 4. For the injection limit, a second Ω_m-interval $I^1_{\Omega_m}$ appears when Γ is increased above the first bifurcation curve, so that $(\Omega_0\theta, \Gamma) \in \mathcal{V}_1 \cap \mathcal{W}_{\text{II}}$, the striped part of figure 4. Here three EFMs exist for $\Omega_m \in I^0_{\Omega_m} \cup I^1_{\Omega_m}$ and one EFM otherwise. There is even a tiny region $\mathcal{V}_1 \cap \mathcal{W}_{\text{III}}$ where a third Ω_m-interval $I^2_{\Omega_m}$ appears. Hence in the sets of Theorem 5.2, the LK-limit has only one EFM whereas in the injection limit there are Ω_m-values for which three EFMs exist.

6 Bifurcation analysis for varying λ

In this section we study bifurcations of EFMs as λ varies from $\lambda = 0$ to $\lambda \to \infty$. As these states are characterised by their Ω-coordinates determined by $f = g$ (2.11), (2.12), we study SN-bifurcations of EFMs as solutions of $f(\Omega; \lambda) = g(\Omega; \lambda)$ and $f_\Omega(\Omega; \lambda) = g_\Omega(\Omega; \lambda)$ (or the same characterisation in one of the alternative independent variables x or y). Solutions of this pair of equations are not easily found, so we alternatively use the following sufficient conditions to characterise SN-bifurcations at which two EFMs are *created* as λ is increased, here stated in the y variable:

At a SN-bifurcation ($y = y_0; \lambda = \lambda_0$) two EFMs are *created* as λ is increased, if $\forall \delta > 0$ small enough one of the following two statements holds:

$$\begin{aligned} &i) \ f(y_0;\lambda_0) = g(y_0;\lambda_0) > 0 \\ &\qquad \text{and } f(y_0;\lambda_0+\delta) > g(y_0;\lambda_0+\delta), \\ &ii) \ f(y_0;\lambda_0) = g(y_0;\lambda_0) < 0 \\ &\qquad \text{and } f(y_0;\lambda_0+\delta) < g(y_0;\lambda_0+\delta). \end{aligned} \tag{6.1}$$

Moreover, for a SN-bifurcation taking place in ($y = y_0; \lambda = \lambda_0$), the equation $f_y(y_0;\lambda_0) = g_y(y_0;\lambda_0)$ also holds.

Two EFMs will vanish under these conditions with $f(y_0;\lambda_0+\delta) < g(y_0;\lambda_0+\delta)$ in case $i)$ or $f(y_0;\lambda_0+\delta) > g(y_0;\lambda_0+\delta)$ in case $ii)$. Note here, that a non-degeneracy condition $f_{yy} \neq g_{yy} = 0$ should also hold to assure a SN-bifurcation. With C and D as before, we have

$$\begin{aligned} &f(y_0;\lambda_0+\delta) = \\ &\quad = -\tfrac{C}{\theta\sqrt{1+y_0^2}} \sin(\lambda_0 y_0\theta + \delta y_0\theta + \arctan y_0 + D) \\ &\quad = f(y_0;\lambda_0)\cos(\delta y_0\theta) + \tfrac{1}{y_0\theta}\tfrac{\partial f}{\partial\lambda}(y_0;\lambda_0)\sin(\delta y_0\theta). \end{aligned} \tag{6.2}$$

This yields, by using $f(y_0;\lambda_0) = g(y_0;\lambda_0)$, the exact expression

$$\begin{aligned} &f(y_0;\lambda_0+\delta) - g(y_0;\lambda_0+\delta) = \\ &f(y_0;\lambda_0)\left[\cos(\delta y_0\theta) - 1\right] + \tfrac{\partial f}{\partial\lambda}(y_0;\lambda_0)\tfrac{\sin(\delta y_0\theta)}{y_0\theta} - \delta y_0 \end{aligned} \tag{6.3}$$

which determines whether EFMs are created or annihilated at (y_0,λ_0). For $\delta \ll 1$, the first term of (6.3) is of order $\mathcal{O}(\delta^2)$ and the second of order $\mathcal{O}(\delta)$. Hence the righthand side of (6.3) is $\mathcal{O}(\delta)$ and the sign of $f(y_0;\lambda_0+\delta) - g(y_0;\lambda_0+\delta)$ depends on the relation between $\frac{\partial f}{\partial\lambda}(y_0;\lambda_0)$ and y_0 as is stated in

Lemma 6.1. *If there exists a constant $K > 0$ such that $\infty > \frac{\partial f}{\partial\lambda}(y_0;\lambda_0) \geq y_0 + K$, then $f(y_0;\lambda_0+\delta) - g(y_0;\lambda_0+\delta) > 0$ for $\delta > 0$ sufficiently small. Analogously $f(y_0;\lambda_0+\delta) - g(y_0;\lambda_0+\delta) < 0$ for $\delta > 0$ sufficiently small, if there exists a constant $K > 0$ such that $-\infty < \frac{\partial f}{\partial\lambda}(y_0;\lambda_0) < y_0 - K$.*

We now compute an *exact* polynomial expression for $\frac{\partial f}{\partial\lambda}(y_0;\lambda_0)$, using the fact that in a bifurcation point $(y_0;\lambda_0)$ both $f(y_0;\lambda_0) = g(y_0;\lambda_0) = \lambda_0 y_0 + \Omega_m$ and $f_y(y_0;\lambda_0) = g_y(y_0;\lambda_0) = \lambda_0$ hold. Substituting the first into the second yields:

$$\begin{aligned} &\lambda_0 = -\tfrac{y_0}{1+y_0^2}(\lambda_0 y_0 + \Omega_m) \\ &-\tfrac{C}{\theta\sqrt{1+y_0^2}}\left[\lambda_0\theta + \tfrac{1}{1+y_0^2}\right]\cos(\lambda_0 y_0\theta + \arctan y_0 + D). \end{aligned} \tag{6.4}$$

This leads to

$$\begin{aligned} \frac{\partial f}{\partial\lambda}(y_0;\lambda_0) &= -y_0\tfrac{C}{\sqrt{1+y_0^2}}\cos(\lambda_0 y_0\theta + \arctan y_0 + D) \\ &= y_0 + \frac{y_0^2\theta(\lambda_0 y_0 + \Omega_m) - y_0}{\lambda_0\theta(y_0^2+1)+1}. \end{aligned} \tag{6.5}$$

by substitution of (6.4). Thus, comparing $\frac{\partial f}{\partial\lambda}(y_0;\lambda_0)$ to y_0, as should be done in order to apply Lemma 6.1, comes down to determining the sign of the second term of (6.5). The result is stated in

Lemma 6.2. *In a bifurcation point (y_0,λ_0), the derivative $\frac{\partial f}{\partial\lambda}(y_0;\lambda_0)$ satisfies:*
$i)\ \forall y_0 \in (-\infty, y_-)\cup(0,y_+)\exists k>0 : \frac{\partial f}{\partial\lambda}(y_0;\lambda_0) = y_0 - k$,
$ii)\ \forall y_0 \in (y_-,0)\cup(y_+,\infty)\exists k>0 : \frac{\partial f}{\partial\lambda}(y_0;\lambda_0) = y_0 + k$,
with $y_\pm = -\frac{\Omega_m}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{\Omega_m^2 + 4\lambda/\theta}$.

Proof. Consider the second term of (6.5). Its denominator is always greater than zero since $\lambda_0\theta > 0$, so $\frac{\partial f}{\partial\lambda}(y_0;\lambda_0)$ is a C^1 function of y_0 on $\mathbf{R}$. The zeroes of its nominator $y^2\theta(\lambda y + \Omega_m) - y$ are $y = 0$ and $y_\pm = -\frac{\Omega_m}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{\Omega_m^2 + 4\lambda/\theta}$, where $y_- < 0 < y_+$. It is easily checked that $y^2\theta(\lambda y + \Omega_m) - y < 0$ for $y \in (-\infty, y_-) \cup (0, y_+)$ and $y^2\theta(\lambda y + \Omega_m) - y > 0$ for $y \in (y_-, 0) \cup (y_+, \infty)$, so by continuity the claim is proved. □

Recalling the statements (6.1), the above results, with $I_y := (-\frac{\Omega_m}{\lambda_0}, 0)$ for $\Omega_m > 0$, $I_y := (0, -\frac{\Omega_m}{\lambda_0})$ for $\Omega_m < 0$, in

Theorem 6.3. *Let α, Γ, θ, Ω_m and Ω_0 be fixed and let $y_\pm = -\frac{\Omega_m}{2\lambda_0} \pm \frac{1}{2\lambda_0}\sqrt{\Omega_m^2 + 4\lambda_0/\theta}$.*
In a bifurcation point (y_0,λ_0) with $\lambda_0 > 0$, two EFMs are created if $y_0 \in (-\infty, y_-) \cup I_y \cup (y_+, \infty)$ when λ is increased from λ_0. Two EFMs disappear at a bifurcation point (y_0,λ_0) if $y_0 \in (y_-, y_+) \setminus I_y$ when λ is increased from λ_0.

Proof. The proof is by combining Lemmas 6.1 and 6.2 and the fact that $f(y_0;\lambda_0) = g(y_0;\lambda_0) > 0$ if $y_0 > -\frac{\Omega_m}{\lambda_0}$ and vice versa. □

Although this theorem gives a characterisation of values of y, Ω, of SN-bifurcations for which the number of EFMs increases or decreases, it is not clear yet whether bifurcations can actually take place in each region of y-values or not. To analyse whether annihilating bifurcations may occur or not, it is useful to observe that the result in Theorem 6.3 is invariant under the transformation $\{\Omega_m \to -\Omega_m, y_0 \to -y_0\}$. Hence we take, without loss of generality, $\Omega_m > 0$ in the sequel.

In the following, we will narrow down the regions of y where SN-bifurcations can occur by looking at f as a function of y. The envelope of f serves as upper and lower bounds: $-\frac{C}{\theta\sqrt{1+y^2}} \leq f(y;\lambda) \leq \frac{C}{\theta\sqrt{1+y^2}}$. In a

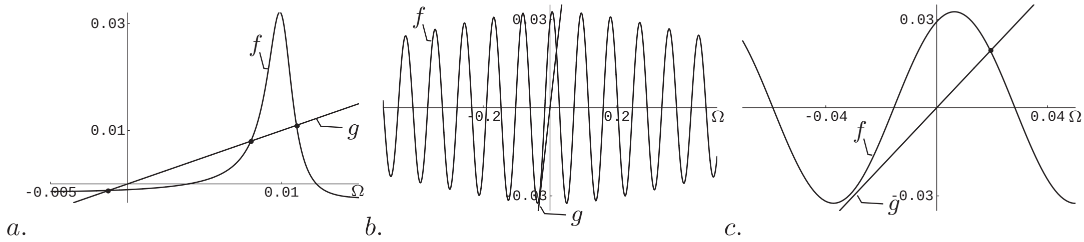

Figure 5. f and g as functions of Ω as λ is increased. a. $\lambda = 0.001$: there are three intersections of f and g so there exist three EFMs. b. $\lambda = 0.5$, with in c. a blowup of the $\lambda = 0.5$ plot. In this case only one EFM exists. The other parameters are $\Gamma = 0.0064$, $\theta = 70$, $\alpha = 5.0, \Omega_0 = 0.0414, \Omega_m = 0.01$.

SN-bifurcation point $(y_0; \lambda_0)$, where $f = g$, this yields the requirement

$$-\frac{C}{\theta\sqrt{1+y_0^2}} \le \lambda_0 y_0 + \Omega_m \le \frac{C}{\theta\sqrt{1+y_0^2}}. \qquad (6.6)$$

An equality at either side gives, after taking squares, $(\lambda_0 y_0 + \Omega_m)^2(1+y_0^2) = \frac{C^2}{\theta^2}$. We therefore define

$$k(y_0) = (\lambda_0 y_0 + \Omega_m)^2(1+y_0^2) - \frac{C^2}{\theta^2}, \qquad (6.7)$$

and analyse its sign depending on y_0. Note that $k(y_0)$ is again invariant under $\{\Omega_m \to -\Omega_m, y_0 \to -y_0\}$ so we only study $\Omega_m > 0$. The inequality (6.6) is only satisfied if $k(y_0) \le 0$, which means that SN-bifurcations can only occur for those y_0 with $k(y_0) \le 0$. The function $k(y_0)$ satisfies

$$k'(y_0) = 0 \text{ for } y_0 = -\frac{\Omega_m}{\lambda_0}$$
$$\text{or } y_0 = y_{p,m} := \frac{1}{4\lambda_0}[-\Omega_m \pm \sqrt{\Omega_m^2 - 8\lambda_0^2}], \qquad (6.8)$$

where $y_{p,m}$ only exist if $\Omega_m^2 \ge 8\lambda_0^2$, and $-\frac{\Omega_m}{\lambda_0} < y_{p,m} < 0$ if $\Omega_m > 0$. Moreover,

$$k(0) = \Omega_m^2 - \frac{C^2}{\theta^2} \text{ and } k(-\frac{\Omega_m}{\lambda_0}) = -\frac{C^2}{\theta^2} < 0. \qquad (6.9)$$

Hence $k(y_0)$ always has a negative (local) minimum in $y_0 = -\frac{\Omega_m}{\lambda_0}$. Moreover, $\lim_{y_0 \to \pm\infty} k(y_0) = \infty$ so k has at least two zeroes $y_k^{\pm}$, $y_k^- < y_k^+$. Turning back to (6.6) we observe that there is a bounded region $[y_k^-, y_k^+]$ outside which the inequality is never satisfied, and that $\lambda_0 y_k^- + \Omega_m < 0$ and $\lambda_0 y_k^+ + \Omega_m > 0$. This region is split into two intervals when the local maximum y_m and minimum y_p exist and satisfy $k(y_m) > 0$ and $k(y_p) < 0$, resulting in two more zeroes $y_{1,2}$ of k with $y_{1,2} \in I_y$.

The relative positions of zeroes of $k(y_0)$ and the points $y_0 = 0$, $-\frac{\Omega_m}{\lambda_0}$ and $y_\pm$, that form the boundaries between creation and annihilation of EFMs in Theorem 6.3, determine whether bifurcations of creating or annihilating nature are possible or not. We compute

$$k(y_\pm) = \tfrac{1}{4}[(\Omega_m \pm \sqrt{\Omega_m^2 + \tfrac{4\lambda_0}{\theta}})^2 + \tfrac{4}{\theta^2}] - \tfrac{C^2}{\theta^2}, \qquad (6.10)$$

from which can be deduced that

$$k(y_+; \Omega_m) = 0 \text{ iff } \Omega_m = \Omega_m^+ \qquad (6.11)$$
$$k(y_-; \Omega_m) = 0 \text{ iff } \Omega_m = \Omega_m^-. \qquad (6.12)$$

Here $\Omega_m^\pm := \frac{\pm 1}{\sqrt{C^2-1}}[\frac{1}{\theta}(C^2-1) - \lambda_0]$. Thus $k(y_\pm) = 0$ is only possible if the remaining parameters satisfy

$$C = \theta\Gamma\sqrt{1+\alpha^2} > 1 \qquad (6.13)$$

(recall that $C, \theta > 0$), *i.e.*, while varying Ω_m a zero of $k(y_0)$ can only pass through $y_\pm$ if (6.13) holds. We now have sufficient information to conclude

Lemma 6.4. *If $C = \theta\Gamma\sqrt{1+\alpha^2} \le 1$, then* any *SN-bifurcation with $y_0 \notin I_y$ results in an annihilation of two EFMs. Moreover, any SN-bifurcation with $\Omega_m < \frac{C}{\theta}$ and $y_0 \in I_y$ or with $\Omega_m > \frac{C}{\theta}$ and $y_0 \in [-\frac{\Omega_m}{\lambda_0}, y_k^+]$ results in a* creation *of two EFMs.*

Proof. For $C \le 1$, the $\Omega_m^\pm$ do not exist so $k(y_\pm)$ has the same sign for every Ω_m. We determine this sign by observing that for $\Omega_m = 0$

$$k(y_\pm; \Omega_m = 0) \begin{cases} > 0 \text{ if } \lambda_0 > \frac{1}{\theta}(C^2-1) \\ < 0 \text{ if } \lambda_0 < \frac{1}{\theta}(C^2-1). \end{cases} \qquad (6.14)$$

From this and the fact that $\lambda_0 > 0$, it is clear that if $C \le 1$, $k(y_\pm) \ge 0$ for all $\Omega_m > 0$. Since SN-bifurcations are only possible within the interval $[y_k^-, y_k^+] \subset [y_-, y_+]$, this, combined with Theorem 6.3, implies that at any bifurcation with $\Omega_m > 0$ and $y_0 \notin I_y$ two EFMs are annihilated. The symmetry yields the same conclusion for $\Omega_m < 0$. □

In the case $C > 1$, the zeroes of k and the points $y_\pm$ do change place as λ_0 and Ω_m are varied. By a careful analysis, see (Hek and Rottschäfer, 2005) for more details, we conclude

Lemma 6.5. *With* $C = \theta\Gamma\sqrt{1+\alpha^2} > 1$*, EFMs are created in a SN-bifurcation for*

$y_0 \in [y_k^-, y_-]$ *if* $0 < \lambda_0 < \frac{1}{\theta}(C^2-1)$,
and if $\lambda_0 > \frac{1}{\theta}(C^2-1)$ *with* $\Omega_m > \Omega_m^-$,
$y_0 \in I_y$ *if* $0 < \Omega_m < \frac{C}{\theta}$,
$y_0 \in [-\frac{\Omega_m}{\lambda_0}, y_k^+]$ *if* $\Omega_m > \frac{C}{\theta}$,
$y_0 \in [y_+, y_k^+]$ *if* $0 < \lambda_0 < \frac{1}{\theta}(C^2-1)$
with $0 < \Omega_m < \Omega_m^+$.

Annihilation of EFMs occurs in a SN-bifurcation for

$y_0 \in [y_-, -\frac{\Omega_m}{\lambda_0}]$ *if* $0 < \lambda_0 < \frac{1}{\theta}(C^2-1)$,
and if $\lambda_0 > \frac{1}{\theta}(C^2-1)$ *with* $\Omega_m > \Omega_m^-$,
$y_0 \in [0, y_k^+]$ *if* $0 < \lambda_0 < \frac{1}{\theta}(C^2-1)$, $\Omega_m^+ < \Omega_m < \frac{C}{\theta}$,
and if $\lambda_0 > \frac{1}{\theta}(C^2-1)$ *with* $\Omega_m < \frac{C}{\theta}$,
$y_0 \in [0, y^+]$ *if* $0 < \lambda_0 < \frac{1}{\theta}(C^2-1)$, *with* $\Omega_m < \Omega_m^+$,
$y_0 \in [y_k^-, -\frac{\Omega_m}{\lambda_0}]$ *if* $\lambda_0 > \frac{1}{\theta}(C^2-1)$
with $0 < \Omega_m < \Omega_m^-$.

In figure 5, plots of the functions f and g show an example of an annihilating bifurcation.

Having thus characterised more precisely for which parameters SN-bifurcations of creating or annihilating nature can occur, it is useful to make to additional remarks about our results in relation to others.

Remark 6.6. Saddle-node bifurcations of EFMs with $y_0 \in I_y$ form a bifurcation sequence that, as λ is increased from 0 to ∞, results in connection of the two 'islands' of fixed points mentioned in (Yousefi and Lenstra, 1999). The $\lambda \to 0$ limiting EFMs close to $\Omega = 0$ and $\Omega = \Omega_m$ (if they exist) are the limiting 'island' configuration, and by Theorem 6.3 all bifurcations in this sequence are of creating nature.

Remark 6.7. The two islands mentioned in remark 6.6 become connected when $g(y)$ and the envelope of $f(y)$ are tangent in some $y_c \in I_y$, *i.e.*, for $\lambda = \lambda_c$ such that $\pm\frac{C}{\theta\sqrt{1+y_c^2}} = \lambda_c y_c + \Omega_m$ and $\mp\frac{C y_c}{\theta}(1+y_c^2)^{-3/2} = \lambda_c$. See (Green and Krauskopf, 2005).

7 Conclusions

In the literature various versions of the equations for an injection laser and the Lang-Kobayashi equations occur. We showed that, if they are rescaled in the same way, they are the $\lambda \to 0$ and $\lambda \to \infty$ limits of the model for a semiconductor laser with filtered external optical feedback. Hence the relation between 'fixed points' in both limits (steady states and ECMs, respectively) can be studied. By a careful analysis we thus characterised saddle-node bifurcations that are creating and annihilating of nature. The presence of the latter was rather surprising, as they were not found before in numerical studies.

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