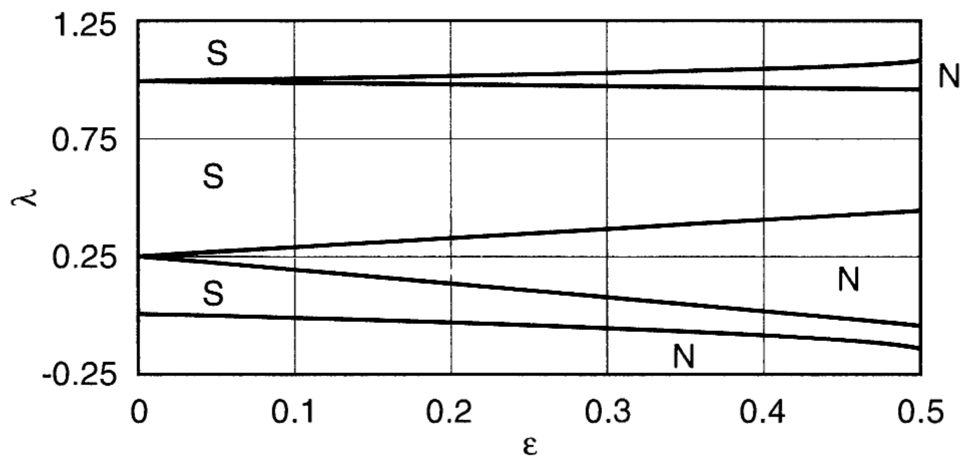
Assignment-set 4 Introduction to Perturbation Methods

Deadline to hand in: 26 May 2015, 11.15u

1.) Mathieu's equation is given by

$$y'' + [\lambda + \varepsilon\cos(t)]\, y = 0, \text{ for } t > 0,$$

where $y(0) = a$ and $y'(0) = b$. Also, λ is a positive constant. This equation describes the small amplitude oscillations of a pendulum whose length varies periodically with time. If the pendulum's natural frequency is a particular multiple of the frequency of the length variation, then instability can occur. This is indicated in the figure below, which shows the regions in the (ε, λ)-plane where the motion is stable (S) and unstable (N). Equations for the boundaries of these regions are derived in this exercise.



(a) Assuming λ is independent of ε, use a regular expansion to show that secular terms appear in the second term of the expansion if $\lambda = \frac{1}{4}$ and in the third term no matter what the value of λ.

(b) In the case where $\lambda = \frac{1}{4}$, use multiple scales to remove the secular term in the second term of the expansion. Use this to explain why the solution can grow exponentially in time and is therefore unstable. By generalising this analysis, it is possible to show that the solutions may be unbounded, and hence unstable, if $\lambda = \frac{n^2}{4}$, where $n = 0, 1, 2, 3, ...$ (you do not need to show this).

(c) Assuming $\lambda \neq \frac{n^2}{4}$, where $n = 0, 1, 2, 3, ...$, use multiple scales to show that

$$y = a_0 \cos\left(\sqrt{\lambda}t + \frac{\varepsilon^2 t}{4\sqrt{\lambda}(1 - 4\lambda)} + \theta_0\right),$$

to leading order, where a_0 and θ_0 are constants. This expression indicates what the solution looks like in the stable regions in the figure.

(d) To investigate what happens for λ values near $\frac{1}{4}$, suppose $\lambda = \frac{1}{4}(1 + 2\varepsilon\lambda_1) + \cdots$. Find a two-term approximation of the solution that is valid for large t. From this, derive a condition for which the solution is bounded if $|\lambda_1| < 1$. Also, show that the solution is always bounded if $|\lambda_1| > 1$. Because of this, the curves $\lambda = \frac{1}{4}(1 \pm 2\varepsilon) + \cdots$ form the stability boundaries of this region.

(e) To investigate what happens near $\lambda = 1$, suppose $\lambda = 1 + \varepsilon^2\lambda_1 + \cdots$ Find a first-term approximation of the solution that is valid for large t. From this, show that the solution may be unbounded, depending on the initial conditions, if $-\frac{1}{12} < \lambda_1 < \frac{5}{12}$.

2.) Use the WKBJ method to find an approximate solution of the following problem

$$\varepsilon y'' + y' + e^x y = 0,$$

for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 1$. Compare your answer with the composite expansion obtained using matched asymptotic expansions.

3.) Consider the initial value problem

$$\varepsilon^2 y'' + e^{-2t} y = 0, \text{ for } 0 < t < \infty,$$

where $y(0) = 0$ and $y'(0) = 1$.

(a) Find a first-term WKBJ approximation of the solution.

(b) Determine the second term in the WKBJ expansion and then discuss the shortcomings of this approximation. Are any of these complications similar to what might be expected near a turning point?

(c) Find the exact solution. On the same axes, plot the solution and the first-term expansion from part (a) for $0 < t < 10$ (take $\varepsilon = 10^{-2}$). Comment on the differences between the curves in conjunction with the results from part (b).

4.) In studying the theory of collective ruin, one comes across a variable $R(x)$, which is the probability of having resources available for emergencies (i.e., a risk reserve). It satisfies the integro-differential equation

$$\varepsilon\beta(x)R' - R + \lambda \int_0^{\frac{x}{\varepsilon}} R(x - \varepsilon y)e^{-\lambda y} dy = 0,$$

for $0 < x < \infty$, where $R(x) \to 1$ as $x \to \infty$. Also, λ is a positive constant and $\beta(x)$ is a smooth, positive function.

(a) Find the exact solution when β is a positive constant. Explain why it's necessary to require $\lambda\beta > 1$.

(b) Based on your observations from part (a), find the first term in a WKBJ expansion of the solution when β is not constant. What conditions must be imposed on λ and $\beta(x)$?

5.) The motion of planetary rings is described using the theory of self- gravitating annuli orbiting a central mass. For circular motion in the plane, with the planet at the origin, one ends up having to find the circumferential velocity $V(r,\theta,t) = v(r)e^{i(\omega t+m\theta)}$. The function $v(r)$ satisfies

$$\frac{d}{dr}\left(r\frac{d}{dr}(rv)\right) = m^2(1-\kappa^2 r^2)v \text{ for } 0 < r < \infty,$$

where $\kappa = \frac{\alpha+\beta m}{m}$. Here r is the radial coordinate and α and β are positive constants. The parameter m is positive and is a mode number. Find a first-term approximation of the solution for large m.