Assignment-set 5 Introduction to Dynamical Systems 2016

Deadline to hand in: 20 January 2017, 17.00u, in mailbox Olfa

1.) Consider for $\alpha \in \mathbf{R}$ the 2-dimensional system

$$\begin{cases} \dot{x} &= -2xy + x^3 + y^3, \\ \dot{y} &= -y + \alpha x^2. \end{cases}$$

Determine the center manifold $W^c((0,0))$ up to and including terms of order three. Determine the (approximate) flow on $W^c((0,0))$ near $(0,0)$. Determine the stability of $(0,0)$ for all $\alpha \in \mathbf{R}$.

2.) Consider for $\beta \in \mathbf{R}$ the 2-dimensional system

$$\begin{cases} \dot{x} &= -x + y^2 + \beta y^4, \\ \dot{y} &= -y^3. \end{cases}$$

(a) Take $\beta = -2$. Determine the stable manifold $W^s((0,0))$ and the center manifold(s) $W^c((0,0))$ *explicitly* by solving the appropriate equations. Is $W^c((0,0))$ uniquely determined? Is it analytic? If so, give an expression of $W^c((0,0))$ in terms of a power series.

(b) Sketch, for β still equal to -2, the phase portrait, including the manifolds $W^s((0,0))$ and $W^c((0,0))$.

(c) Consider the general case $\beta \in \mathbf{R}$. What can you say about $W^c((0,0))$? Is it unique? Is it analytic? Can you give an explicit expression, or a power series expansion?

3.) Consider the 3-dimensional system

$$\begin{cases} \dot{x} &= y + 2z + (x+z)^2 + xy - y^2, \\ \dot{y} &= (x+z)^2, \\ \dot{z} &= -2z - (x+z)^2 + y^2. \end{cases}$$

(a) Find a transformation to write the system in the form (5.34).

(b) Determine the center manifold $W^c((0,0,0))$ up to and including terms of order two.

(c) Determine the stability of the critical point $(0,0,0)$.

4.) Consider the system

$$\begin{cases} \dot{x} &= \lambda x - y + xy^2, \\ \dot{y} &= x + \lambda y + y^3. \end{cases} \tag{1}$$

(a) Determine the stability of the point at the origin. Denote the value of λ where the origin goes from stable to unstable by λ_c.
Next, we analyse what happens to orbits close to the origin when λ is varied around λ_c.

(b) Write the system in polar coordinates.

(c) Determine the Poincaré map defined on the positive x-axis explicitly. For this you can use that Bernoulli's equation

$$\dot{z} = a(t)z + b(t)z^{n+1}$$

is transformed to a linear equation by the change of variables $w = z^{-n}$.
For which values of λ do there exist periodic solutions and what is their radius?
Determine their stability.
Sketch the value of the radius of the periodic solution as function of λ.

(d) Choose λ such that it is close to λ_c. Conclude from the above what happens to the orbit close to the origin as λ is varied from $\lambda < \lambda_c$ to $\lambda > \lambda_c$.