Assignment-set 1 Introduction to Dynamical Systems 2016

Deadline to hand in: 10 October 2016, 11:15u

1.) (a) Consider $g : [0, \infty) \to \mathbf{R}$ given by

$$g(t) = \frac{\cos t^2}{t+2}.$$

Show that $\lim_{t\to\infty} g(t)$ exists, while $\lim_{t\to\infty} \dot{g}(t)(=\frac{dg}{dt}(t))$ does not.

(b) Consider the autonomous ODE $\dot{x} = f(x)$, $x \in \mathbf{R}^n$, with initial condition $x(0) = x_0$ and $f : \mathbf{R}^n \to \mathbf{R}^n$ (at least) continuously differentiable. Let $\phi(t; x_0)$ be a solution such that

$$\lim_{t\to\infty} \phi(t; x_0) = a$$

for a certain $a \in \mathbf{R}^n$. Prove that a must be a critical point of the system.
Warning: Be aware of functions that behave like $g(t)$ in (a).

(c) Explain why the function $g(t)$ that is given in (a) cannot be a solution of a system as described in (b) (with $n = 1$).

2.) Consider the non-autonomous equation,

$$\dot{x} = t^2 + [\sin(x+t)]x, \quad \text{with } \; x(0) = x_0, \tag{1}$$

and its autonomous equivalent,

$$\begin{cases} \dot{x} &= y^2 + [\sin(x+y)]x, \\ \dot{y} &= 1, \end{cases} \quad \text{with } \; (x(0), y(0)) = (x_0, 0). \tag{2}$$

Note that it is clear from the theory of chapter 3 in the book that equation (1)/system (2) must have a uniquely defined solution on a certain time interval.

(a) Explain why we cannot conclude from Theorems 4.3 and 4.5 (in the book) that equation (1)/system (2) defines a complete flow.

(b) Use (1) to prove that $|x(t)| \leq |x(0)| + \frac{1}{3}t^3 + \int_0^t |x(s)|ds$.

(c) Prove that, for some constant $K > 0$, $|x(t)| \leq Ke^t$ for all $t \geq 0$.
Hint: Introduce $z(t) \geq 0$ and $\alpha(t) \geq 0$ by $|x| = z(t) - \alpha(t)$ and substitute this into the estimate of (b). Construct an explicit function $\alpha(t)$ in such a way that Grönwall's Lemma (Lemma 3.13 in the book) can be applied to z.

(d) Prove that equation (1)/system (2) defines a complete flow.

3.) Consider the two-dimensional system,

$$\begin{cases} \frac{du}{d\xi} &= v \\ \frac{dv}{d\xi} &= Av - u(1-u), \end{cases} \tag{3}$$

with parameter $A > 0$.

(a) Determine the critical points E_1 and E_2 of (3) and their character (as function of $A > 0$). Sketch the local linearized phase portraits near the critical points E_1 and E_2, depending on A.

The aim of this exercise is to establish the existence of a *positive heteroclinic orbit* $(u_h(\xi), v_h(\xi))$ that connects the critical point E_1 to E_2, i.e. a solution $(u_h(\xi), v_h(\xi))$ of (3) that satisfies $\lim_{\xi \to -\infty}(u_h(\xi), v_h(\xi)) = E_1$ and $\lim_{\xi \to +\infty}(u_h(\xi), v_h(\xi)) = E_2$, while $u_h(\xi), v_h(\xi) > 0$ for all $\xi \in \mathbf{R}$.

(b) Explain that $A \geq 2$ is a necessary condition for the existence of such an orbit $(u_h(\xi), v_h(\xi))$. Is $(u_h(\xi), v_h(\xi))$ uniquely determined (if it exists)?

To construct $(u_h(\xi), v_h(\xi))$, we consider the ODE (3) in 'backwards time' $\tilde{\xi} = -\xi$ and consider the well-defined orbit $(u_s(\tilde{\xi}), v_s(\tilde{\xi}))$ (by the nature of E_2) that satisfies $\lim_{\tilde{\xi} \to -\infty}(u_s(\tilde{\xi}), v_s(\tilde{\xi})) = E_2$. Within this framework, proving the existence of the positive heteroclinic orbit $(u_h(\xi), v_h(\xi))$ is equivalent to establishing that $\lim_{\tilde{\xi} \to \infty}(u_s(\tilde{\xi}), v_s(\tilde{\xi})) = E_1$ (while $u_s(\tilde{\xi}), v_s(\tilde{\xi}) > 0$ for all $\tilde{\xi} \in \mathbf{R}$).

(c) Formulate the equivalent of (3) in terms of $\tilde{\xi} = -\xi$ and show that for $\alpha > \frac{1}{4A}$, $(u_s(\tilde{\xi}), v_s(\tilde{\xi}))$ can only leave the rectangular region with vertices $(0, 0)$, $(1, 0)$, $(1, \alpha)$ and $(0, \alpha)$ through the edge between $(0, 0)$ and $(0, \alpha)$.

(d) Prove the existence of a positive heteroclinic orbit $(u_h(\xi), v_h(\xi))$ for every $A \geq 2$. *Hint:* Show that there exists a $k > 0$ such that $(u_s(\tilde{\xi}), v_s(\tilde{\xi}))$ cannot cross through the (half)line $\{v = ku, u > 0\}$ and apply exercise 1.

(e) A (positive) traveling wave solution to the PDE

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + U(1 - U)$$

with $U(x, t) : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$, is a positive bounded solution of the PDE that is stationary in a co-moving frame that travels with speed $c \in \mathbf{R}$ – the latter implies that $U(x, t)$ can be written as $u(x - ct)$ for a certain $c \in \mathbf{R}$. The function $U(x, t) = u_h(\xi)$ defines such a traveling wave. Explain! What is the relation between ξ and (x, t), and between A and c? Sketch the traveling wave $U(x, t) = u_h(\xi)$ for several values of t and A or c.