LAGUERRE POLYNOMIALS

The Laguerre polynomials appear naturally in many branches of pure and applied mathematics and mathematical physics.

First associated Laguerre polynomials $L_n^{(1)}$ are given by

$$L_n^{(1)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{x^k}{k!}.$$

The associated Laguerre polynomials are orthogonal over $[0,\infty)$ with respect to the weight xe^{-x}:

$$\int_0^\infty L_n^{(1)}(x) L_m^{(1)}(x)\, xe^{-x}\, dx = \begin{cases} 0, & n \neq m, \\ n+1, & n = m. \end{cases}$$

In fact, Laguerre polynomials $\{L_n^{(1)}\}_{n\geq 0}$ form also a *complete* system in $L^2([0,\infty),\mu)$, where measure μ has density xe^{-x} with respect to the Lebesgue measure. Namely, any function f in $L^2([0,\infty),\mu)$ can be represented as a Fourier-Laguerre series

$$f(t) = \sum_{n=0}^{\infty} f_n^\dagger L_n^{(1)}(t),$$

where

$$f_n^\dagger = \int_0^\infty f(t)\, \frac{L_{n+1}^{(1)}(t)}{n+1}\, te^{-t} dt.$$

The map $f \mapsto (f_n^\dagger)_{n\geq 0}$ is called Laguerre transform.

Objectives:

- Study the properties of the Laguerre polynomials and the Laguerre transform.
- Investigate properties of a class of functions with non-negative Laguerre coefficients.
- Investigate whether there exist non-negative sequence $\{c_{2n}\}_{n\geq 0}$ with $\sum c_{2n} = 1$ such that for all $j \geq 0$,

$$c_{2j} = \int_0^\infty \frac{t^{2j+1}}{2^{2j+1}(2j+1)!} \frac{1}{\sinh(t/2)} \Big[\sum_{n=0}^{\infty} c_{2n} L_{2n}^{(1)}(2t)\Big] e^{-t} dt$$

References

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[2] Ushio Sumita, *The Laguerre transform and a family of functions with nonnegative Laguerre coefficients*, Math. Oper. Res. **9** (1984), no. 4, 510–521, DOI 10.1287/moor.9.4.510. MR769390 (86i:60050)

Evgeny Verbitskiy: evgeny@math.leidenuniv.nl