PRIME FACTORS OF ARITHMETIC PROGRESSIONS AND BINOMIAL COEFFICIENTS

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Part 0 INTRODUCTION

Sylvester [Syl] proved in 1892 that a product of k consecutive positive integers $x, x+1, \ldots, x+k-1$ greater than k is divisible by a prime exceeding k. It is a generalization of Bertrand's Postulate that there is a prime among $k+1, k+2, \ldots, 2k$. (Take $x = k+1$.) The assumption $x > k$ can not be removed since $x = 1$ should be excluded. In the present paper we discuss generalizations and variations of Sylvester's theorem.

In Part 1 we treat blocks of consecutive integers. First we consider estimates for the greatest prime factor of the product of k consecutive integers. Next we give an upper bound for the distances between positive integers with a prime factor $> k$. Thereafter we consider bounds for the number of distinct prime factors of a product of k consecutive integers. Furthermore we discuss the corresponding problems for binomial coefficients.

In fact Sylvester proved a more general result implying that if x, d, k are positive integers with $x \geq d + k$, then $P(x(x+d)(x+2d)\ldots(x+(k-1)d)) > k$. The above mentioned result is the case $d = 1$. In Part 2 we study results on the prime factors of $x(x+d)(x+2d)\ldots(x+(k-1)d)$ where d is an integer > 1.

In Part 3 we consider prime factors of $\prod_{i=1}^{k} f(x+i)$ where $f(x)$ is a given polynomial with integer values on $\mathbb{Z}$

Finally, in Part 4, we mention some applications, first connected with the squarefree part and the greatest squarefree divisor of a number, and then some on irreducibility of polynomials.

For the early history of the subject we refer to two surveys of Langevin [Lan75a], [Lan76].

0.1 Notation

Let x, k be integers with $x \geq 1, k > 2, d > 1$ and $\gcd(x,d) = 1$. We consider the prime factors of

$$\triangle_1 := \triangle_1(x,k) := x(x+1)(x+2)\ldots(x+k-1)$$

and

$$\triangle := \triangle(x,k,d) := x(x+d)(x+2d)\ldots(x+(k-1)d).$$

For an integer ν with $|\nu| > 1$, we denote by $P(\nu)$ the greatest prime factor of ν and by $\omega(\nu)$ the number of distinct prime divisors of ν. Further we put $P(1) = P(-1) = 1$ and $\omega(1) = \omega(-1) = 0$. The numbers C appearing in this article can vary at every appearance and are effectively computable. The latter means that each C can be determined explicitly in terms of the various parameters under consideration. Constants c may be ineffective, but they may have different values at different places too. We use $\gg$ and $\ll$ to denote the Vinogradov symbols.

0.2 Small and large primes

The basic technique for most results in this paper is to distinguish between small and large primes. A positive number is small if it is less than k and large otherwise. (In some papers k is considered as a small number.) Usually there is not much information available on the large primes and the argument is based on a study of the small primes. For example, by an argument of Erdős (cf. [ST86] or [ST90b]) the product of the small prime factors of $(x+1)\cdots(x+k-1)$ is at most $k!(x+k-1)^{\pi(k)}$. For large x the factor $(x+k-1)^{\pi(k)}$ in this estimate can be considerably reduced by using estimates in linear forms in logarithms. Since the product of the k consecutive integers is at least $(x+1)^k$, the product of the large primes can be bounded from below. When we deal with binomial coeffficients $\binom{x+k}{k}$ the factor $k!$ is divided out.

This is well illustrated by a result of Ecklund, Eggleton, Erdős and Selfridge [EEES]. They wrote $\binom{x}{k} = uv = UV$ with u (U) the product of the primes $< k$ $(\leq k)$ and v (V) the product of the prime factors $\geq k$ $(> k$, respectively). They proved that there are exactly 12 pairs (x, k), explicitly given, with $u > v$ and only finitely many cases (conjecturally 19 pairs) with $U > V$. For other illustrations of the combinatorial approach we refer to Section 1.4 and Part 4.

Part 1: BLOCK OF INTEGERS

1.1 The greatest prime factor of $\triangle_1$

We assume $x > k$. It is likely that when x increases $P(\triangle_1)$ grows on average too. Therefore we conjecture, in accordance with a conjecture of Cramer on the gaps between consecutive primes, that $P(\triangle_1)$ exceeds $2k - C\log^2 k$ for all k and suitable C. Improving on results of Sylvester [Syl] and Hanson [Han], Laishram and Shorey [LS06a] proved that $P(\triangle_1) > 1.95k$ if $x > k$ except for an explicitly given finite set of exceptions. Furthermore, they proved that there is no exception when $k > 270$ or $x > k + 11$. Observe that $P(\triangle_1) \geq 2k$ cannot be true in general, since $x = k + 1$ is permitted.

If x is small, then $P(\triangle_1)$ is the greatest prime among $x, \dots, x+k-1$. For such values of x we apply estimates on the distances between consecutive primes. For explicit estimates of gaps between consecutive primes we refer to Ramaré and Saouter [RrS]. Hoheisel [Hoh] proved that there exists a constant $C < 1$ such that $p_{n+1} - p_n \ll p_n^C$ with $C = 1 - 1/33000$. After him many researchers improved the constant C. The best result up to now is $C = .525$ obtained by Baker, Harman and Pintz [BHP]. We conclude that $P(\triangle_1)$ equals the greatest prime among $x, x+1, x+2, \dots, x+k-1$

for $x < k^{1.9}$ if k is sufficiently large. In particular, $P(\triangle_1) > x$ in this range if x is sufficiently large.

For $x < k^{2-\epsilon}$ with $0 < \epsilon \leq 0.1$ the best result up to now seems to be due to Jia and M.-C. Liu [JL], viz. $P(\triangle_1) \gg_\epsilon k^{\frac{25}{13}-\epsilon}$. For $x = k^2$ H.-Q. Liu and Wu [LW] proved that $P(\triangle_1) \gg k^{1.476}$. According to R.C. Baker (Review MR 2003e:11096) the exponent has been improved to 1.482 by Harman.

For $x > k^2$ only weaker estimates for $P(\triangle_1)$ are available. Using Vaughan's identity and Vinogradov's mean value theorem for exponential sums Sander [San] derived the following improvement of a result of Jutila. For $\lambda > 0$ there exist positive constants δ and k_0 such that if $k > k_0$ and $k^{1+\lambda} \leq x \leq k^{\delta(\log k/\text{loglog } k)^{1/2}}$, then $P(\triangle_1) \gg_\lambda k^{1+c(\log k/\log x)^2}$ for some positive constant c. In particular $P(\triangle_1) \gg k \exp((\log k)^{1/3})$ if $k^2 < x \leq \exp((\log k)^{4/3})$.

For $\exp((\log k)^{\frac{3}{2}-\epsilon}) < k < \exp(\exp((\text{loglog } k/\text{logloglog } k)^2))$ only rather weak lower bounds for $\triangle_1$ are known. By a special linear form estimate Shorey [Sho] proved that $P(\triangle_1) \gg k \log k \text{ loglog } k/(\text{logloglog } k)$ for $\exp(\log k \text{ loglog } k) < x < \exp((\log k)^L)$ with $L = (\log k)^{1/4}$.

For $x \geq \exp(\exp((\text{loglog } k/\text{logloglog } k)^2))$ other estimates obtained by using estimates for linear forms give better bounds. Ramachandra [Rc73] proved that, for $x > k^{\text{loglog } k}$,

$$P(\triangle_1) \gg \min(k^2, k \log k \left(\frac{\text{loglog } x}{\text{loglog } k + \text{logloglog } x}\right)^{1/2}).$$

Hence $P(\triangle_1) \gg k(\log k)^{9/8}$ for $x > \exp((\log k)^L)$. In [ST90a] the authors proved that $P(\triangle_1) \gg_\epsilon k \text{ loglog } x$ if $x > k^{1+\epsilon}$. This yields the best known bound if $x > \exp(\exp((\log k)^2/(\text{loglog } k)))$. For x large enough the constant can be arbitrarily close to 1 according to Langevin [Lan75b].

From the above estimates we conclude that for all k and $x > k(\log k)^2$ we have

$$P(\triangle_1) \gg k \log k \frac{\text{loglog } k}{\text{logloglog } k}. \tag{1}$$

1.2 The distance between integers having a prime factor $> k$.

Let k be a positive integer and $n_1, n_2, \ldots$ be the sequence of positive integers which have a prime factor $> k$. It follows from Sylvester's result that $n_{i+1} - n_i \leq k$ for $i = 1, 2, \ldots$. Put

$$f(k) = \max_{i=1,2,\ldots} (n_{i+1} - n_i).$$

Hence $f(k) \leq k$. Erdős [Erd] obtained $f(k) \leq (3 + o(1))\frac{k}{\log k}$ and remarked that $f(k) < \pi(k)$ for large k would already be hard to prove. On the other hand, the best known lower bound reads

$$f(k) \gg \log k \text{ loglog } k \ \frac{\text{loglogloglog } k}{(\text{logloglog } k)^2}$$

due to Rankin [Ran]. It is not unlikely that the actual order of magnitude of $f(k)$ is $(\log k)^2$. The small values of $f(k)$ were computed by Utz [Utz] and Lehmer [Leh]: $f(2) = 2, f(3) = f(4) = 3, f(k) = 4$ for $5 \leq k < 13, f(k) = 6$ for $13 \leq k < 41, f(41) = 7$.

Ramachandra [Rc70] proved that $f(k) \leq (1 + o(1))\frac{k}{\log k}$, and, by a refinement of an argument of Ramachandra, Tijdeman [Tij] improved upon this to $f(k) \leq (\frac{1}{2} + o(1))\frac{k}{\log k}$, which implies $f(k) < \pi(k)$ for all large k. Subsequently Ramachandra and Shorey [RcS] got $f(k) = O(\frac{k}{\log k}(\frac{\text{loglogloglog } k}{\text{logloglog } k})^{1/2})$ in 1973 and Shorey [Sh74]

$$f(k) = O(\frac{k}{\log k}\frac{\text{logloglog } k}{\text{loglog } k})$$

in 1974. This bound is a direct consequence of the result (1) from the previous subsection. No further progress seems to have been made since 1974. Corresponding with a conjecture of Cramer on the distance between consecutive primes we conjecture that $f(k) = O((\log k)^2)$ as $k \to \infty$.

1.3 The number of distinct prime factors of $\triangle_1$.

Now we turn to lower bounds for $\omega(\triangle_1)$. We see that $k!$ divides $\triangle_1(x, k)$ and therefore, Sylvester's theorem can be reformulated as

$$\omega(\triangle_1) > \pi(k) \text{ if } x > k.$$

A well-known conjecture states that $2^p - 1$ is prime for infinitely many primes, which would imply that $\omega(\triangle_1) = 2$ for infinitely many x when $k = 2$. Assume $k \geq 3$. Saradha and Shorey [SS03] improved Sylvester's theorem to

$$\omega(\triangle_1) \geq \pi(k) + [\frac{1}{3}\pi(k)] + 2 \text{ if } x > k$$

except for an explicitly given finite set of possibilities. This is the best known estimate for $3 \leq k \leq 18$. For $k \geq 19$ Laishram and Shorey [LS04] sharpened it to

$$\omega(\triangle_1) \geq \pi(k) + [\frac{3}{4}\pi(k)] - 1 \text{ if } x > k$$

except for finitely many explicitly given possibilities. All the exceptions satisfy $\omega(\triangle_1) \geq \pi(2k) - 1$. Thus

$$\omega(\triangle_1) \geq \min(\pi(k) + [\frac{3}{4}\pi(k)] - 1, \pi(2k) - 1) \text{ if } x > k.$$

The proof depends on explicit estimates for $\pi(x)$ due to Dusart [Dus].

This inequality can be sharpened if x is large. Then estimates for linear forms in logarithms come into play. By using them, information on the prime factors of primes $\leq k$ can be obtained. A typical result is that if $\log x \gg (\log k)^2$ then the number of integers among $x, x+1, \ldots, x+k-1$ which are entirely composed of primes at most k is at most $\pi(k)$ (cf. [RST]).

It has been shown in [LS04] that for every $\epsilon > 0$ there exists $k_0 = k_0(\epsilon)$ depending only on ϵ such that

$$\omega(\triangle_1) \geq (1-\epsilon)\pi(2k) \quad \text{for} \quad k \geq k_0.$$

On the other hand, $\omega(\triangle_1(k+1,k)) = \pi(2k)$ and there exist infinitely many k such that $\omega(\triangle_1) = \pi(2k) - 1$. Further $\omega(\triangle_1(74,57)) = \pi(2k) - 2, \omega(\triangle_1(3936,3879)) = \pi(2k) - 3, \omega(\triangle_1(1304,1239)) = \pi(2k) - 4$ and $\omega(\triangle_1(3932,3880)) = \pi(2k) - 5$. We refer to [LS04] for more such pairs. It would be interesting to know the supremum of the values r such that $\omega(\triangle_1(x,k)) = \pi(2k) - r$ has at least one solution x (has infinitely many solutions x, respectively).

For larger values of x better estimates for $\omega(\triangle_1)$ exist. In 1976 Ramachandra, Shorey and Tijdeman [RST] proved that $\omega(\triangle_1) \geq k$ for $1 \leq k \leq \exp(C(\log x)^{1/2})$. Already in 1918 Pólya [Pol] derived from the Thue-Siegel theorem that $\omega(\triangle_1) \geq k + \pi(k) - 1$ if $x > c(k)$ for some ineffective constant c. The authors [ST92a] proved that

$$\omega(\triangle_1) \geq \left[\frac{k\log(1+x/k)}{\log(x+k)}\right]$$

for all x, that for $\epsilon > 0$ we have $\omega(\triangle_1) \geq k + (1-\epsilon)\pi(k)$ provided that $k > C$ and $\log x \geq (\log k)^C$ for suitable values of C depending on ϵ, and that $\omega(\triangle_1) \geq k + \pi(k) - 1$ provided that $k \geq C$ and $\log\log x \geq Ck$ for suitable values of C.

1.4 Binomial coefficients

We consider binomial coefficients $\binom{n}{k}$ with $n \geq k \geq 0$ integers. Without loss of generality we may assume that $n \geq 2k$. It is obvious that, putting $x := n-k+1$, we have $x > k$, hence

$$P(\binom{n}{k}) = P(\binom{x+k-1}{k}) = P(\triangle_1(x,k))$$

by Sylvester's theorem. So for the greatest prime factor of binomial coefficients all the estimates in Section 1.1 apply. In particular, $P(\binom{n}{k}) > 1.95k$ for all n, k with $n \geq 2k > 0$ and

$$P(\binom{n}{k}) \gg k\log k\frac{\log\log k}{\log\log\log k} \text{ for } n > k((\log k)^2+1).$$

We now consider $\omega_B := \omega(\binom{n}{k})$. Already in 1980 Langevin [Lan80] gave various estimates on $P(\binom{n}{k})$ and $\omega(\binom{n}{k})$. For every prime p dividing $\binom{n}{k}$ choose $i_p \in \{0,1,2,\ldots,k-1\}$ such that $n - i_p$ contains the highest power of p among $n, n-1, \ldots, n-k+1$. We know that for every prime $p \leq k$ and every $j \geq 0$ there are outside $n - i_p$ at most $\lfloor \frac{k}{p^j} \rfloor$ integers among $n, n+1, \ldots, n+k-1$ having j factors p. Hence the total contribution is at most $k!$ and each number $n-i$ except for $n-i_p$ has a p-part at most k. Now divide out $k!$ by dividing out first all factors p outside $n-i_p$ and then the remaining factors out of $n-i_p$. If the quotient still contains factors p, then $\binom{n}{k}$ is divisible by p, otherwise not. We do this for all primes $p \leq k$. If $n-i$

after the division reduces to a number > 1, then $n-i$ contributes a prime to ω_B, otherwise not. So $\omega_B \geq k-s$ where s is the number of 1's after the division by $k!$. If $n-i$ reduces to 1, then the division factor is $n-i > n-k$. So the total product exceeds $(n-k)^s$. But since we divided out $k!$, we conclude that $(n-k)^s \leq k!$. Thus $s \leq \frac{\log(k!)}{\log(n-k)}$, hence $\omega_B \geq k - \frac{\log(k!)}{\log(n-k)}$.

Let now $n > e^k$. By our way of dividing out $k!$ in the previous argument the maximal factor by which a number $n-i$ is divided out by powers of some prime p is at most k. Hence, if 1 remains after division, then $k^{\pi(k)} > n-k$. Thus if $n \geq k^{\pi(k)}+k$, then no number reduces to 1 and $\omega_B \geq k$. So we have proved the following theorem.

Theorem 1. (i) *Let n, k be positive integers with $n > 2k \geq 0$. Then*

$$\omega(\binom{n}{k}) \geq k - \frac{\log(k!)}{\log(n-k)}.$$

(ii) If $n \geq k^{\pi(k)} + k$ then $\omega(\binom{n}{k}) \geq k$.

By the prime number theorem $k^{\pi(k)}+k$ is about e^k. Since we do not know whether the binomial coefficient is divisible by primes $< k$, we are not able to derive better lower bounds for ω_B by using linear forms in logarithms.

PART 2. ARITHMETIC PROGRESSIONS

Now we consider variants of Sylvester's theorem for a product of terms in arithmetic progression. For positive integers x and $d \geq 2$ and $k \geq 3$ with x and d coprime, put

$$\triangle := \triangle(x,d,k) := x(x+d)\cdots(x+(k-1)d),\ \ \chi := x+(k-1)d.$$

We observe that $P(\triangle(x,d,2)) = 2$ if (and only if) $x=1$ and $d+1$ is a power of 2 which justifies the condition $k \geq 3$. Furthermore there is no loss of generality in assuming that $d > 2$, since the case $d=2$ is similar to that of $d=1$ already considered, because of the restriction $\gcd(n,d)=1$.

2.1 The greatest prime factor

Sylvester [Syl] proved a result that implies that $P(\triangle) > k$ if $x \geq d+k$. Langevin [Lan77] replaced the assumption $x \geq d+k$ by $x > k$. Thereafter the authors [ST90b] showed that, for all x, d and k,

$$P(\triangle) > k \ \text{ unless } \ (x,d,k) = (2,7,3).$$

Laishram and Shorey [LS06b] proved that

$$P(\triangle(x,d,k)) > 2k \ \text{ for } \ d > 2$$

unless $k=3, (x,d) = (1,4),(1,7),(2,3),(2,7),(2,23),(2,79),(3,61),(4,23),$
$(5,11),(18,7); k=4, (x,d) = (1,3),(1,13),(3,11); k=10, (x,d)=(1,3)$. In these exceptional cases $P(\triangle) < 2k$. We conjecture that, for every positive integer a,

$$P(\triangle) > ak \ \text{ for } \ d > a$$

with only finitely many exceptions. Thus the conjecture has been confirmed for $a = 1, 2$ according to the above inequalities.

If lower bounds on χ are imposed, better bounds can be derived. In [ST90a] the authors generalized a result of Langevin [Lan81] by proving that $P(\triangle) >> \text{loglog } \chi$ if $\chi > k^{1+\epsilon}$ for some positive ϵ. This result does not hold under the weaker condition $\chi > k$. They derived $P(\triangle) \gg k \text{ loglog } \frac{\chi}{k}$ as an unconditional bound. They [ST92b] further proved that, for every ϵ,

$$P(\triangle) \gg_\epsilon k \text{ loglog}\chi \text{ if } \chi > k(\log k)^\epsilon$$

By observing $\chi \geq P$, we see that the assumption $\chi > k(\log k)^\epsilon$ can not be replaced by $\chi > k(\text{loglog } k)^\epsilon$ with $0 < \epsilon < 1$.

2.2 The number of distinct prime factors of $\triangle$

Now we turn to giving lower bounds for $\omega(\triangle)$. We keep the notation of the previous subsection. Improving upon results on Sylvester [Syl] and Langevin [Lan77] the authors [ST89] proved that $\omega(\triangle) \geq \pi(k)$ unconditionally. Moree [Mor] showed that $\omega(\triangle) > \pi(k)$ unless $x = 1, d = 2, k = 5$. A well-known conjecture, known as Schinzel's Hypothesis H, implies that there are infinitely many d such that $1 + d, 1 + 2d, 1 + 3d, 1 + 4d$ are all primes. Thus Hypothesis H implies that the estimate of Moree is best possible for $k = 4, 5$. For $k \geq 6$, Saradha, Shorey and Tijdeman [SST] sharpened and extended the preceding inequality by showing $\omega(\triangle) > \frac{6}{5}k$ for $k \geq 6$ with finitely many explicitly given exceptions. Laishram and Shorey [LS06b] proved that

$$\omega(\triangle) \geq \pi(2k) - 1 \text{ unless } (x, d, k) = (1, 3, 10),$$

confirming a conjecture of Moree [Mor]. This is best possible when $d = 2$ as we see by considering $\omega(\triangle(k+1, 2, k)) = \pi(2k)-1$. The proofs depend on explicit estimates for the number of primes in arithmetic progressions due to Ramaré and Rumely [RrR].

For large χ better bounds for $\omega(\triangle)$ can be constructed. In [ST92a] it can be found that

$$\omega(\triangle) \geq \left[k \, \frac{\log(\chi/k)}{\log \chi} \right].$$

This is not far from best possible when $\chi < k^C$ for some constant C as is clear from the following result from [ST92a]: For every k and prime d, there exists $x < d$ with $\gcd(x, d) = 1$ and

$$\omega(\triangle) \leq k\log\frac{\log \chi}{\log k} + C\frac{k}{\log k}$$

for some constant C. The above lower bound can be improved when χ is large as compared with k. It has been shown in [ST92a] that $\omega(\triangle) \geq k - 1$ when

$$k \geq C, \quad \log \chi \geq k^{4/3}(\log k)^C \tag{2}$$

for some absolute constants C and that $\omega(\triangle) \geq k$ if (2) holds together with $x \geq k$. Recently it has been proved by Green and Tao [GT] that there are arbitrary long chains of primes in arithmetic progressions. Hence we can not have estimates better than $\omega(\triangle) \geq k$. It is an open question what the minimal function f is such that if $\chi \geq f(k)$ then $\omega(\triangle) \geq k$.

Part 3. CONSECUTIVE POLYNOMIAL VALUES

Let $f(X)$ be a polynomial with integer coefficients and at least two distinct roots. Let $x \geq 100$. In 1918 Pólya [Pol] used a result of Thue to show that if f is quadratic, then $P(f(x)) \to \infty$ as $x \to \infty$. In 1967 Schinzel [Sch] used a result of Gelfond on linear forms in two logarithms to give an effective version of Pólya's result yielding that $f(x) \gg \text{loglog}x$ as $x \to \infty$. In 1971 Keates [Kea] did so for cubic polynomials. The combination of the results of Sprindzhuk [Spr] and Kotov [Kot1], [Kot2] yields $P(f(x)) \gg \text{loglog}x$ for all polynomials $f \in \mathbb{Z}[x]$ with at least two distinct roots. A refinement was established in [ST76] where it is proved that, for sufficiently large x, we have that $\omega(f(x))$ is at least constant times $(\text{loglog } x)/\text{logloglog } x$ whenever $\log(P(f(x))) \leq (\text{loglog}x)^2$. In [SPTS] it is conjectured that $\log P(f(x)) \gg \text{loglog } x$ as $x \to \infty$ for all polynomials under consideration.

In 1976 the authors [ST76] considered $\max_{1 \leq i \leq k} P(f(x+i)) = \prod_{i=1}^k f(x+i)$. They proved that given $B > 0$ for any integers $x > 100$ and k with $\log k \leq (\text{loglog}x)^B$ we have

$$P(\prod_{i=1}^{k} f(x+i)) \gg_{B,f} k \frac{\text{loglog } x}{\text{logloglog } x}(\log k + \text{logloglog } x).$$

The right-hand side is at least k loglog$x$. So if the condition on x holds, then this is a lower bound for $\max_{1 \leq i \leq k} P(f(x+i))$. But if the condition does not hold, then the greatest prime factor itself already exceeds the right-hand side. Consequently $\max_{1 \leq i \leq k} P(f(x+i)) \gg_f k \text{ loglog}x$.

Part 4. APPLICATIONS

A lower bound for the number of distinct prime factors of a block of integers has been used to answer a question of V.G. Sprindzhuk ([Spr82] p.240) on the squarefree part $S(\triangle_1)$ of $\triangle_1$. Sprindzhuk wondered whether there exist a constant c and infinitely many pairs of positive integers (x, k) with $k < (\log x)^c$ for which $S(\triangle_1) < k^k$. Khodzhaev [Kho] observed that the result of Ramachandra, Shorey, Tijdeman cited in Section 1.3 combined with the Prime Number Theorem implies that the answer to Sprindzhuk's question is negative. He further showed that there exist positive constants c_1, c_2, c_3 such that, if $x \geq c_1$, then Sprindzhuk's inequality does not hold when $1 \leq k \leq c_2\sqrt{x}$, but it holds for all $k \geq c_3\sqrt{x}$.

If N, A, B are positive integers with $N = AB^2$ where A is squarefree, then A is the squarefree part $S(N)$ of N and AB is the greatest squarefree divisor $Q(N)$ of N. If $p_1, p_2, \ldots$ denotes the sequence of primes then we know by the Prime Number Theorem that $\prod_{i=1}^r p_r \gg_\epsilon r^{(1-\epsilon)r}$ for any $\epsilon > 0$. Therefore lower bounds for $\omega(N)$ give immediately lower bounds for $Q(N)$. Similar bounds can be derived from conditional statements obained by applying linear forms estimates as we have indicated at the end of Part 3.

Examples of the technique indicated in Section 0.2 can be found in a paper by Allen and Filaseta [AF]. In this paper the authors determine for various prime values of k, for which n the inequality

$$\prod_{p^r \| \binom{n+1}{k}, p>k} p^r > n+1$$

holds. This means that the product of large prime factors of $\binom{n+1}{k}$ exceeds $n+1$, or equivalently, that the product of the small prime factors of $\binom{n+1}{k}$ is less than $\binom{n+1}{k}/(n+1)$. Filaseta, partly with collaborators, has applied such results in numerous papers to prove that certain polynomials are irreducible. See e.g. [AF] and [FT].

References

[AF] M. Allen and M. Filaseta, *A generalization of a second irreducibility theorem of I. Schur*, Acta Arith. 109 (2003), 65–79.

[BHP] R.C. Baker, G. Harman and J. Pintz, *The difference between consecutive primes II*, Proc. London Math. Soc. (3) 83 (2001), 532-562.

[Dus] P. Dusart, *Inégalitiés explicites pour $\psi(X), \theta(X), \pi(X)$ et les nombres premiers*, C. R. Math. Rep. Acad. Sci. Canada, 21 (1999), 53-59.

[EEES] E.F. Ecklund Jr, R.B. Eggleton, P. Erdős and J.L.; Selfridge, *On the prime factorization of binomial coefficients*, J. Austral. Math. Soc. Ser. A 26 (1978), 257–269.

[Erd] P. Erdős, *On the difference of consecutive primes*, Nieuw Arch. Wisk. 3 (1955), 124-128.

[FT] M. Filaseta and O. Trifonov, *The irreducibility of the Bessel polynomials*, J. reine angew. Math. 550 (2002), 125–140.

[GT] B. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. Math., to appear.

[Han] D. Hanson, *On a theorem of Sylvester and Schur*, Canadian Math. Bull. 16 (1973), 195-199.

[Hoh] G. Hoheisel, *Primzahlprobleme in der Analysis*, Sitz. preuss. Akad. Wiss. 2 (1930), 1-13.

[JL] C. Jia and M.-C. Liu, *On the largest prime factor of integers*, Acta Arith. 95 (2000), 17-48.

[Kea] M. Keates, *On the greatest prime factor of a polynomial*, Proc. Edinburgh Math. Soc. (2) 16 (1968/1969) 301–303.

[Kho] N.M. Khodzhaev, *The problem of V. G. Sprindzhuk on the square-free part of the product of consecutive natural numbers,* Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1997, no. 1, 3-7, 78; transl.: Moscow Univ. Math. Bull. 52 (1997), no. 1, 1-5.

[Kot1] S.V. Kotov, *The greatest prime factor of a polynomial* (Russian), Mat. Zametki 13 (1973), 515–522.

[Kot2] S.V. Kotov, *The Thue-Mahler equation in relative fields* (Russian), Acta Arith. 27 (1975), 293–315.

[LS04] S. Laishram and T.N. Shorey, *Number of prime divisors in a product of consecutive integers*, Acta Arith. 113 (2004), 327-341.

[LS06a] S. Laishram and T.N. Shorey, *The greatest prime divisor of a product of consecutive integers*, Acta Arith., to appear.

[LS06b] S. Laishram and T.N. Shorey, *The greatest prime divisor of a product of terms in an arithmetic progression*, Indag. Math., to appear.

[Lan75a] M. Langevin, *Sur la fonction plus grand facteur premier. I-VI.* Sém. Delange-Pisot-Poitou, 16e année (1974/75), Exp. Nos. G3, G7, G10, G12, G22, 29 pp., Paris, 1975.

[Lan75b] M. Langevin, *Plus grand facteur premier d'entiers voisins*, C. R. Acad. Sci. Paris 281 (1975), A491–A493.

[Lan76] M. Langevin, *Méthodes élementaires en vue d'une théorème de Sylvester*, Sém. Delange-Pisot-Poitou, 17e année: 1975/76, Exp. No. G2, 9 pp., Paris, 1977.

[Lan77] M. Langevin, *Plus grand facteur premier d'entiers en progression arithmétique*, Sém. Delange - Pisot - Poitou, 18e année, 1976/77, Exp. No. 3, 6 pp., Paris, 1977.

[Lan80] M. Langevin, *Facteurs premiers des coefficients binomiaux*, Sém. Delange-Pisot-Poitou, 20e année: 1978/79, Exp. No. 27, 15 pp., Paris, 1980.

[Leh] D.H. Lehmer, *The prime factors of consecutive integers*, Amer. Math. Monthly 72 no. 2, part II (1965), 19–20.

[LW] H.-Q. Liu and J. Wu, *Numbers with a large prime factor*, Acta Arith. 89 (1999), 163-187.

[Mor] P. Moree, *On arithmetical progressions having only few different prime factors in comparison with their length*, Acta Arith. 70 (1995), 295-312.

[Pol] G. Pólya, *Zur arithmetischen Untersuchung der Polynome*, Math. Z. 1 (1918), 143-148.

[Rc70] K.Ramachandra, *A note on numbers with a large prime factor II*, J. Indian Math. Soc. 34 (1970), 39-48.

[Rc73] K. Ramachandra, *Application of Baker's theory to two problems considered by Erdős and Selfridge*, J. Indian Math. Soc. (N.S.) 37 (1973), 25–34 (1974).

[RcS] K. Ramachandra and T.N. Shorey, *On gaps between numbers with a large prime factor*, Acta Arith. 24 (1973), 99-111.

[RST] K. Ramachandra, T.N. Shorey and R. Tijdeman, *On Grimm's problem relating to factorisation of a block of consecutive integers II*, J. reine angew. Math. 288 (1976), 192-201.

[RrR] O. Ramaré and R. Rumely, *Primes in arithmetic progression*, Math. Comp. 65 (1996), 397-425.

[RrS] O. Ramaré and Y. Saouter, *Short effective intervals containing primes*, J. Number Th. 98 (2003), 10-33.

[Ran] Rankin, *The difference between consecutive prime numbers*, J. London Math. Soc. 13 (1938), 242-247.

[San] J.W. Sander, *On numbers with a large prime factor*, Acta Math. Hungar. 63 (1994), 149-165.

[SS03] N. Saradha and T.N. Shorey, *Almost squares and factorisations in consecutive integers*, Compositio Math. 138 (2003), 113-124.

[SST] N. Saradha, T.N. Shorey and R. Tijdeman, *Some extensions and refinements of a theorem of Sylvester*, Acta Arith. 102 (2002), 167-181.

[Sch] A. Schinzel, *On two theorems of Gelfond and some of their applications*, Acta Arith 13 (1967/1968), 177–236.

[Sho] T.N. Shorey, *On gaps between numbers with a large prime factor II*, Acta Arith. 25 (1974), 365-373.

[SPTS] T.N. Shorey, A.J. van der Poorten, R. Tijdeman and A. Schinzel, *Applications of the Gel'fond-Baker method to Diophantine equations*, Transcendence Theory: Advances and Applications, ed. by A. Baker and D.W. Masser, Academic Press, London (1977), 59-77.

[ST76] T.N. Shorey and R. Tijdeman, *On the greatest prime factors of polynomials at integer points*, Compositio Math. 33 (1976), 187-195.

[ST86] T.N. Shorey and R. Tijdeman, *Exponential Diophantine equations*, Cambridge Tracts in Mathematics, 87. Cambridge University Press, 1986.

[ST89] T.N. Shorey and R. Tijdeman, *On the number of prime factors of an arithmetical progression*, Jour. Sichuan Univ. 26 (1989), 72-74.

[ST90a] T.N. Shorey and R. Tijdeman *On the greatest prime factor of an arithmetical progression (II)*, Acta Arith. 53 (1990), 499-504.

[ST90b] T.N. Shorey and R. Tijdeman, *On the greatest prime factor of an arithmetical progression*, A Tribute to Paul Erdős, ed. A. Baker, B. Bollobas and A. Hajnal, Cambridge University Press (1990), 385-389.

[ST92a] T.N. Shorey and R. Tijdeman *On the number of prime factors of a finite arithmetical progression*, Acta Arith. 61 (1992), 375-390.

[ST92b] T.N. Shorey and R. Tijdeman, *On the greatest prime factor of an arithmetical progression (III),* Diophantine Approximation and Transcendental Numbers, Luminy 1990, ed. Ph. Philippon, Walter de Gruyter, New York (1992), 275-280.

[Spr71] V.G. Sprindzhuk, *The greatest prime divisor of a binary form* (Russian), Dokl. Akad. Nauk BSSR 15 (1971), 389–391.

[Spr82] V.G. Sprindzhuk, *Classical Diophantine equations in two unknowns* (Russian), Nauka, Moscow, 1982.

[Syl] J.J. Sylvester, *On arithmetical series*, Messenger Math. 21 (1892), 1-19, 87-120.

[Tij] R. Tijdeman, *On the maximal distance of numbers with a large prime factor*, J. London Math. Soc. 5 (1972), 313-320.

[Utz] W.R. Utz, *A conjecture of Erdős concerning consecutive integer* Amer. Math. Monthly 68 (1961), 896–897.

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