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Jacobian nullwerte and algebraic equations ☆

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Abstract

We present two applications of jacobian nullwerte, both related with the resolution of algebraic equations of any degree. We give a very simple expression of the roots of a polynomial of arbitrary degree in terms of derivatives of hyperelliptic theta functions. This expression can be understood as an explicit proof of Torelli's theorem in the hyperelliptic case. We also give geometrical expressions of the discriminant of a polynomial. Both applications are based on a jacobian version of Thomae's formula. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

In the last decades of the 19th century, Rosenhain [1], Thomae [2], Riemann [3] and Frobenius [4] began the study of determinants of derivatives of odd theta functions, in order to find generalizations of Jacobi's derivative formula [5]:

$$\theta'_{11}(0,\tau) = -\pi\theta_{00}(0,\tau)\theta_{01}(0,\tau)\theta_{10}(0,\tau).$$

They discovered fascinating formulae, expressing the value at zero of these jacobians ("*jacobian nullwerte*") as products of zero values of even Theta functions ("*Thetanullwerte*"). For much of the twentieth century, these jacobian

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nullwerte seemed to be forgotten, until Weil asked Igusa whether any jacobian nullwerte could be expressed as a polynomial in the even Thetanullwerte, or at least as a rational function of them. Igusa showed that the second assertion is true, but not the first. He also used the jacobian nullwerte to show the non-integral closeness of the ring of theta constants (cf. [6,7]).

In this paper we present two applications of jacobian nullwerte, both related with the resolution of algebraic equations.

It is well known that it is possible to express the roots and the discriminant of a polynomial in terms of hyperelliptic theta functions and hyperelliptic integrals (cf. [8,9]). These formulae are rather intricate, and the geometry behind them is not evident. Their proofs rely on the classical formula of Thomae, which relates some products of differences between the roots of a polynomial with a given Thetanullwerte.

We present here a more geometrical approach to the resolution of algebraic equations. Using Riemann's theorems on the theta function, we are able to give a very simple expression of the roots of a polynomial in terms of derivatives of odd theta functions. Geometrically speaking, we give an effective method to recover a hyperelliptic curve from its jacobian, that is, we provide an explicit proof of Torelli's theorem in the hyperelliptic case.

As a second application of the jacobian Thetanullwerte, we discuss the expression of the discriminant of a polynomial as a value of a Siegel modular form. We propose formulas which give a geometrical content to the classical definition of the discriminant as a modular form.

2. Preliminaries

2.1. Notations

Throughout the paper, C will be a smooth curve of genus $g > 1$ defined over the complex field $\mathbb{C}$. Given a divisor D on C, $[D]$ will denote its linear equivalence class; we will write $l(D) = h^0(C, O(D))$, and the sign $\equiv$ will be used to express the linear equivalence of divisors on C.

Given a separable polynomial $f(X) \in \mathbb{C}[X]$ of degree n, we will denote by C_f the genus $g = \lfloor (n-2)/2 \rfloor$ hyperelliptic curve given by

$$C_f : Y^2 = f(X) = a_n X^n + \cdots + a_0 = a_n (X - \alpha_1) \cdots (X - \alpha_n).$$

In fact, the hyperelliptic curves that we will work with will always be of this type, so that the symbol C_f will always stand for a hyperelliptic curve associated with certain polynomial $f(X)$. We will denote by $W_i = (\alpha_i, 0),\ i = 1, \ldots, n$ its Weierstrass points.

For $\alpha_1, \ldots, \alpha_g \in \mathbb{C}$, let $V(\alpha_1, \ldots, \alpha_g)$ denote the Vandermonde matrix:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_g \\ \vdots & \vdots & & \vdots \\ \alpha_1^{g-1} & \alpha_2^{g-1} & \cdots & \alpha_g^{g-1} \end{pmatrix},$$

whose determinant is

$$\left|V(\alpha_1, \ldots, \alpha_g)\right| = (-1)^{g(g-1)/2} \prod_{j<k} (\alpha_j - \alpha_k).$$

The symbol $\operatorname{diag}(x_1, \ldots, x_n)$ will stand for the matrix whose diagonal entries are $x_1, \ldots, x_n$ and remaining entries are zero.

We begin by recalling the main classical results that will be used in the paper. There are many excellent books [5,10–12] which can be used as general references for the rest of the paper.

2.2. *The Abel–Jacobi map*

Let C be a complex smooth curve of genus $g > 1$. We pick a symplectic basis $\gamma_1, \ldots, \gamma_{2g}$ of the singular homology $H_1(C, \mathbb{Z})$ of C, and a basis $\omega_1, \ldots, \omega_g$ of $H^0(C, \Omega^1)$, the space of holomorphic differentials on C. The period matrix of C with respect to these bases is the $g \times 2g$ matrix

$$\Omega = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_{2g}} \omega_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega_g & \cdots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}.$$

We write $\Omega = (\Omega_1 | \Omega_2)$ to denote by Ω_1 the first half of Ω (that is, the matrix formed by the first g columns of Ω), and by Ω_2 the second half. With these data, we can consider a second basis of $H^0(C, \Omega^1)$, given by

$$\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_g \end{pmatrix} = \Omega_1^{-1} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix}. \tag{1}$$

The period matrix of C with respect to this new basis is $(1_g | Z)$, where 1_g denotes the $g \times g$ identity matrix and Z belongs to the Siegel upper half space $\mathbb{H}_g$, i.e., it is a symmetric matrix whose imaginary part is positive definite (these are called the Riemann conditions). A matrix satisfying the Riemann conditions is called a normalized period matrix. The jacobian variety $J(C)$ of the curve C is the complex torus $J(C) := \mathbb{C}^g / (1_g | Z)$; the Riemann conditions ensure that this torus is projective, which implies that $J(C)$ is an abelian variety.

Let us denote by C_d the d-fold symmetric product of C, which can be identified with the set of effective divisors of degree d on C. We fix once for all a Weierstrass point $W \in C$ as base point for the Abel–Jacobi map in degree d, which is defined by

$$\begin{aligned} u_d : C_d &\to J(C), \\ D = \sum_i m_i P_i &\mapsto u_d(D) := \sum_i m_i \int_W^{P_i} (\eta_1, \ldots, \eta_g). \end{aligned}$$

Of course, the values of the integrals depend on the path chosen to integrate, but the class of $u_d(D)$ on $\mathbb{C}^g/(1_g|Z)$ is well-defined. Unless explicitly stated, the choice of the base point will not affect any of the results, so we will make no further reference to it.

It is usual to extend the Abel–Jacobi map u_d to non-effective divisors: if

$$D = \sum_{j=1}^{d+r} P_j - \sum_{k=1}^{r} Q_k,$$

we take

$$u_d(D) = \sum_{j=1}^{d+r} \int_W^{P_i} (\eta_1, \ldots, \eta_g) - \sum_{k=1}^{r} \int_W^{Q_k} (\eta_1, \ldots, \eta_g).$$

Abel's theorem tell us that the Abel–Jacobi map is invariant through linear equivalence:

$$D_1 \equiv D_2 \quad \Longrightarrow \quad u_d(D_1) = u_d(D_2).$$

2.3. Riemann theorems

For $z \in \mathbb{C}^g$, we define the *Riemann theta function* of $J(C)$ by

$$\theta(z) = \theta(z; Z) = \sum_{n \in \mathbb{Z}^g} \exp\bigl(\pi i\, {}^t n.Z.n + {}^t n.z\bigr).$$

This function is quasi-periodic with respect to the columns of the period matrix $(1_g|Z)$. Although it cannot be considered as a function on $J(C)$, its zero locus is a well-defined divisor Θ in $J(C)$, called the theta divisor. Two fundamental results of Riemann relate the geometry of Θ with the geometry of C.

Theorem 2.1 (Riemann theorem). *The image of the Abel–Jacobi map in degree $g-1$ is a translate of the theta divisor*: *for a suitable $\kappa \in J(C)$ we have $u_{g-1}(C_{g-1}) = \Theta - \kappa$.*

The constant κ, which is usually called the *Riemann vector*, of course depends on the base point W. We define a new map $\Pi : C_{g-1} \to J(C)$, given by $\Pi(D) =$

$u_{g-1}(D) + \kappa$. This map satisfies the symmetry condition: $\Pi(K - D) = -\Pi(D)$, where K denotes a canonical divisor. The map Π is extended to non-effective divisors of degree $g - 1$ in the same way as the map u_{g-1}.

Abel's theorem assures that around an effective divisor D of degree $g - 1$ such that $l(D) = 0$, the Abel–Jacobi map u_{g-1} is one-to-one, so by Riemann theorem, the theta divisor is smooth at the point $u_{g-1}(D)$. The Riemann singularity theorem explains what happens for *exceptional* divisors of degree $g - 1$.

Theorem 2.2 (Riemann singularity theorem). *For every effective divisor D of degree $g - 1$, the multiplicity of the theta divisor Θ at the point $\Pi(D)$ is $l(D)$*:

$$\mathrm{mult}_{\Pi(D)}\Theta = l(D).$$

We are mainly interested in the case of effective divisors D with $l(D) = 1$. For such a divisor, Riemann singularity theorem says that the function θ vanishes at point $\Pi(D)$, and that at least one of the first derivatives is non-zero at $\Pi(D)$.

3. The canonical map

The choice of the basis $\omega_1, \dots, \omega_g$ of the holomorphic differential forms on C provides a map from C to $\mathbb{P}^{g-1} = \mathbb{P}H^0(C, \Omega^1)^*$, given by

$$\begin{aligned} \phi : C &\to \mathbb{P}^{g-1}, \\ P &\mapsto \phi(P) = \big(\omega_1(P), \dots, \omega_g(P)\big). \end{aligned}$$

This is the *canonical map* of C. Note that if the curve C (and the differential forms ω_j) is defined over a number field K, the canonical map is also defined over this field.

Proposition 3.1. *Let $P_1, \dots, P_{g-1} \in C(\overline{K})$ such that the divisor $D = P_1 + \cdots + P_{g-1}$ satisfies $l(D) = 1$. The equation*:

$$H_D(X_1, \dots, X_g) := \left(\frac{\partial\theta}{\partial z_1}\big(\Pi(D)\big), \dots, \frac{\partial\theta}{\partial z_g}\big(\Pi(D)\big)\right)\Omega_1^{-1}\begin{pmatrix} X_1 \\ \vdots \\ X_g \end{pmatrix} = 0$$

determines a hyperplane H_D of $\mathbb{P}^{g-1}$, which cuts the divisor $\phi(D)$ on the curve $\phi(C)$.

Proof. In [10, p. 228] is seen that the points P_i satisfy the equation

$$\sum_{k=1}^{g} \frac{\partial\theta}{\partial z_k}\big(\Pi(D)\big)\eta_k(P) = 0,$$

and the proposition follows from the relation (1). □

Corollary 3.1. *If the curve C is defined over a number field K, any quotient of the coefficients of the equation in Proposition* 3.1 *is algebraic over the field* $K(P_1, \dots, P_{g-1})$.

Proof. The hyperplane $H_D \subset \mathbb{P}^{g-1}$ determined by the points $P_1, \dots, P_{g-1}$ is defined over the field $K(P_1, \dots, P_{g-1})$, so that any equation having a coefficient equal to one must have all its coefficients in this field. $\square$

4. Theta characteristics on hyperelliptic curves. Two-torsion points

A semicanonical divisor on a curve of genus g is a divisor D of degree $g-1$ such that $2D$ is linearly equivalent to a canonical divisor. A *geometric theta characteristic* is the class of a semicanonical divisor. It is well known that there are 4^g different theta characteristics. A semicanonical divisor D (and the corresponding geometric theta characteristic) is called *even* or *odd* according to the parity of $l(D)$.

For a hyperelliptic curve, all these classes can be described with divisors supported on the Weierstrass points.

Lemma 4.1 [10]. *Let C_f be a hyperelliptic curve of genus g defined over an algebraically closed field, with Weierstrass points $W_1, \dots, W_{2g+2}$. Given a set of indices $I = \{k, i_1, \dots, i_{g-1-2m}\}$, denote by D_I the divisor $2mW_k + W_{i_1} + \cdots + W_{i_{g-1-2m}}$.*

- (a) *Any semicanonical divisor on C_f is linearly equivalent to one of the form D_I for some set of indices I with $-1 \leqslant m \leqslant (g-1)/2$ and $i_r \neq i_s$ for $r \neq s$.*
- (b) *Two semicanonical divisors $2mW_k + W_{i_1} + \cdots + W_{i_{g-1-2m}}$ and $2nW_{k'} + W_{j_1} + \cdots + W_{j_{g-1-2n}}$ are linearly equivalent if and only if $m = n = -1$ and $\{i_1, \dots, i_{g+1}, j_1, \dots, j_{g+1}\} = \{1, 2, \dots, 2g+2\}$.*
- (c) $l(D_I) = m + 1$.
- (d) *If $l(D_I) = 1$ then D_I is the sum of $g-1$ Weierstrass points on C_f.*

We introduce the following definition.

Definition 4.1. A Weierstrass divisor on C_f is a semicanonical divisor satisfying condition (d) of Lemma 4.1.

The set $J(C_f)[2]$ of 2-torsion points in $J(C_f) = \mathbb{C}^g/(1_g|Z)$ can be described as

$$J(C_f)[2] = \left\{ z_m = \frac{1}{2} Z.m' + \frac{1}{2} m'' \;\middle|\; m = \begin{pmatrix} m' \\ m'' \end{pmatrix},\ m', m'' \in \{0,1\}^g \right\}.$$

The point z_m is said to be *even* (respectively *odd*) if $m'.m'' \equiv 0 \pmod 2$ (respectively $\equiv 1 \pmod 2$). The set of 2-torsion points of $J(C)$ splits into two disjoint subsets Σ^+ and Σ^- of even and odd 2-torsion points, respectively.

The image of a geometric theta characteristic $[D]$ through the Abel–Jacobi map Π introduced in Section 2.3 is a 2-torsion point of the jacobian of the curve, since $-\Pi(D) = \Pi(K - D) = \Pi(D)$ for any canonical divisor K. In fact, the Abel–Jacobi map establishes a bijection between the set of theta characteristics and the set of 2-torsion points of the jacobian, sending even (respectively odd) geometric theta characteristics to even (respectively odd) 2-torsion points.

Definition 4.2. A Weierstrass 2-torsion point is an element of $J(C_f)[2]$ which is the image through the Abel–Jacobi map of the class of a Weierstrass divisor.

Riemann theorem, combined with part (d) of Lemma 4.1, gives a complete characterization of the Weierstrass 2-torsion points: they are the odd 2-torsion points $w \in J(C_f)[2]$ such that

$$\left(\frac{\partial\theta}{\partial z_1}(w), \dots, \frac{\partial\theta}{\partial z_g}(w)\right) \neq (0, \dots, 0).$$

The bijection between geometric theta characteristics and 2-torsion points of the jacobian can be made completely explicit, once we have fixed an ordering of the roots of the polynomial $f(X)$ which defines the hyperelliptic curve C_f. This is explained in [12, pp. 3.75–3.88]. For the sake of completeness, we will briefly outline the process. We take the *standard basis* for $H_1(C_f, \mathbb{Z})$ described in *loc. cit.*, p. 3.76. We compute the period matrix $\Omega = (\Omega_1|\Omega_2)$ of some basis of $H^0(C_f, \Omega)$ with respect to this homology basis, and take the corresponding normalized period matrix $Z = \Omega_1^{-1}\Omega_2$ (observe the change of notation with respect to Mumford). For $i = 1, \dots, g+1$, let

$$\eta_{2i-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} {}^t\eta'_{2i-1} \\ {}^t\eta''_{2i-1} \end{pmatrix},$$

$$\eta_{2i} = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} {}^t\eta'_{2i} \\ {}^t\eta''_{2i} \end{pmatrix},$$

where the non-zero entry in the first row is at the ith column in both cases (and $\eta_{2g+2} = \left(\begin{smallmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{smallmatrix}\right)$). Now, for any set $S \subset \{1, 2, \dots, 2g+2\}$ we define $\eta_S := \sum_{i \in S} \eta_i \pmod 2$ and

$$z_i := \frac{1}{2} Z.\eta'_i + \frac{1}{2}\eta''_i, \qquad z_S := \sum_{i \in S} z_i.$$

We take the point W_{2g+2} as base point for the Abel–Jacobi map. With these choices, the image of a semicanonical divisor $D_I = 2mW_k + W_{i_1} + \cdots + W_{i_{g-1-2m}}$ through the Abel–Jacobi map is

$$u_{g-1}(D_I) = z_I,$$

and the Riemann vector is

$$\kappa = z_{\{1,3,\dots,2g+1\}} = \frac{1}{2}Z.\begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} g \\ g-1 \\ \vdots \\ 1 \end{pmatrix},$$

so that

$$w_I := \Pi(D_I) = z_{I \circ U}.$$

Finally, we have the following relation:

$$\Pi(D_I) + \Pi(D_J) = \Pi(D_{I \circ J \circ U}), \tag{2}$$

where $U = \{1, 3, \dots, 2g+1\}$ and $S \circ T$ denotes the symmetric difference of two sets S, T.

For the rest of the paper, we will assume that we are working in this frame, in particular with the choice of the standard basis for $H_1(C_f, \mathbb{Z})$.

5. Jacobian nullwerte. Fundamental systems

We will call *analytic theta characteristic* a $2 \times g$ matrix $\eta = \binom{{}^t\eta'}{{}^t\eta''}$ with entries in $\{0, 1\}$. We say that η is even (respectively odd) if $\eta'.\eta'' \equiv 0 \pmod 2)$ (respectively $\equiv 1 \pmod 2$). We denote by TC_g (respectively TC_g^+, TC_g^-) the set of all (respectively even, odd) analytic theta characteristics of dimension g.

Every analytic theta characteristic η defines a *theta function with characteristic*:

$$\theta[\eta](z; Z) := \exp\left(\frac{1}{4}\pi i\, {}^t\eta'.Z.\eta' + \frac{1}{2}\pi i \eta'.(z + \eta'')\right) \times \theta\left(z + \frac{1}{2}Z.\eta' + \frac{1}{2}\eta''; Z\right).$$

The values of these functions at $z = 0$ are called *Thetanullwerte*. It is common to write $\theta[\eta](Z)$ instead of $\theta[\eta](0; Z)$. Thetanullwerte have been widely studied, and there are many excellent references for them. But we are now interested on the values at zero of the derivatives of the theta functions with characteristics:

Definition 5.1 [7]. A *jacobian nullwerte* is a determinant of the form:

$$[\eta_1,\dots,\eta_g](Z) := \det\begin{pmatrix} \frac{\partial\theta[\eta_1]}{\partial z_1}(0;Z) & \cdots & \frac{\partial\theta[\eta_1]}{\partial z_g}(0;Z) \\ \vdots & & \vdots \\ \frac{\partial\theta[\eta_g]}{\partial z_1}(0;Z) & \cdots & \frac{\partial\theta[\eta_g]}{\partial z_g}(0;Z) \end{pmatrix},$$

where $\eta_1,\dots,\eta_g$ are different odd analytic theta characteristics.

Jacobian nullwerte can be thought of as (modular) functions over the Siegel upper half space. Rosenhain, Thomae, Frobenius and Riemann obtained beautiful results for jacobian nullwerte in low genera. Most of these results can be understood as generalizations of Jacobi's derivative formula. Igusa [6,7] studied the Jacobian nullwerte for general genus, in connection with a problem raised by Weil about the integral closeness of the ring of the Thetanullwerte. We recall one of Igusa's results about Jacobian nullwerte.

For an analytic theta characteristic η, we put $e(\eta)=1$ if η is even, and $e(\eta)=-1$ if η is odd. Given three analytic theta characteristics η_1,η_2,η_3 we define

$$e(\eta_1,\eta_2) = e(\eta_1)e(\eta_2)e(\eta_1+\eta_2),$$

$$e(\eta_1,\eta_2,\eta_3) = e(\eta_1)e(\eta_2)e(\eta_3)e(\eta_1+\eta_2+\eta_3).$$

We say that the triplet $\{\eta_1,\eta_2,\eta_3\}$ is azygetic (respectively zygetic) if $e(\eta_1,\eta_2,\eta_3)$ is -1 (respectively 1). A *fundamental system* of analytic theta characteristics is a set $S=\{\eta_1,\dots,\eta_{2g+2}\}$ of $2g+2$ analytic theta characteristics such that $\eta_1,\dots,\eta_g$ are odd, $\eta_{g+1},\dots,\eta_{2g+2}$ are even and every triplet $\{\eta_i,\eta_j,\eta_k\}\subset S$ is azygetic. The last condition is equivalent to $e(\eta_1+\eta_i,\eta_1+\eta_j)=-1$ for every pair $i\neq j$. The analytic theta characteristics in a fundamental system are always *essentially independent*, which means that every sum of an even number of them is non-zero.

Let $\mathbb{C}[\theta]$ be the "ring of Thetanullwerte", which is the $\mathbb{C}$-algebra generated by the functions $\theta[\eta](Z)$, where η runs over the set of even analytic theta characteristics.

Theorem 5.1 [6]. *Let $\eta_1,\dots,\eta_g$ be odd analytic theta characteristics such that the function $[\eta_1,\dots,\eta_g](Z)$ is not identically zero and is contained in the ring of Thetanullwerte $\mathbb{C}[\theta]$. Then $\eta_1,\dots,\eta_g$ can be completed to form a fundamental system, and*

$$[\eta_1,\dots,\eta_g](Z) = \pi^g \sum_{\{\eta_{g+1},\dots,\eta_{2g+2}\}\in\mathcal{S}} \pm \prod_{i=g+1}^{2g+2} \theta[\eta_i](0;Z),$$

where $\mathcal{S}$ is the set of all $(g+2)$-tuples $\{\eta_{g+1},\dots,\eta_{2g+2}\}$ of even theta characteristics such that $\{\eta_1,\dots,\eta_g,\ \eta_{g+1},\dots,\eta_{2g+2}\}$ form a fundamental system. If Z is the period matrix of a hyperelliptic curve, there is exactly one non-zero term in the sum of the right hand side of the equality.

We are especially interested in the values of jacobian nullwerte at hyperelliptic period matrices, i.e., at matrices $Z_f \in \mathbb{H}_g$ which are the period matrix of a hyperelliptic genus g curve C_f. The complex abelian variety $\mathbb{C}^g/(1_g|Z)$ is then the jacobian $J(C_f)$ of C_f, and the analytic theta characteristics can be naturally tied with geometric theta characteristics, following the process explained in Section 4. When we work in this context, we apply the terminology introduced for analytic theta characteristics to semicanonical divisors and to 2-torsion points on $J(C_f)$. For instance, the parity of a 2-torsion point $w \in J(C_f)$ is denoted by $e(w)$, and $e(w_1,w_2,w_3)$ is defined in the obvious way for three 2-torsion points $w_1,w_2,w_3 \in J(C_f)$. A fundamental system of 2-torsion points in $J(C_f)$ (respectively of semicanonical divisors in C_f) is a set of 2-torsion points (respectively of semicanonical divisors) whose corresponding analytic theta characteristic form a fundamental system. With this in mind, we define the following specific jacobian nullwerte:

Definition 5.2. Let $D_1,\dots,D_g$ be g Weierstrass divisors on a hyperelliptic genus g curve C_f, with period matrix $\Omega=(\Omega_1|\Omega_2)$. Let $Z_f=\Omega_1^{-1}\Omega_2$ and write $w_i=\Pi(D_i)$.

$$J(D_1,\dots,D_g)=J(w_1,\dots,w_g):=\begin{pmatrix} \frac{\partial\theta}{\partial z_1}(w_1;Z_f) & \cdots & \frac{\partial\theta}{\partial z_g}(w_1;Z_f) \\ \vdots & & \vdots \\ \frac{\partial\theta}{\partial z_1}(w_g;Z_f) & \cdots & \frac{\partial\theta}{\partial z_g}(w_g;Z_f) \end{pmatrix}.$$

6. Fundamental systems for hyperelliptic curves

Let C_f be a hyperelliptic curve of genus g, with Weierstrass points $W_1,\dots,W_{2g+2}$. We will show a geometric way of constructing fundamental systems on $J(C_f)$. We pick a set of g Weierstrass points, which we assume to be $W_1,\dots,W_g$ to simplify. We define

$$\begin{aligned} D &= \sum_{i=1}^{g} W_i, \\ D_i &= D - W_i, \quad i=1,\dots,g, \\ D_i &= D + W_i - 2W_{2g+2}, \quad i=g+1,\dots,2g+1, \\ w_i &= \Pi(D_i). \end{aligned}$$

Proposition 6.1. *The set $w_1, \ldots, w_{2g+2}$ is a fundamental system of 2-torsion points in $J(C_f)$.*

Proof. By Proposition 4.1, the divisors D_i have the right parity, so we have only to check that $e(w_1 + w_i, w_1 + w_j) = -1$ for every pair $i \neq j$. Let $U = \{1, 3, \ldots, 2g+1\}$. Applying formula (2) we find that $w_i + w_j = \Pi(D_i) + \Pi(D_j) = \Pi(D_{\{i,j\} \circ U})$. Hence, $e(w_i + w_j) = (-1)^{i+j+1}$, and $e(w_1 + w_i, w_1 + w_j) = (-1)^{2i+2j+1} = -1$. □

7. Jacobian nullwerte on hyperelliptic curves

Let

$$C_f : Y^2 = f(X) = a_n(X - \alpha_1) \cdots (X - \alpha_n) \in \mathbb{C}$$

be a complex hyperelliptic curve of genus $g = \lfloor (n-2)/2 \rfloor$. We fix

$$\omega_1 = \frac{\mathrm{d}x}{y}, \qquad \omega_2 = \frac{x\,\mathrm{d}x}{y}, \qquad \ldots, \qquad \omega_g = \frac{x^{g-1}\,\mathrm{d}x}{y}$$

as the basis of the holomorphic differential forms and denote by $\Omega = (\Omega_1 | \Omega_2)$ the period matrix of C_f with respect to this basis of $H^0(C_f, \Omega^1)$ and *some* symplectic basis of $H_1(C_f, \mathbb{Z})$. The canonical map of C_f with respect to $\omega_1, \ldots, \omega_g$ has a very simple expression:

$$\begin{aligned} \phi : C_f &\to \mathbb{P}^{g-1}, \\ P = (x, y) &\mapsto \phi(P) = \big(1, x, \ldots, x^{g-1}\big). \end{aligned}$$

Theorem 7.1. *Let $W_1 = (\alpha_1, 0), \ldots, W_g = (\alpha_g, 0) \in C_f(\mathbb{C})$ be g different Weierstrass points on C_f. Take $D_i = \sum_{j \neq i} W_j$, $w_i = \Pi(D_i)$ and let H_{D_i} be the hyperplane associated to D_i by Proposition* 3.1. *Then*

(a) $$J(w_1, \ldots, w_g)\Omega_1^{-1} V(\alpha_1, \ldots, \alpha_g) = \mathrm{diag}\big(H_{D_1}(\alpha_1), \ldots, H_{D_g}(\alpha_g)\big). \tag{3}$$

(b) *For any Weierstrass point $P = (\alpha, 0) \notin \{W_1, \ldots, \widehat{W}_i, \ldots, W_g\}$, the quotients*

$$\frac{1}{H_{D_i}}(\alpha)\left(\frac{\partial\theta}{\partial z_1}\big(\Pi(D_i)\big) \cdots \frac{\partial\theta}{\partial z_g}\big(\Pi(D_i)\big)\right)\Omega_1^{-1}$$

belong to the field $\mathbb{Q}(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_g, \alpha)$.

Proof. The divisors D_k satisfy $l(D_k) = 1$, so that the corresponding hyperplanes H_{D_i} are well-defined, and cut the divisor $\phi(D_k)$ on $\phi(C_f)$, by Proposition 3.1. This is the content of (a). As for (b), it is true for $P = W_i$ since

$$\mathrm{diag}\big(H_{D_1}(\alpha_1)^{-1}, \ldots, H_{Dg}(\alpha_g)^{-1}\big) J(w_1, \ldots, w_g)\Omega_1^{-1} = V(\alpha_1, \ldots, \alpha_g)^{-1},$$

which is a matrix with entries in the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_g)$. If we replace W_i by any Weierstrass point different from the remaining W_k, the divisor D_i will not change, and hence the hyperplane H_{D_i} will be exactly the same. □

Taking determinants in formula (3), we obtain the following corollary, which could be considered as a *jacobian version* of Thomae's formula (cf. Section 11).

Corollary 7.1.

$$\left|V(\alpha_1, \ldots, \alpha_g)\right| = \det \Omega_1 \left(\prod_{j=1}^{g} H_{D_j}\left(1, \alpha_j, \ldots, \alpha_j^{g-1}\right) \right) \Big/ \left|J(w_1, \ldots, w_g)\right|.$$

8. Solving algebraic equations with jacobian nullwerte

Let $f(X) = a_n(X - \alpha_1) \cdots (X - \alpha_n) \in \mathbb{C}[X]$ be a polynomial of arbitrary degree $n > 4$ with different roots. We will express the roots $\alpha_1, \ldots, \alpha_n$ of f in terms of jacobian nullwerte on the jacobian of the hyperelliptic curve

$$C_f : Y^2 = f(X).$$

As in the previous section, we fix

$$\omega_1 = \frac{\mathrm{d}x}{y}, \qquad \omega_2 = \frac{x\,\mathrm{d}x}{y}, \qquad \ldots, \qquad \omega_g = \frac{x^{g-1}\,\mathrm{d}x}{y}$$

as basis of $H^0(C_f, \Omega^1)$. Let $\Omega = (\Omega_1|\Omega_2)$ the period matrix of $\omega_1, \ldots, \omega_g$ with respect to the *standard* symplectic basis of $H_1(C_f, \mathbb{Z})$ (cf. [12, p. 3.76]), and let $Z_f = \Omega_1^{-1}\Omega_2$ be the corresponding normalized period matrix, so that $J(C_f) = \mathbb{C}^g/(1_g|Z_f)$.

Note that we have $\binom{2g-1}{g-2}$ Weierstrass D divisors which contain a given Weierstrass point $W_k = (\alpha_k, 0)$. For any of these divisors, the hyperplane H_D given by Proposition 3.1 passes through the point $\phi(W) = (1 : \alpha_k : \alpha_k^2 : \cdots : \alpha_k^{g-1})$. Now, if we take $g - 1$ different Weierstrass divisors containing W and intersect the corresponding hyperplanes (which amounts to solving a linear system) we recover the coordinates of $\phi(W)$, and hence we find the root α_k of the polynomial $f(x)$. We thus obtain the following theorem, which can be seen as a completely explicit version of Torelli's theorem for hyperelliptic curves:

Theorem 8.1. *Let $W_1 = (\alpha_1, 0), \ldots, W_n = (\alpha_n, 0)$ be g different Weierstrass points on C_f. Let $D_i = \Sigma_{j \neq i} W_j$, and take $w_i = \Pi(D_i)$. If $(x_1, \ldots, x_g)$ is* *a solution of the linear system of equations*:

$$\begin{pmatrix} \frac{\partial\theta}{\partial z_1}(w_1) & \cdots & \frac{\partial\theta}{\partial z_g}(w_1) \\ \vdots & & \vdots \\ \frac{\partial\theta}{\partial z_1}(w_{i-1}) & \cdots & \frac{\partial\theta}{\partial z_g}(w_{i-1}) \\ \frac{\partial\theta}{\partial z_1}(w_{i+1}) & \cdots & \frac{\partial\theta}{\partial z_g}(w_{i+1}) \\ \vdots & & \vdots \\ \frac{\partial\theta}{\partial z_1}(w_g) & \cdots & \frac{\partial\theta}{\partial z_g}(w_g) \end{pmatrix} \Omega_1^{-1} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

then $\alpha_i = x_2/x_1$.

In some situations, one has a period matrix of C_f with respect to a symplectic basis which is not the *standard* one. As will be seen below, this has no effect for genus 2 and 3. For higher genus, this may present a problem, since we do not know a priori which odd 2-torsion points we must take to find the roots of f. If we take a 2-torsion point which does not come from a Weierstrass divisor we will detect it immediately, since we will obtain

$$\left(\frac{\partial\theta}{\partial z_1}(w_1), \dots, \frac{\partial\theta}{\partial z_g}(w_1)\right) = (0, \dots, 0).$$

It is more difficult to know which combinations of Weierstrass torsion points we must take. In this case, we can use the following approach.

Theorem 8.2. *Let $w \in J(C_f)$ be a Weierstrass 2-torsion point. Let*

$$H_w(X_1, \dots, X_g) := \left(\frac{\partial\theta}{\partial z_1}(w) \cdots \frac{\partial\theta}{\partial z_g}(w)\right) \Omega_1^{-1} \begin{pmatrix} X_1 \\ \vdots \\ X_g \end{pmatrix} = \sum_{j=1}^{g} h_j X_j.$$

The polynomial $f_w(X) = h_1 + h_2 X + \cdots + h_g X^{g-1}$ has $g-1$ common roots with the polynomial f.

Proof. Let $W_1 = (\alpha_1, 0), \dots, W_{g-1} = (\alpha_{g-1}, 0)$ be the Weierstrass points on C_f such that $w = \Pi(W_1 + \cdots + W_{g-1})$. The α_j satisfy the equality

$$H_w\left(1, \alpha_j, \alpha_j^2, \dots, \alpha_j^{g-1}\right) = 0,$$

so they are roots of f_w. □

In the following two sections we give specific versions of Theorems 8.1 and 8.2 for genus 2 and 3, where they are especially simple.

9. The quintic and sextic equations

Let $f(x) = a_6X^6 + a_5X^5 + a_4X^4 + a_3X^3 + a_2X^2 + a_1X + a_0 \in \mathbb{C}[X]$ be a separable polynomial of degree 5 or 6. Consider the associated genus 2 hyperelliptic curve C_f defined by $Y^2 = f(X)$, which has 6 Weierstrass points $W_1 = (\alpha_1, 0),\ \ldots,\ W_6 = (\alpha_6, 0)$ given by the roots α_k of the polynomial f (we take $\alpha_6 = \infty$ if $\deg f = 5$). The Weierstrass divisors are precisely these six points, $D_k = W_k,\ k = 1, \ldots, 6$. The hyperplanes $H_k := H_{D_k}$ intersect the canonical curve $\phi(C_f)$ exactly on the images $\phi(W_k)$ of the Weierstrass points through the canonical map. The jacobian has exactly 6 odd 2-torsion points. Hence, any odd 2-torsion point is a Weierstrass 2-torsion point. In this situation, Theorem 8.1 reads:

Theorem 9.1. *Let*

$$f(x) = a_6X^6 + a_5X^5 + a_4X^4 + a_3X^3 + a_2X^2 + a_1X + a_0 \in \mathbb{C}[X]$$

be a separable polynomial of degree 5 *or* 6*. The six roots of the polynomial f are the ratios $x_{k,2}/x_{k,1}$, given by the solutions $(x_{k,1}, x_{k,2})$ of the homogeneous equations*:

$$H_k(X_1, X_2) := \left(\frac{\partial\theta}{\partial z_1}(w_k)\ \frac{\partial\theta}{\partial z_2}(w_k)\right)\Omega_1^{-1}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 0, \quad 1 \leqslant k \leqslant 6.$$

Here $\Omega = (\Omega_1|\Omega_2)$ is the period matrix of the curve C_f with respect to the basis $\omega_1 = \mathrm{d}x/y$, $\omega_2 = x\,\mathrm{d}x/y$ of $H^0(C_f, \Omega^1)$ and any *symplectic basis of $H_1(C_f, \mathbb{Z})$, $Z_f = \Omega_1^{-1}\Omega_2$, and*

$$w_1 = \frac{1}{2}Z_f\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad w_2 = \frac{1}{2}Z_f\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$w_3 = \frac{1}{2}Z_f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad w_4 = \frac{1}{2}Z_f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$w_5 = \frac{1}{2}Z_f\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad w_6 = \frac{1}{2}Z_f\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

are the six odd 2*-torsion points of $J(C_f)$.*

Note that we do not need to use any particular homology basis, since the only effect of a symplectic change of basis will be a reordering of the values of the ratios $x_{k,1}/x_{k,2}$.

10. The septic and octic equations

Let us consider now the case of a hyperelliptic curve C_f of genus 3, or equivalently, the problem of solving a separable equation of degree 7 or 8, $f(x) =$

$a_8X^8 + \cdots + a_0$. We have now 8 Weierstrass points $W_1 = (\alpha_1, 0), \ldots, W_8 = (\alpha_8, 0)$ (one of the α_i may be ∞) which give rise to 28 Weierstrass divisors $D_{ij} = W_i + W_j$, $1 \leqslant i < j \leqslant 28$. The images $w_{ij} = \Pi(D_{ij})$ of these divisors through the Abel–Jacobi map are exactly all the 28 odd 2-torsion points in $J(C_f)$, so any odd 2-torsion point is a Weierstrass 2-torsion point. The specific version of Theorem 8.2 for this case is the following corollary.

Corollary 10.1. *Let $f(x) = a_8X^8 + \cdots + a_0$ be a separable polynomial of degree* 7 *or* 8. *Let $\Omega = (\Omega_1|\Omega_2)$ be the period matrix of the curve C_f with respect to the basis*

$$\omega_1 = \frac{\mathrm{d}x}{y}, \qquad \omega_2 = \frac{x\,\mathrm{d}x}{y}, \qquad \omega_3 = \frac{x^2\,\mathrm{d}x}{y} \quad \text{of } H^0\big(C_{f,}\Omega^1\big)$$

and any *symplectic basis of $H_1(C_f, \mathbb{Z})$, and let $Z_f = \Omega_1^{-1}\Omega_2$. Let $w \in J(C_f)[2]$ be any odd 2-torsion point, and denote by $H_w(X_1, X_2, X_3) = MX_1 + NX_2 + PX_3 = 0$ the line of $\mathbb{P}^2$ given by*

$$H_w(X_1, X_2, X_3) := \left(\frac{\partial\theta}{\partial z_1}(w)\ \frac{\partial\theta}{\partial z_2}(w)\ \frac{\partial\theta}{\partial z_3}(w)\right)\Omega_1^{-1}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = 0.$$

The x-coordinates of the Weierstrass points $W' = (\alpha, 0)$, $W'' = (\beta, 0)$ such that $w = \Pi(W' + W'')$ are the roots of $Px^2 + Nx + M = 0$.

11. The discriminant

For a hyperelliptic curve of genus g

$$\begin{aligned} C_f : Y^2 = f(X) &= a_{2g+2}X^{2g+2} + \cdots + a_0 \\ &= a_{2g+2}(X - \alpha_1) \cdots (X - \alpha_{2g+2}), \end{aligned}$$

there are two objects that deserve to be called *discriminant* of C_f. First we have the *algebraic discriminant* of C_f, $\Delta_{\mathrm{alg}}(C_f)$, which is the discriminant of the polynomial f defining the curve:

$$\Delta_{\mathrm{alg}}(C_f) := \Delta(f) = a_{2g+2}^{4g+2}\prod_{j<k}(\alpha_j - \alpha_k)^2.$$

On the other side, we have the *analytic discriminant*, whose definition is slightly more intricate (cf. [13]). In order to simplify, we are assuming that $\deg f = 2g+2$ (unlike in *loc. cit.*; we give the adapted versions of formulas appearing there). We denote by $\mathcal{T}_k$ be the collection of subsets $T \subset \{1, \ldots, 2g+2\}$ of cardinality k.

Definition 11.1 [13]. The *discriminant modular form* on $\mathbb{H}_g$ is

$$\delta(Z) = \prod_{T\in\mathcal{T}_{g+1}} \theta[\eta_{T\circ U}](Z)^8.$$

Here $U = \{1, 3, \ldots, 2g+1\}$ and the analytic theta characteristic $\eta_{T\circ U}$ has been introduced in Section 4.

This product is never zero on hyperelliptic period matrices, since the analytic theta characteristics $\eta_{T\circ U}$ are always associated with geometric theta characteristics given by non-effective semicanonical divisors.

It is well known that these two discriminants agree up to a multiplicative constant for genus 1 curves. Grant [14] proved this relation for genus 2. In the general case we have the following statement.

Proposition 11.1 (Lockhart [13]). *Let $C_f : Y^2 = f(X)$ be a genus g hyperelliptic curve given by a polynomial f of degree $2g+2$ (not necessarily monic), and let Z_f be a normalized period matrix for it. Then*

$$\Delta_{\mathrm{alg}}(C_f)^{2n} = (2\pi)^{4rg}(\det \Omega_1)^{-4r}\delta(Z_f)^2,$$
$$\text{where} \quad r = \binom{2g+2}{g+1}, \quad n = \binom{2g}{g+1}.$$

(Note that the result that appears in [13] is for monic polynomials of degree $2g+1$.) The proof of this proposition is based on the classical formula of Thomae, which we recall in the general case of a non-monic polynomial.

Theorem 11.1 (Thomae's formula [12]). *For any set $S \subset \{1, \ldots, 2g+2\}$ of even cardinality,*

$$\theta[\eta_S](Z)^4 = \pm\big(4\pi^2 a_{2g+2}\big)^g \det \Omega_1^2 \prod_{\substack{i<j \\ i,j\in S\circ U}} (\alpha_i - \alpha_j) \prod_{\substack{i<j \\ i,j\notin S\circ U}} (\alpha_i - \alpha_j).$$

We will now give another modular approach to the discriminant, using our jacobian version of Thomae's formula (Theorem 7.1).

So far, we have been using a slightly different definition of the Thetanullwerte and the jacobian nullwerte: we have overpassed the exponential factor, since it was completely unnecessary for the resolution of algebraic equations. But now we want to combine our results with the classical ones, so we introduce some notations to make all the definitions compatible.

Given a semicanonical divisor D, we will write $\theta[D]$ to denote the Thetanullwerte $\theta[\eta](Z_f)$ given by the analytic theta characteristic η corresponding to D. For a Weierstrass divisor D, we define $H[D]$ to be the hyperplane given by the equation:

$$H[D](X_1, \ldots, X_g) := \left(\frac{\partial\theta[\eta]}{\partial z_1}(0, Z_f) \cdots \frac{\partial\theta[\eta]}{\partial z_g}(0, Z_f)\right)\Omega_1^{-1}\begin{pmatrix} X_1 \\ \vdots \\ X_g \end{pmatrix} = 0.$$

Finally, given g semicanonical divisors $D_1, \ldots, D_g$, the symbol $J[D_1, \ldots, D_g]$ will stand for $[\eta_1, \ldots, \eta_g](Z_f)$, where the η_i are the analytic theta characteristics associated with the D_i.

In Section 6 we have explained how to associate with every $I = \{i_1, \ldots, i_g\} \in \mathcal{T}_g$ a fundamental system $D_I = \{D_{i_1}, \ldots, D_{i_g}, D_{i_{g+1}}, \ldots, D_{i_{2g+2}}\}$ of semicanonical divisors (where $I \cup \{i_{g+1}, \ldots, i_{2g+2}\} = \{1, \ldots, 2g+2\}$). This fundamental system of semicanonical divisors gives in turn a fundamental system $w_{i_1}, \ldots, w_{i_{2g+2}}$ of 2-torsion points in $J(C_f)$. We will denote by $J[D_I]$ the jacobian nullwerte $J[D_{i_1}, \ldots, D_{i_g}]$ of this fundamental system. Note that $J[D_I]$ and $J(w_{i_1}, \ldots, w_{i_g})$ differ only by an exponential factor.

Each subset $J = \{j_1, \ldots, j_{g-1}\} \in \mathcal{T}_{g-1}$ gives a Weierstrass divisor $D_J = \sum_{j \in J} W_j$. We define

$$H(J) := \prod_{j \notin J} H[D_J](W_j),$$

where $H[D_J](W_j) = H[D_J](1, \alpha_j, \ldots, \alpha_j^{g-1})$.

Theorem 11.2. *Let* $s = \binom{2g}{g-2}$, $m = \binom{2g+2}{g} = |\mathcal{T}_g|$. *Then*

$$\Delta_{\mathrm{alg}}(C_f)^s = a_{2g+2}^{(4g+2)s} \det \Omega_1^{2m} \prod_{J \in \mathcal{T}_{g-1}} H(J)^2 \prod_{I \in \mathcal{T}_g} \bigl|J[D_I]\bigr|^{-2}.$$

Proof. We first rewrite Theorem 7.1 with the notations just introduced. It looks exactly as before, since the exponential factors that appear in $H[D_j]$ and $|J[D_1, \ldots, D_g]|$ are cancelled:

$$\bigl|V(\alpha_1, \ldots, \alpha_g)\bigr| = \det \Omega_1 \left(\prod_{j=1}^{g} H[D_j](W_j) \right) \Big/ \bigl|J[D_1, \ldots, D_g]\bigr|.$$

Now, we take the product of these equalities over all fundamental systems of the form D_I, and raise it to the square. Every factor $(\alpha_i - \alpha_j)$ appearing in the definition of $\Delta_{\mathrm{alg}}(C_f)$ appears once for every D_I such that $\{i, j\} \subset I$, and hence it appears $2\binom{2g}{g-2}$ times in the final product. For every fundamental system D_I we also obtain the product

$$\prod_{\substack{J \in \mathcal{T}_{g-1}, \\ J \cup \{j\} = I}} H[D_J]\bigl(1, \alpha_j, \ldots, \alpha_j^g\bigr),$$

and thus, in the final product every form $H[D_J](X_1, \ldots, X_g)$ appears evaluated at every α_j with $j \notin J$. $\square$

We can now combine Lockhart's result with our theorem, to find relations between jacobian nullwerte and products of theta nullwerte. We will make this for low genera, where a geometrical interpretation of these relations becomes clear.

12. Geometric Jacobi identity and discriminant for genus 2

For curves of genus 2, there is a classical relation between jacobian nullwerte and products of theta nullwerte: Rosenhain's identity (cf. [1,4,7]). Let us denote by $\eta_1, \ldots, \eta_6$ the six odd analytic theta characteristics of dimension 2. Rosenhain's identity states that for every pair η_j, η_k:

$$[\eta_j, \eta_k](Z_f) = \pm\pi^2 \prod_{r \neq j,k} \theta[\eta_j + \eta_k - \eta_r](Z_f),$$

where the sum $\eta_j + \eta_k - \eta_r$ is taken modulus 2. This formula has a geometrical counterpart: let C_f a hyperelliptic curve of genus 2, with Weierstrass points $W_1 = (\alpha_1, 0), \ldots, W_6 = (\alpha_6, 0)$. For every pair W_j, W_k,

$$\bigl|J[W_j, W_k]\bigr| = \pm\pi^2 \prod_{r \neq i,j} \theta[W_j + W_k - W_r]. \tag{4}$$

This relation, combined with Theorem 7.1, gives the following formula, which can be seen as a *geometric Jacobi identity* for genus 2.

Proposition 12.1.

$$\begin{aligned} &H[W_j](1, \alpha_k) H[W_k](1, \alpha_j) \\ &\quad = \pm\pi^2 \det \Omega_1^{-1} (\alpha_k - \alpha_j) \prod_{r \neq i,j} \theta[W_j + W_k - W_r]. \end{aligned}$$

For genus 2 we also have a specific formula for the discriminant [14]:

$$\Delta_{\mathrm{alg}}(C_f) = \pm(2\pi)^{20} \det \Omega_1^{-10} \prod_{\eta \in TC_2^+} \theta[\eta](Z_f)^2 \tag{5}$$

$$\qquad\qquad = \pm(2\pi)^{20} \det \Omega_1^{-10} \prod_{i<j<k} \theta[W_i + W_j - W_k]. \tag{6}$$

If we take the product of $|J[W_j, W_k]|$ over all pairs $j < k$ and apply Rosenhain's identity in each factor we obtain:

$$\begin{aligned} \prod_{j<k} \bigl|J[W_j, W_k]\bigr|^2 &= \pi^{60} \prod_{i<j<k} \theta[W_i + W_j - W_k]^6 \\ &= 2^{-120} \pi^{-60} \det \Omega_1^{60} \Delta_{\mathrm{alg}}(C_f)^6. \end{aligned}$$

Finally, Theorem 11.2 for genus 2 states that

$$\Delta_{\mathrm{alg}}(C_f) = a_6^{10} \det \Omega_1^{30} \frac{\prod_{j<k} H[W_j](W_k)^2}{\prod_{j<k} |J[W_j, W_k]|^2}.$$

Combining the last two equalities we arrive at the following proposition.

Proposition 12.2.

$$\Delta_{\mathrm{alg}}(C_f)^7 = 2^{120} a_6^{10} \pi^{60} \det \Omega_1^{-30} \prod_{j<k} H[W_j](W_k)^2.$$

13. Geometric Jacobi identity and discriminant for genus 3

In order to obtain an analogue of Proposition 12.1 for genus 3, we need a version of Rosenhain's identity for genus 3. We find this identity in [4]. We provide a geometric version of Frobenius' formula.

Proposition 13.1. *Let C_f be a hyperelliptic curve of genus* 3*, with Weierstrass points $W_1, \ldots, W_8$. The following equality holds for every triplet W_i, W_j, W_k:*

$$\left|J[W_i + W_k, W_i + W_j, W_j + W_k]\right| = \pi^3 \prod_{r \neq i,j,k} \theta[W_i + W_j + W_k - W_r].$$

Proof. Applying formula (4.2), we see that the sum

$$\begin{aligned}&\Pi(W_i + W_k) + \Pi(W_i + W_j) + \Pi(W_j + W_k)\\ &\quad = \Pi(D_{\{j,k\}\circ\{1,3,5,7\}}) + \Pi(W_j + W_k) = \Pi(D_{\{1,3,5,7\}}) = \kappa\end{aligned}$$

is always even, and hence, by [7, Corollary to Theorem 2], the jacobian Thetanullwerte $|J[W_i + W_k, W_i + W_j, W_j + W_k]|$ is a product of five even Thetanullwerte $\theta[\widetilde{D}_1] \cdots \theta[\widetilde{D}z_5]$. By Theorem 5.1, the divisors $\widetilde{D}_r$ are the only ones which form a fundamental system with $W_i + W_k$, $W_i + W_j$, $W_j + W_k$ and $\theta[\widetilde{D}_r] \neq 0$. We have seen in Proposition 6.1 that the divisors $\widetilde{D}_r = W_i + W_j + W_k - W_r$, $r \neq i, j, k$ form a fundamental system with $W_i + W_k$, $W_i + W_j$, $W_j + W_k$ and by Riemann's singularity theorem, $\theta[\widetilde{D}_r] \neq 0$, because $l(W_i + W_j + W_k - W_r) = 0$ by Lemma 4.1. □

With this in hand, we can now proceed exactly as in the previous section. We obtain the following results.

Proposition 13.2 (Geometric Jacobi identity for genus 3). *For every triplet of Weierstrass points W_i, W_j, W_k,*

$$\begin{aligned}&H[W_i + W_j](W_k) H[W_i + W_k](W_j) H[W_j + W_k](W_i)\\ &\quad = \pm \pi^3 \det \Omega_1^{-1} \left|V(\alpha_i, \alpha_j, \alpha_k)\right| \prod_{r \neq i,j,k} \theta[W_i + W_j + W_k - W_r].\end{aligned}$$

Proposition 13.3.

(a) $$\Delta_{\mathrm{alg}}(C_f)^{30} = 2^{840}\pi^{504}a_8^{420}\det\Omega_1^{-280} \times \prod_{i<j<k} J[W_i+W_k, W_i+W_j, W_j+W_k].$$

(b) $$\Delta_{\mathrm{alg}}(C_f)^{36} = a_8^{84}\pi^{504}2^{840}\det\Omega_1^{-168}\prod_{j<k}\prod_{r\neq j,k} H[W_j+W_k](W_r).$$

14. Some conjectural formulas

At this point, it is natural to ask whether it is possible to generalize the formulas in the previous sections for arbitrary genus g. The necessary tool is a generalization of Rosenhain and Frobenius formulae (that is, a generalization of Jacobi's triple product identity). Igusa's Theorem 5.1 shows that in genus $g \geqslant 6$ this generalization is not possible for a general matrix $Z \in \mathbb{H}_g$. But Frobenius [4, p. 254] showed that for any hyperelliptic matrix Z_f and any fundamental system $\eta_1,\ldots,\eta_{2g+2}$, there is a constant $\varepsilon \in \mathbb{C}$ such that

$$[\eta_1,\ldots,\eta_g](Z_f) = \varepsilon\pi^g \prod_{i=g+1}^{2g+2}\theta[\eta_i](0;Z_f).$$

Taking into account Igusa's theorem, it seems reasonable to conjecture that this constant ε is equal to ± 1. This would imply the following (conjectural) formulas.

Conjecture 14.1. *Let C_f be a hyperelliptic curve of genus g, with Weierstrass points $W_1=(\alpha_1,0),\ldots,W_{2g+2}=(\alpha_{2g+2},0)$. Let $D=\sum_{i=1}^g W_i$, $D_i=D-W_i$, for $i=1,\ldots,g$, and $D_i=D+W_i-2W_{2g+2}$, for $i=g+1,\ldots,2g+1$. Then*

(a) $$\bigl|J[D_1,\ldots,D_g]\bigr| = \pm\pi^g\prod_{k=g+1}^{2g}\theta[D_k].$$

(b) $$\prod_{i=1}^{g} H[D_i](W_i) = \pm\pi^g\det\Omega_1^{-1}\bigl|V(\alpha_1,\ldots,\alpha_g)\bigr|\prod_{r\notin\{1,\ldots,g\}}\theta[D-W_r].$$

(c) *Let $m=\binom{2g+2}{g+1}$. Then*

$$\prod_{I\in\mathcal{T}_g}\bigl|J[D_I]\bigr|^8 = \pi^{mg}\delta(Z_f)^{g+1}.$$

(d) *Let $r=\binom{2g+2}{g+1}$, $n=2\binom{2g}{g+1}$. Then*

$$\Delta_{\mathrm{alg}}(C_f)^{n(g+1)} = (2\pi)^{4g(g+1)r}\pi^{-2mg}(\det\Omega_1)^{-4r(g+1)}\prod_{I\in\mathcal{T}_g}\bigl|J[D_I]\bigr|^{16}.$$

We observe that formulas (a) and (b) are equivalent by Theorem 7.1 and they imply formulas (c) and (d).

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