An Introduction to Stochastic Processes in Continuous Time

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adaptation of the text by Harry van Zanten
to be used at your own expense

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Chapter 1

Stochastic Processes

1.1 Introduction

Loosely speaking, a stochastic process is a phenomenon that can be thought of as evolving in time in a random manner. Common examples are the location of a particle in a physical system, the price of stock in a financial market, interest rates, mobile phone networks, internet traffic, etc.

A basic example is the erratic movement of pollen grains suspended in water, so-called Brownian motion. This motion was named after the English botanist R. Brown, who first observed it in 1827. The movement of pollen grain is thought to be due to the impacts of water molecules that surround it. Einstein was the first to develop a model for studying the erratic movement of pollen grains in an article in 1926. We will give a sketch of how this model was derived. It is more heuristically than mathematically sound.

The basic assumptions for this model (in dimension 1) are the following:

1) the motion is continuous.

Moreover, in a time-interval $[t, t + \tau]$, $\tau$ small,

2) particle movements in two non-overlapping time intervals of length $\tau$ are mutually independent;

3) the relative proportion of particles experiencing a displacement of size between $\delta$ and $\delta + d\delta$ is approximately $\phi(\delta)$ with

- the probability of some displacement is 1: $\int_{-\infty}^{\infty} \phi(\delta)d\delta = 1$;
- the average displacement is 0: $\int_{-\infty}^{\infty} \delta\phi(\delta)d\delta = 0$;
- the variation in displacement is linear in the length of the time interval: $\int_{-\infty}^{\infty} \delta^2\phi(\delta)d\delta = D\tau$, where $D \geq 0$ is called the diffusion coefficient.

Denote by $f(x, t)$ the density of particles at position $x$, at time $t$. Under differentiability assumptions, we get by a first order Taylor expansion that

$$f(x, t + \tau) \approx f(x, t) + \tau \frac{\partial f}{\partial t}(x, t).$$
On the other hand, by a second order expansion

\[
    f(x, t + \tau) = \int_{-\infty}^{\infty} f(x - \delta, t) \phi_\tau(\delta) d\delta \\
    \approx \int_{-\infty}^{\infty} [f(x, t) - \delta \frac{\partial f}{\partial x}(x, t) + \frac{1}{2}\delta^2 \frac{\partial^2 f}{\partial x^2}(x, t)] \phi_\tau(\delta) d\delta \\
    \approx f(x, t) + \frac{1}{2}D^2 \frac{\partial^2 f}{\partial x^2}(x, t).
\]

Equating gives rise to the heat equation in one dimension:

\[
    \frac{\partial f}{\partial t} = \frac{1}{2}D \frac{\partial^2 f}{\partial x^2},
\]

which has the solution

\[
    f(x, t) = \frac{\text{#particles}}{\sqrt{4\pi D t}} \cdot e^{-x^2/4Dt}.
\]

So \( f(x, t) \) is the density of a \( N(0, 4Dt) \)-distributed random variable multiplied by the number of particles.

**Side remark.** In section 1.5 we will see that under these assumptions paths of pollen grain through liquid are non-differentiable. However, from physics we know that the velocity of a particle is the derivative (to time) of its location. Hence pollen grain paths must be differentiable. We have a conflict between the properties of the physical model and the mathematical model. What is wrong with the assumptions? Already in 1926 editor R. Fürth doubted the validity of the independence assumption (2). Recent investigation seems to have confirmed this doubt.

Brownian motion will be one of our objects of study during this course. We will now turn to a mathematical definition.

**Definition 1.1.1** Let \( T \) be a set and \((E, \mathcal{E})\) a measurable space. A **stochastic process** indexed by \( T \), with values in \((E, \mathcal{E})\), is a collection \( X = (X_t)_{t \in T} \) of measurable maps from a (joint) probability space \((\Omega, \mathcal{F}, P)\) to \((E, \mathcal{E})\). The space \((E, \mathcal{E})\) is called the state space of the process.

The index \( t \) is a time parameter, and we view the index set \( T \) as the set of all observation instants of the process. In these notes we will usually have \( T = \mathbb{Z}_+ = \{0, 1, \ldots\} \) or \( T = \mathbb{R}_+ = [0, \infty) \) (or \( T \) is a sub-interval of one these sets). In the former case, we say that time is **discrete**, in the latter that time is **continuous**. Clearly a discrete-time process can always be viewed as a continuous-time process that is constant on time-intervals \([n, n + 1)\).

The state space \((E, \mathcal{E})\) will generally be a Euclidian space \( \mathbb{R}^d \), endowed with its Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^d) \). If \( E \) is the state space of the process, we call the process **\( E \)-valued**.

For every fixed observation instant \( t \in T \), the stochastic process \( X \) gives us an \( E \)-valued random element \( X_t \) on \((\Omega, \mathcal{F}, P)\). We can also fix \( \omega \in \Omega \) and consider the map \( t \rightarrow X_t(\omega) \) on \( T \). These maps are called the **trajectories** or **sample paths** of the process. The sample paths are functions from \( T \) to \( E \) and so they are elements of the function space \( E^T \). Hence, we can view the process \( X \) as an \( E^T \)-valued random element. Quite often, the sample paths belong to a nice subset of this space, e.g. the continuous or right-continuous functions, alternatively
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called the path space. For instance, a discrete-time process viewed as the continuous-time process described earlier, is a process with right-continuous sample paths.

Clearly we need to put an appropriate $\sigma$-algebra on the path space $E^T$. For consistency purposes it is convenient that the marginal distribution of $X_t$ be a probability measure on the path space. This is achieved by ensuring that the projection $x \rightarrow x_t$, where $t \in T$, is measurable. The $\sigma$-algebra $E^T$, described in BN §2, is the minimal $\sigma$-algebra with this property.

The mathematical model of the physical Brownian motion is a stochastic process that is defined as follows.

**Definition 1.1.2** The stochastic process $W = (W_t)_{t \geq 0}$ is called a (standard) Brownian motion or Wiener process, if

i) $W_0 = 0$, a.s.;

ii) $W_t - W_s$ is independent of $(W_u, u \leq s)$ for all $s \leq t$, that is, $\sigma(W_t - W_s)$ and $\sigma(W_u, u \leq s)$ are independent;

iii) $W_t - W_s \overset{d}{=} \mathcal{N}(0, t - s)$;

iv) almost all sample paths are continuous.

In these notes we will abbreviate ‘Brownian motion’ as BM. Property (i) tells that standard BM starts at 0. A process with property (ii) is said to have independent increments. Property (iii) implies that the distribution of the increment $W_t - W_s$ only depends on $t - s$. This is called stationarity of the increments. A stochastic process with property (iv) is called a continuous process. Similarly, a stochastic process is said to be right-continuous if almost all of its sample paths are right-continuous functions. Finally, the acronym *cadlag* (continu à droite, limites à gauche) is used for processes with right-continuous sample paths having finite left-hand limits at every time instant.

Simultaneously with Brownian motion we will discuss another fundamental process: the Poisson process.

**Definition 1.1.3** A real-valued stochastic process $N = (N_t)_{t \geq 0}$ is called a Poisson process if

i) $N$ is a counting process, i.o.w.

   a) $N_t$ takes only values in $\mathbb{Z}_+$, $t \geq 0$;

   b) $t \mapsto N_t$ is increasing, i.o.w. $N_s \leq N_t$, $t \geq s$.

   c) (no two occurrences can occur simultaneously) $\lim_{s \uparrow t} N_s \leq \lim_{s \uparrow t} N_s + 1$, for all $t \geq 0$.

ii) $N_0 = 0$, a.s.;

iii) (independence of increments) $\sigma(N_t - N_s)$ and $\sigma(N_u, u \leq s)$ are independent;

iv) (stationarity of increments) $N_t - N_s \overset{d}{=} N_{t-s}$ for all $s \leq t$;

**Note:** so far we do not know yet whether a BM process and a Poisson process exist at all! The Poisson process can be constructed quite easily and we will do so first before delving into more complex issues.
Construction of the Poisson process  The construction of a Poisson process is simpler than the construction of Brownian motion. It is illustrative to do this first.

Let a probability space \((\Omega, \mathcal{F}, P)\) be given. We construct a sequence of i.i.d. \((\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\)-measurable random variables \(X_n, n = 1, \ldots\), on this space, such that \(X_n \overset{d}{=} \exp(\lambda)\). This means that

\[
P\{X_n > t\} = e^{-\lambda t}, \quad t \geq 0.
\]

Put \(S_0 = 0\), and \(S_n = \sum_{i=1}^{n} X_i\). Clearly \(S_n, n = 0, \ldots\) are in increasing sequence of random variables. Since \(X_n\) are all \(\mathcal{F}/\mathcal{B}(\mathbb{R}_+)\)-measurable, so are \(S_n\). Next define

\[
N_t = \max\{n \mid S_n \leq t\}.
\]

We will show that this is a Poisson process. First note that \(N_t\) can be described alternatively as

\[
N_t = \sum_{n=1}^{\infty} 1_{\{S_n \leq t\}}.
\]

\(N_t\) maybe infinite, but we will show that it is finite with probability 1 for all \(t\). Moreover, no two points \(S_n\) and \(S_{n+1}\) are equal. Denote by \(\mathcal{E}\) the \(\sigma\)-algebra generated by the one-point sets of \(\mathbb{Z}_+\).

**Lemma 1.1.4** There exists a set \(\Omega^* \in \mathcal{F}\) with \(P\{\Omega^*\} = 1\), such that \(N_t(\omega) < \infty\) for all \(t \geq 0\), \(\omega \in \Omega\), and \(S_n(\omega) < S_{n+1}(\omega), n = 0, \ldots\).

**Proof.** From the law of large numbers we find a set \(\Omega'\), \(P\{\Omega'\} = 1\), such that \(N_t(\omega) < \infty\) for all \(t \geq 0\), \(\omega \in \Omega'\). It easily follows that there exists a subset \(\Omega^* \subset \Omega', P\{\Omega^*\} = 1\), meeting the requirements of the lemma. Measurability follows from the fact that \(1_{\{S_n \leq t\}}\) is measurable. Hence a finite sum of these terms is measurable. The infinite sum is then measurable as well, being the monotone limit of measurable functions. QED

Since \(\Omega^* \in \mathcal{F}\), we may restrict to this smaller space without further ado. Denote the restricted probability space again by \((\Omega, \mathcal{F}, P)\).

**Theorem 1.1.5** For the constructed process \(N\) on \((\Omega, \mathcal{F}, P)\) the following hold.

i) \(N\) is a \((\mathbb{Z}_+, \mathcal{E})\)-measurable stochastic process that has properties (i,...,iv) from Definition 1.1.3. Moreover, \(N_t\) is \(\mathcal{F}/\mathcal{E}\)-measurable, it has a Poisson distribution with parameter \(\lambda t\) and \(S_n\) has a Gamma distribution with parameters \((n, \lambda)\). In particular \(E N_t = \lambda t, \text{ and } EN_n^2 = \lambda t + (\lambda t)^2\).

ii) All paths of \(N\) are cadlag.

**Proof.** The second statement is true by construction, as well as are properties (i,ii). The fact that \(N_t\) has a Poisson distribution with parameter \(\lambda t\), and that \(S_n\) has \(\Gamma(n, \lambda)\) distribution is standard.

We will prove property (iv). It suffices to show for \(t \geq s\) that \(N_t - N_s\) has a Poisson \((\lambda(t-s))\) distribution. Clearly

\[
P\{N_t - N_s = j\} = \sum_{i \geq 0} P\{N_s = i, N_t - N_s = j\} = \sum_{i \geq 0} P\{S_i \leq s, S_{i+1} > s, S_{i+j} \leq t, S_{i+j+1} > t\}. \tag{1.1.1}
\]
1.1. INTRODUCTION

First let \( i, j > 1 \). Recall the density \( f_{n, \lambda} \) of the \( \Gamma(n, \lambda) \) distribution:

\[
    f_{n, \lambda}(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)}, \quad n \neq 1
\]

where \( \Gamma(n) = (n-1)! \). Then, with a change of variable \( u = s_2 - (s - s_1) \) in the third equation,

\[
    \begin{align*}
        &\mathbb{P}\{N_t - N_s = j, N_s = i\} = \mathbb{P}\{S_i \leq s, S_{i+1} > s, S_{i+j} \leq t, S_{i+j+1} > t\} \\
        &= \int_0^s \int_0^{t-s_1} e^{-\lambda(t-s_3)} f_{j_i-1, \lambda}(s_3) ds_3 \lambda e^{-\lambda s_2} ds_2 f_{i, \lambda}(s_1) ds_1 \\
        &= \int_0^s \int_0^{t-s} e^{-\lambda(t-s-u)} f_{j_i-1, \lambda}(s_3) ds_3 \lambda e^{-\lambda u} du \cdot e^{-\lambda(s_1)} f_{i, \lambda}(s_1) ds_1 \\
        &= \mathbb{P}\{S_j \leq t - s, S_{j+1} > t - s\} \cdot \mathbb{P}\{S_i \leq s, S_{i+1} > s\} \\
        &= \mathbb{P}\{N_{t-s} = j\} \mathbb{P}\{N_s = i\}.
    \end{align*}
\]

(1.1.2)

For \( i = 0, 1, j = 1 \), we get the same conclusion. (1.1.1) then implies that \( \mathbb{P}\{N_t - N_s = j\} = \mathbb{P}\{N_{t-s} = j\} \), for \( j > 0 \). By summing over \( j > 0 \) and substracting from 1, we get the relation for \( j = 0 \) and so we have proved property (iv).

Finally, we will prove property (iii). Let us first consider \( \sigma(N_u, u \leq s) \). This is the smallest \( \sigma \)-algebra that makes all maps \( \omega \mapsto N_u(\omega), u \leq s, \) measurable. Section 2 of BN studies its structure. It follows that (see Exercise 1.1) the collection \( \mathcal{I} \), with

\[
    I = \left\{ A \in \mathcal{F} \mid \exists n \in \mathbb{Z}_+, t_0 \leq t_1 < t_2 < \cdots < t_n, \ i_l \in [0, s], \ i_l \in \mathbb{Z}_+, \ l = 0, \ldots, n, \right. \\
    \left. \text{such that } A = \{N_{t_i} = i_l, \ l = 0, \ldots, n\} \right\}
\]

a \( \pi \)-system for this \( \sigma \)-algebra.

To show independence property (iii), it therefore suffices show for each \( n \), for each sequence \( 0 \leq t_0 < \cdots < t_n \), and \( i_0, \ldots, i_n \), that

\[
    \mathbb{P}\{N_{t_i} = i_l, \ l = 0, \ldots, n, \ N_t - N_s = i\} = \mathbb{P}\{N_{t_i} = i_l, \ l = 0, \ldots, n\} \cdot \mathbb{P}\{N_t - N_s = i\}.
\]

This is analogous to the proof of (1.1.2). QED

A final observation. We have constructed a mapping \( N : \Omega \rightarrow \Omega' \subset \mathbb{Z}_+^{[0, \infty)} \), with \( \mathbb{Z}_+^{[0, \infty)} \) the space of all integer-valued functions. The space \( \Omega' \) consists of all integer valued paths \( \omega' \), that are right-continuous and non-decreasing, and have the property that \( \omega'_t \leq \lim \inf_{s \uparrow t} \omega'_s + 1 \).

It is desirable to consider \( \Omega' \) as the underlying space. One can then construct a Poisson process directly on this space, by taking the identity map. The \( \sigma \)-algebra to consider, is then the minimal \( \sigma \)-algebra \( \mathcal{F}' \) that makes all maps \( \omega' \mapsto \omega'_t \) measurable, \( t \geq 0 \). It is precisely \( \mathcal{F}' = \mathcal{E}^{[0, \infty)} \cap \Omega' \).

Review BN §2 on measurability issues for a description of \( \mathcal{E}^{[0, \infty)} \).

Then \( \omega \mapsto N(\omega) \) is measurable as a map \( \Omega \rightarrow \Omega' \). On \( (\Omega', \mathcal{F}') \) we now put the induced probability measure \( \mathbb{P}' \) by \( \mathbb{P}'\{A\} = \mathbb{P}\{\omega \mid N(\omega) \in A\} \).

We will next discuss a procedure to construct a stochastic process, with given marginal distributions.
1.2 Finite-dimensional distributions

In this section we will recall Kolmogorov’s theorem on the existence of stochastic processes with prescribed finite-dimensional distributions. We will use the version that is based on the fact that $T$ is ordered. It allows to prove the existence of a process with properties (i,ii,iii) of Definition 1.1.2.

**Definition 1.2.1** Let $X = (X_t)_{t \in T}$ be a stochastic process. The distributions of the finite-dimensional vectors of the form $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$, $t_1 < t_2 < \cdots < t_n$, are called the finite-dimensional distributions (fdd’s) of the process.

It is easily verified that the fdd’s of a stochastic process form a consistent system of measures in the sense of the following definition.

**Definition 1.2.2** Let $T \subset \mathbb{R}$ and let $(E, \mathcal{E})$ be a measurable space. For all $n \in \mathbb{Z}_+$ and all $t_1 < \cdots < t_n$, $t_i \in T$, $i = 1, \ldots, n$, let $\mu_{t_1, \ldots, t_n}$ be a probability measure on $(E^n, \mathcal{E}^n)$. This collection of measures is called consistent if it has the property that

$$
\mu_{t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n}(A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) = \mu_{t_1, \ldots, t_n}(A_1 \times \cdots \times A_{i-1} \times E \times A_{i+1} \times \cdots \times A_n),
$$

for all $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \in \mathcal{E}$.

The Kolmogorov consistency theorem states that, conversely, under mild regularity conditions, every consistent family of measures is in fact the family of fdd’s of some stochastic process. Some assumptions are needed on the state space $(E, \mathcal{E})$. We will assume that $E$ is a Polish space. This is a topological space, on which we can define a metric that consistent with the topology, and which makes the space complete and separable. As $\mathcal{E}$ we take the Borel-$\sigma$-algebra, i.e. the $\sigma$-algebra generated by the open sets. Clearly, the Euclidian spaces $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ fit in this framework.

**Theorem 1.2.3 (Kolmogorov’s consistency theorem)** Suppose that $E$ is a Polish space and $\mathcal{E}$ its Borel-$\sigma$-algebra. Let $T \subset \mathbb{R}$ and for all $n \in \mathbb{Z}_+$, $t_1 < \cdots < t_n \in T$, let $\mu_{t_1, \ldots, t_n}$ be a probability measure on $(E^n, \mathcal{E}^n)$. If the measures $\mu_{t_1, \ldots, t_n}$ form a consistent system, then there exists a probability measure $\mathbb{P}$ on $E^T$, such that the co-ordinate variable process $(X_t)_t$ on $(\Omega = E^T, \mathcal{F} = \mathcal{E}^T, \mathbb{P})$, defined by

$$
X(\omega) = \omega, \quad X_t(\omega) = \omega_t,
$$

has fdd’s $\mu_{t_1, \ldots, t_n}$.

The proof can for instance be found in Billingsley (1995). Before discussing this theorem, we will discuss its implications for the existence of BM.

Review BN §4 on multivariate normal distributions

**Corollary 1.2.4** There exists a probability measure $\mathbb{P}$ on the space $(\Omega = \mathbb{R}^{[0,\infty)}, \mathcal{F} = \mathcal{B}(\mathbb{R})^{[0,\infty)})$, such that the co-ordinate process $W = (W_t)_{t \geq 0}$ on $(\Omega = \mathbb{R}^{[0,\infty)}, \mathcal{F} = \mathcal{B}(\mathbb{R})^{[0,\infty)}, \mathbb{P})$ has properties (i,ii,iii) of Definition 1.1.2.
1.3. Kolmogorov’s Continuity Criterion

Proof. First show that for $0 \leq t_0 < t_1 < \cdots < t_n$, there exist multivariate normal distributions with covariance matrices

$$
\Sigma = \begin{pmatrix}
    t_0 & 0 & \cdots & \cdots & 0 \\
    0 & t_1 - t_0 & 0 & \cdots & 0 \\
    0 & 0 & t_2 - t_1 & \ddots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & t_n - t_{n-1}
\end{pmatrix},
$$

and

$$
\Sigma_{t_0,\ldots,t_n} = \begin{pmatrix}
    t_0 & t_0 & \cdots & \cdots & t_0 \\
    t_0 & t_1 & t_1 & \cdots & t_1 \\
    t_0 & t_1 & t_2 & \cdots & t_2 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    t_0 & t_1 & t_2 & \cdots & t_n
\end{pmatrix}.
$$

Next, show that this collection is a consistent system of probability measures. Then show that a stochastic process $W$ has properties (i, ii, iii) if and only if for all $n \in \mathbb{Z}$, $0 \leq t_0 < \ldots < t_n$ the vector $(W_{t_0}, W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}) \overset{d}{=} \mathcal{N}(0, \Sigma)$. Finally, show that a stochastic process $W$ has properties (i, ii, iii) if and only if for all $n \in \mathbb{Z}$, $0 \leq t_0 < \ldots < t_n$ the vector $(W_{t_0}, W_{t_1}, \ldots, W_{t_n}) \overset{d}{=} \mathcal{N}(0, \Sigma_{t_0,\ldots,t_n})$. Then finish the proof. QED

The drawback of Kolmogorov’s Consistency Theorem is, that in principle all functions on the positive real line are possible sample paths. Our aim is the show that we may restrict to the subset of continuous paths in the Brownian motion case.

However, the set of continuous paths is not even a measurable subset of $\mathcal{B}([0, \infty))$, and so the probability that the process $W$ has continuous paths is not well defined. The next section discussed how to get around the problem concerning continuous paths.

1.3 Kolmogorov’s Continuity Criterion

Why do we really insist on Brownian motion to have continuous paths? First of all, the connection with the physical process. Secondly, without regularity properies like continuity, or weaker right-continuity, events of interest are not ensured to measurable sets. An example is: $\{\sup_{t \geq 0} W_t \leq x\}, \inf\{t \geq 0 \mid W_t = x\}$.

The idea to address this problem, is to try to modify the constructed process $W$ in such a way that the resulting process, $\tilde{W}$ say, has continuous paths and satisfies properties (i, ii, iii), in other words, it has the same fdd’s as $W$. To make this idea precise, we need the following notions.

**Definition 1.3.1** Let $X$ and $Y$ be two stochastic processes, indexed by the same set $T$ and with the same state space $(E, \mathcal{E})$, defined on probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ respectively. The processes are called *versions* of each other, if they have the same fdd’s. In other words, if for all $n \in \mathbb{Z}$, $t_1, \ldots, t_n \in T$ and $B_1, \ldots, B_n \in \mathcal{E}$

$$
P\{X_{t_1} \in B_1, X_{t_2} \in B_2, \ldots, X_{t_n} \in B_n\} = P'\{Y_{t_1} \in B_1, Y_{t_2} \in B_2, \ldots, Y_{t_n} \in B_n\}.$$

CHAPTER 1. STOCHASTIC PROCESSES

$X$ and $Y$ are both $(E, \mathcal{E})$-valued stochastic processes. They can be viewed as random elements with values in the measurable path space $(E^T, \mathcal{E}^T)$. $X$ induces a probability measure $P_X$ on the path space with $P_X\{A\} = P\{X^{-1}(A)\}$. In the same way $Y$ induces a probability $P_Y$ on the path space. Since $X$ and $Y$ have the same fdd, it follows for each $n \in \mathbb{Z}^+$ and $t_1 < \cdots < t_n \in T$, and $A_1, \ldots, A_n \in \mathcal{E}$ that

$$P_X\{A\} = P_Y\{A\},$$

for $A = \{x \in E^T \mid x_i \in A_i, i = 1, \ldots, n\}$. The collection of sets $B$ of this form are a $\pi$-system generating $\mathcal{E}^T$ (cf. remark after BN Lemma 2.1), hence $P_X = P_Y$ on $(E^T, \mathcal{E}^T)$ by virtue of BN Lemma 1.2(i).

**Definition 1.3.2** Let $X$ and $Y$ be two stochastic processes, indexed by the same set $T$ and with the same state space $(E, \mathcal{E})$, defined on the same probability space $(\Omega, \mathcal{F}, P)$.

i) The processes are called **modifications** of each other, if for every $t \in T$

$$X_t = Y_t, \quad \text{a.s.}$$

ii) The processes are called **indistinguishable**, if there exists a set $\Omega^* \in \mathcal{F}$, with $P\{\Omega^*\} = 1$, such that for every $\omega \in \Omega^*$ the paths $t \to X_t(\omega)$ and $t \to Y_t(\omega)$ are equal.

The third notion is stronger than the second notion, which in turn is clearly stronger than the first one: if processes are indistinguishable, then they are modifications of each other. If they are modifications of each other, then they certainly are versions of each other. The reverse is not true in general (cf. Exercises 1.3, 1.6). The following theorem gives a sufficient condition for a process to have a continuous modification. This condition (1.3.1) is known as **Kolmogorov’s continuity condition**.

**Theorem 1.3.3 (Kolmogorov’s continuity criterion)** Let $X = (X_t)_{t \in [0,T]}$ be an $\mathbb{R}^d$-valued process on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that there exist constants $\alpha, \beta, K > 0$ such that

$$E\|X_t - X_s\|^{\alpha} \leq K|t - s|^{1+\beta},$$

(1.3.1)

for all $s, t \in [0, T]$. Then there exists a (everywhere!) continuous modification of $X$, i.o.w. $X(\omega)$ is a continuous function on $[0, T]$ for each $\omega \in \Omega$.

**Note:** $\beta > 0$ is needed for the continuous modification to exist. See Exercise 1.5.

**Proof.** The proof consists of the following steps:

1. (1.3.1) implies that $X_t$ is continuous in probability on $[0, T]$;
2. $X_t$ is a.s. uniformly continuous on a countable, dense subset $D \subset [0, T]$;
3. ‘Extend’ $X$ to a continuous process $Y$ on all of $[0, T]$.
4. Show that $Y$ is a well-defined stochastic process, and a continuous modification of $X$. 


Without loss of generality we may assume that $T = 1$.

**Step 1** Apply Chebychev’s inequality to the r.v. $Z = |X_t - X_s|$ and the function $\phi : \mathbb{R} \to \mathbb{R}^+$ given by

$$\phi(x) = \begin{cases} 0, & x \leq 0 \\ x^n, & x > 0. \end{cases}$$

Since $\phi$ is non-decreasing, non-negative and $E(\phi(Z)) < \infty$, it follows for every $\epsilon > 0$ that

$$P\{ |X_t - X_s| > \epsilon \} \leq \frac{E|X_t - X_s|^\alpha}{\epsilon^\alpha} \leq \frac{K|t - s|^{1+\beta}}{\epsilon^\alpha}. \quad (1.3.2)$$

Let $t, t_1, \ldots \in [0, 1]$ with $t_n \to t$ as $n \to \infty$. By the above,

$$\lim_{n \to \infty} P\{ \|X_t - X_{t_n}\| > \epsilon \} = 0,$$

for any $\epsilon > 0$. Hence $X_{t_n} \overset{P}{\to} X_t$, $n \to \infty$. In other words, $X_t$ is continuous in probability.

**Step 2** As the set $D$ we choose the dyadic rationals. Let $D_n = \{k/2^n \mid k = 0, \ldots, 2^n \}$. Then $D_n$ is an increasing sequence of sets. Put $D = \cup_n D_n = \lim_{n \to \infty} D_n$. Clearly $D = [0, 1]$, i.e. $D$ is dense in $[0, 1]$.

Fix $\gamma \in (0, \beta/\alpha)$. Apply Chebychev’s inequality (1.3.2) to obtain

$$P\{ \|X_{k/2^n} - X_{(k-1)/2^n}\| > 2^{-\gamma n} \} \leq \frac{K2^{-n(1+\beta)}}{2^{-\gamma n}} = K2^{-n(1+\beta-\alpha\gamma)}.$$  

It follows that

$$P\{ \max_{1 \leq k \leq 2^n} \|X_{k/2^n} - X_{(k-1)/2^n}\| > 2^{-\gamma n} \} \leq \sum_{k=1}^{2^n} P\{ \|X_{k/2^n} - X_{(k-1)/2^n}\| > 2^{-\gamma n} \} \leq 2^n K2^{-n(1+\beta-\alpha\gamma)} = K2^{-n(\beta-\alpha\gamma)}.$$  

Define the set $A_n = \{ \max_{1 \leq k \leq 2^n} \|X_{k/2^n} - X_{(k-1)/2^n}\| > 2^{-\gamma n} \}$. Then

$$\sum_n P(A_n) \leq \sum_n K2^{-n(\beta-\alpha\gamma)} = K \frac{1}{1 - 2^{-(\beta-\alpha\gamma)}} < \infty,$$

since $\beta-\alpha\gamma > 0$. By virtue of the first Borel-Cantelli Lemma this implies that $P(\limsup_m A_m) = P(\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n) = 0$. Hence there exists a set $\Omega^* \subset \Omega$, $\Omega^* \in \mathcal{F}$, with $P(\Omega^*) = 1$, such that for each $\omega \in \Omega^*$ there exists $N_\omega$, for which $\omega \notin \cup_{n \geq N_\omega} A_n$, in other words

$$\max_{1 \leq k \leq 2^n} \|X_{k/2^n}(\omega) - X_{(k-1)/2^n}(\omega)\| \leq 2^{-\gamma n}, \quad n \geq N_\omega. \quad (1.3.3)$$

Fix $\omega \in \Omega^*$. We will show the existence of a constant $K'$, such that

$$\|X_t(\omega) - X_s(\omega)\| \leq K'|t - s|^{\gamma}, \quad \forall s, t \in D, 0 < t - s < 2^{-N_\omega}. \quad (1.3.4)$$

Indeed, this implies uniform continuity of $X_t(\omega)$ for $t \in D$, for $\omega \in \Omega^*$. Step 2 will then be proved.
Let $s, t$ satisfy $0 < t - s < 2^{-N_\omega}$. Hence, there exists $n \geq N_\omega$, such that $2^{-(n+1)} \leq t - s < 2^{-n}$.

Fix $n \geq N_\omega$. For the moment, we restrict to the set of $s, t \in \bigcup_{m \geq n+1} D_m$, with $0 < t - s < 2^{-n}$. By induction to $m \geq n + 1$ we will first show that

$$\|X_t(\omega) - X_s(\omega)\| \leq 2 \sum_{k=n+1}^{m} 2^{-\gamma k},$$

if $s, t \in D_m$.

Suppose that $s, t \in D_{n+1}$. Then $t - s = 2^{-(n+1)}$. Thus $s, t$ are neighbouring points in $D_{n+1}$, i.e. there exists $k \in \{0, \ldots, 2^{n+1} - 1\}$, such that $t = k/2^{n+1}$ and $s = (k + 1)/2^{n+1}$. (1.3.5) with $m = n + 1$ follows directly from (1.3.3). Assume that the claim holds true upto $m \geq n + 1$. We will show its validity for $m + 1$.

Put $s' = \min\{u \in D_m \mid u > s\}$ and $t' = \max\{u \in D_m \mid u \leq t\}$. By construction $s \leq s' \leq t'$, and $s' - s, t - t' \leq 2^{-(m+1)}$. Then $0 < t' - s' \leq t - s < 2^{-n}$. Since $s', t' \in D_m$, they satisfy the induction hypothesis. We may now apply the triangle inequality, (1.3.3) and the induction hypothesis to obtain

$$|X_t(\omega) - X_s(\omega)| \leq |X_t(\omega) - X_{t'}(\omega)| + |X_{t'}(\omega) - X_{s'}(\omega)| + |X_{s'}(\omega) - X_s(\omega)| \leq 2^{-\gamma (m + 1)} + 2 \sum_{k=n+1}^{m} 2^{-\gamma k} + 2^{-\gamma (m + 1)} = 2 \sum_{k=n+1}^{m+1} 2^{-\gamma k}.$$ 

This shows the validity of (1.3.5). We prove (1.3.4). To this end, let $s, t \in D$ with $0 < t - s < 2^{-N_\omega}$. As noted before, there exists $n > N_\omega$, such that $2^{-(n+1)} \leq t - s < 2^{-n}$. Then there exists $m \geq n + 1$ such that $t, s \in D_m$. Apply (1.3.5) to obtain

$$|X_t(\omega) - X_s(\omega)| \leq 2 \sum_{k=n+1}^{m} 2^{-\gamma k} \leq \frac{2}{1 - 2^{-\gamma}} 2^{-\gamma (n+1)} \leq \frac{2}{1 - 2^{-\gamma}} |t - s|^\gamma.$$ 

Consequently (1.3.4) holds with constant $K' = 2/(1 - 2^{-\gamma})$.

**Step 3** Define a new stochastic process $Y = (Y_t)_{t \in [0,1]}$ on $(\Omega, \mathcal{F}, P)$ as follows: for $\omega \notin \Omega^*$, we put $Y_t = 0$ for all $t \in [0,1]$; for $\omega \in \Omega^*$ we define

$$Y_t(\omega) = \begin{cases} 
X_t(\omega), & \text{if } t \in D, \\
\lim_{t_n \to t} X_{t_n}(\omega), & \text{if } t \notin D.
\end{cases}$$

For each $\omega \in \Omega^*$, $t \to X_t(\omega)$ is uniformly continuous on the dense subset $D$ of $[0,1]$. It is a theorem from Analysis, that $t \to X_t(\omega)$ can be uniquely extended as a continuous function on $[0,1]$. This is the function $t \to Y_t(\omega)$, $t \in [0,1]$.

**Step 4** Uniform continuity of $X$ implies that $Y$ is a well-defined stochastic process. Since $X$ is continuous in probability, it follows that $Y$ is a modification of $X$ (Exercise 1.4). See BN §5 for a useful characterisation of convergence in probability.

QED

The fact that Kolmogorov’s continuity criterion requires $K|t - s|^{1+\beta}$ for some $\beta > 0$, guarantees uniform continuity of a.a. paths $X(\omega)$ when restricted to the dyadic rationals, whilst it does not so for $\beta = 0$ (see Exercise 1.5). This uniform continuity property is precisely the basis of the proof of the Criterion.
1.4. GAUSSIAN PROCESSES

Corollary 1.3.4 Brownian motion exists.

Proof. By Corollary 1.2.4 there exists a process $W = (W_t)_{t \geq 0}$ that has properties (i,ii,iii) of Definition 1.1.2. By property (iii) the increment $W_t - W_s$ has a $N(0, t - s)$-distribution for all $s \leq t$. This implies that $\mathbb{E}(W_t - W_s)^4 = (t - s)^2 \mathbb{E}Z^4$, with $Z$ a standard normally distributed random variable. This means the Kolmogorov’s continuity condition (1.3.1) is satisfied with $\alpha = 4$ and $\beta = 1$. So for every $T \geq 0$, there exists a continuous modification $W^T = (W_t^T)_{t \in [0,T]}$ of the process $(W_t)_{t \in [0,T]}$. Now define the process $X = (X_t)_{t \geq 0}$ by

$$X_t = \sum_{n=1}^{\infty} W_t^n 1_{[n-1,n)}(t).$$

In Exercise 1.7 you are asked to show that $X$ is a Brownian motion process. QED

Lemma 1.3.5 allows us to restrict to continuous paths.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Define the outer measure

$$\mathbb{P}^*\{A\} = \inf_{B \in \mathcal{F}, B \supset A} \mathbb{P}\{B\}.$$ 

Lemma 1.3.5 Suppose that $A$ is a subset of $\Omega$ with $\mathbb{P}^*\{A\} = 1$. Then for any $F \in \mathcal{F}$, one has $\mathbb{P}^*\{F\} = \mathbb{P}\{F\}$. Moreover, $(A, \mathcal{A}, \mathbb{P}^*)$ is a probability space, where $\mathcal{A} = \{A \cap F | F \in \mathcal{F}\}$.

Kolmogorov’s continuity criterion applied to BM implies that the outer measure of the set $\mathcal{C}[0,\infty)$ of continuous paths equals 1. The BM process after modification is the canonical process on the restricted space $(\mathbb{R}^{(0,\infty)} \cap \mathcal{C}[0,\infty), \mathcal{B}^{(0,\infty)} \cap \mathcal{C}[0,\infty), \mathbb{P}^*)$, with $\mathbb{P}^*$ the outer measure associated with $\mathbb{P}$.

Note that one can always construct the canonical process to have a desired distribution. Given any $(E, \mathcal{E})$-valued stochastic process $X$ on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then $X : (\Omega, \mathcal{F}) \rightarrow (E^T, \mathcal{E}^T)$ is a measurable map inducing a probability measure $\mathbb{P}_X$ on the path space $(E^T, \mathcal{E}^T)$. The canonical map on $(E^T, \mathcal{E}^T, \mathbb{P}_X)$ now has the same distribution as $X$ by construction.

Suppose that there exists a subset $\Gamma \subset E^T$, such that $X : \Omega \rightarrow \Gamma \cap E^T$. That is, the paths of $X$ have a certain structure. Then $X$ is $\mathcal{F}/\Gamma \cap \mathcal{E}^T$-measurable, and induces a probability measure $\mathbb{P}_X$ on $(\Gamma, \mathcal{E} \cap \mathcal{E}^T)$. Again, we may consider the canonical process on this restricted probability space $(\Gamma, \mathcal{E} \cap \mathcal{E}^T, \mathbb{P}_X)$.

1.4 Gaussian processes

Brownian motion is an example of a so-called Gaussian process. The general definition is as follows.
Definition 1.4.1 A real-valued stochastic process is called Gaussian of all its fdd’s are Gaussian, in other words, if they are multivariate normal distributions.

Let $X$ be a Gaussian process indexed by the set $T$. Then $m(t) = \mathbb{E}X_t$, $t \in T$, is the mean function of the process. The function $r(s, t) = \text{cov}(X_s, X_t)$, $(s, t) \in T \times T$, is the covariance function. By virtue of the following uniqueness lemma, fdd’s of Gaussian processes are determined by their mean and covariance functions.

Lemma 1.4.2 Two Gaussian processes with the same mean and covariance functions are versions of each other.

Proof. See Exercise 1.8. QED

Brownian motion is a special case of a Gaussian process. In particular it has $m(t) = 0$ for all $t \geq 0$ and $r(s, t) = s \wedge t$, for all $s \leq t$. Any other Gaussian process with the same mean and covariance function has the same fdd’s as BM itself. Hence, it has properties (i,ii,iii) of Definition 1.1.2. We have the following result.

Lemma 1.4.3 A continuous or a.s. continuous Gaussian process $X = (X_t)_{t \geq 0}$ is a BM process if and only if it has the same mean function $m(t) = \mathbb{E}X_t = 0$ and covariance function $r(s, t) = \mathbb{E}X_sX_t = s \wedge t$.

The lemma looks almost trivial, but provides us with a number extremely useful scaling and symmetry properties of BM!

Theorem 1.4.4 Let $W$ be a BM process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then the following are BM processes as well:

i) time-homogeneity for every $s \geq 0$ the shifted process $W(s) = (W_{t+s} - W_s)_{t \geq 0}$;

ii) symmetry the process $-W = (-W_t)_{t \geq 0}$;

iii) scaling for every $a > 0$, the process $W^a$ defined by $W^a_t = a^{-1/2}W_{at}$;

iv) time inversion the process $X = (X_t)_{t \geq 0}$ with $X_0 = 0$ and $X_t = tW_{1/t}$, $t > 0$.

If $W$ has (a.s.) continuous paths then $W(s)$, $-W$ and $W^a$ have (a.s.) continuous paths and $X$ has a.s. continuous paths. There exists a set $\Omega^* \in \mathcal{F}$, such that $X$ has continuous paths on $(\Omega^*, \mathcal{F} \cap \Omega^*, \mathbb{P})$.

Proof. We would like to apply Lemma 1.4.3. To this end we have to check that (i) the defined processes are Gaussian; (ii) that (almost all) sample paths are continuous and (iii) that they have the same mean and covariance functions as BM. In Exercise 1.9 you are asked to show this for the processes in (i,ii,iii). We will only prove (iv).

Clearly the processes $X$ in (iv) is a stochastic process. We will show that almost all sample paths of $X$ are continuous. A simple application of Lemma 1.4.2 then finishes the proof.

So let us show that almost all sample paths of $X$ are continuous. By time inversion, it is immediate that $(X_t)_{t \geq 0}$ (a.s.) has continuous sample paths if $W$ has. We only need show a.s. continuity at $t = 0$, that is, we need to show that $\lim_{t \downarrow 0} X_t = 0$, a.s.
Let $\Omega^* = \{\omega \in \Omega \mid (W_t(\omega))_{t \geq 0} \text{ continuous}, W_0(\omega) = 0\}$. By assumption $P(\Omega^*) = 1$. Further, $(X_t)_{t \geq 0}$ has continuous paths on $\Omega^*$.

Let $\omega \in \Omega^*$. Then $\lim_{t \to 0} X_t(\omega) = 0$ if and only if for all $\epsilon > 0$ there exists $\delta_\omega > 0$ such that $|X_t(\omega)| < \epsilon$ for all $t \leq \delta_\omega$. This is true if and only if for all integers $m \geq 1$, there exists an integer $n_\omega$, such that $|X_q(\omega)| < 1/m$ for all $q \in Q$ with $q < 1/n_\omega$, because of continuity of $X_t(\omega)$, $t > 0$. Check that this implies

$$\{\omega : \lim_{t \to 0} X_t(\omega) = 0\} \cap \Omega^* = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{q \in (0,1/n) \cap Q} \{\omega : |X_q(\omega)| < 1/m\} \cap \Omega^*.$$ 

The fdd’s of $X$ and $W$ are equal. It follows that (cf. Exercise 1.10) the probability of the latter equals

$$P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{q \in (0,1/n) \cap Q} \{\omega : |W_q(\omega)| < 1/m\} \cap \Omega^*\right) = P(\Omega^*) = 1,$$

since $(W_t)_{t \geq 0}$ has continuous paths on $\Omega^*$. Hence $P(\{\omega : \lim_{t \to 0} X_t(\omega) = 0\} \cap \Omega^*) = 1$, and so $P(\omega : \lim_{t \to 0} X_t(\omega) = 0) = 1$.

QED

These scaling and symmetry properties can be used to show a number of properties of Brownian motion. The first is that Brownian motion sample paths oscillate between $+\infty$ and $-\infty$.

**Corollary 1.4.5** Let $W$ be a BM, with the property that all paths are continuous. Then

$$P\{\sup_{t \geq 0} W_t = \infty, \inf_{t \geq 0} W_t = -\infty\} = 1.$$ 

**Proof.** It is sufficient to show that

$$P\{\sup_{t \geq 0} W_t = \infty\} = 1. \quad (1.4.1)$$

Indeed, the symmetry property implies

$$\sup_{t \geq 0} W_t \overset{d}{=} -\inf_{t \geq 0} W_t.$$ 

Hence (1.4.1) implies that $P(\inf_{t \geq 0} W_t = -\infty) = 1$. As a consequence, the probability of the intersection equals 1 (why?).

First of all, notice that $\sup_t W_t$ is well-defined. We need to show that $\sup_t W_t$ is a measurable function. This is true (cf. BN Lemma 1.2) if $\{\sup_t W_t \leq x\}$ is measurable for all $x \in \mathbb{R}$ (if $Q$ is sufficient of course). This follows from

$$\{\sup_t W_t \leq x\} = \bigcap_{q \in Q} \{W_q \leq x\}.$$ 

Here we use that all paths are continuous. We cannot make any assertions on measurability of $\{W_q \leq x\}$ restricted to the set of discontinuous paths, unless $\mathcal{F}$ is $P$-complete.
By the scaling property we have for all \( a > 0 \)
\[
\sup_t W_t \overset{d}{=} \sup_t \frac{1}{\sqrt{a}} W_{at} = \frac{1}{\sqrt{a}} \sup_t W_t.
\]
It follows for \( n \in \mathbb{Z}_+ \) that
\[
P\{\sup_t W_t \leq n\} = P\{n^2 \sup_t W_t \leq n\} = P\{\sup_t W_t \leq 1/n\}.
\]
By letting \( n \) tend to infinity, we see that
\[
P\{\sup_t W_t < \infty\} = P\{\sup_t W_t \leq 0\}.
\]
Thus, for (1.4.1) it is sufficient to show that \( P\{\sup_t W_t \leq 0\} = 0. \) We have
\[
P\{\sup_t W_t \leq 0\} \leq P\{W_1 \leq 0, \sup_{t \geq 1} W_t \leq 0\} \leq P\{W_1 \leq 0, \sup_{t \geq 1} W_t - W_1 < \infty\} = P\{W_1 \leq 0\} P\{\sup_{t \geq 1} W_t - W_1 < \infty\},
\]
by the independence of Brownian motion increments. By the time-homogeneity of BM, the latter probability equals the probability that the supremum of BM is finite. We have just showed that this equals \( P\{\sup_t W_t \leq 0\} \). And so we find
\[
P\{\sup_t W_t \leq 0\} \leq \frac{1}{2} P\{\sup_t W_t \leq 0\}.
\]
This shows that \( P\{\sup_t W_t \leq 0\} = 0 \) and so we have shown (1.4.1).

QED

Since BM has a.s. continuous sample paths, this implies that almost every path visits every point of \( \mathbb{R} \) with probability 1. This property is called recurrence. With probability 1 it even visits every point infinitely often. However, we will not further pursue this at the moment and merely mention the following statement.

**Corollary 1.4.6** BM is recurrent.

An interesting consequence of the time inversion property is the following strong law of large numbers for BM.

**Corollary 1.4.7** Let \( W \) be a BM. Then
\[
\frac{W_t}{t} \overset{a.s.}{\to} 0, \quad t \to \infty.
\]

*Proof.* Let \( X \) be as in part (iv) of Theorem 1.4.4. Then
\[
P\{\frac{W_t}{t} \to 0, \quad t \to \infty\} = P\{X_{1/t} \to 0, \quad t \to \infty\} = 1.
\]

QED
1.5 Non-differentiability of the Brownian sample paths

We have already seen that the sample paths of \( W \) are continuous functions that oscillate between \( +\infty \) and \( -\infty \). Figure 1.1 suggests that the sample paths are very rough. The following theorem shows that this is indeed the case.

**Theorem 1.5.1** Let \( W \) be a BM defined on the space \((\Omega, \mathcal{F}, \mathbb{P})\). There is a set \( \Omega^* \) with \( \mathbb{P}\{\Omega^*\} = 1 \), such that the sample path \( t \to W(\omega) \) is nowhere differentiable, for any \( \omega \in \Omega^* \).

**Proof.** Let \( W \) be a BM. Consider the upper and lower right-hand derivatives

\[
D^W(t, \omega) = \limsup_{h \downarrow 0} \frac{W_{t+h}(\omega) - W_t(\omega)}{h},
\]

\[
D_W(t, \omega) = \liminf_{h \downarrow 0} \frac{W_{t+h}(\omega) - W_t(\omega)}{h}.
\]

Let

\[ A = \{ \omega \mid \text{there exists } t \geq 0 \text{ such that } D^W(t, \omega) \text{ and } D_W(t, \omega) \text{ are finite } \} \]

Note that \( A \) is not necessarily a measurable set. We will therefore show that \( A \) is contained in a measurable set \( B \) with \( \mathbb{P}\{B\} = 0 \). In other words, \( A \) has outer measure 0.

To define the set \( B \), first consider for \( k, n \in \mathbb{Z}_+ \) the random variable

\[
X_{n,k} = \max \{ |W_{(k+1)/2^n} - W_{k/2^n}|, |W_{(k+2)/2^n} - W_{(k+1)/2^n}|, |W_{(k+3)/2^n} - W_{(k+2)/2^n}| \}.
\]

Define for \( n \in \mathbb{Z}_+ \)

\[
Y_n = \min_{k \leq n/2^n} X_{n,k}.
\]

A the set \( B \) we choose

\[ B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{ Y_k \leq k2^{-k} \} \]

We claim that \( A \subseteq B \) and \( \mathbb{P}\{B\} = 0 \).

To prove the inclusion, let \( \omega \in A \). Then there exists \( t = t_\omega \), such that \( D_W(t, \omega), D^W(t, \omega) \) are finite. Hence, there exists \( K = K_\omega \), such that

\[-K < D_W(t, \omega) \leq D^W(t, \omega) < K.\]

As a consequence, there exists \( \delta = \delta_\omega \), such that

\[
|W_s(\omega) - W_t(\omega)| \leq K \cdot |s-t|, \quad s \in [t, t+\delta]. \quad (1.5.1)
\]

Now take \( n = n_\omega \in \mathbb{Z}_+ \) so large that

\[
\frac{4}{2^n} < \delta, \quad 8K < n, \quad t < n. \quad (1.5.2)
\]

Next choose \( k \in \mathbb{Z}_+ \), such that

\[
\frac{k-1}{2^n} \leq t < \frac{k}{2^n}. \quad (1.5.3)
\]
By the first relation in (1.5.2) we have that
\[
\left| \frac{k + 3}{2^n} - t \right| \leq \left| \frac{k + 3}{2^n} - \frac{k - 1}{2^n} \right| \leq \frac{4}{2^n} < \delta,
\]
so that \(k/2^n, (k + 1)/2^n, (k + 2)/2^n, (k + 3)/2^n \in [t, t + \delta]\). By (1.5.1) and the second relation in (1.5.2) we have our choice of \(n\) and \(k\) that
\[
X_{n,k}(\omega) \leq \max \{ |W_{(k+1)/2^n} - W_t| + |W_t - W_{k/2^n}|, |W_{(k+2)/2^n} - W_t| + |W_t - W_{(k+1)/2^n}|, |W_{(k+3)/2^n} - W_t| + |W_t - W_{(k+2)/2^n}| \} \leq 2K \frac{4}{2^n} < \frac{n}{2^n}.
\]
The third relation in (1.5.2) and (1.5.3) it holds that \(k - 1 \leq t2^n < n2^n\). This implies \(k \leq n2^n\) and so \(Y_n(\omega) \leq X_{n,k}(\omega) \leq n/2^n\), for our choice of \(n\).

Summarising, \(\omega \in A\) implies that \(Y_n(\omega) \leq n/2^n\) for all sufficiently large \(n\). This implies \(\omega \in B\). We have proved that \(A \subseteq B\).

In order to complete the proof, we have to show that \(P\{B\} = 0\). Note that \(|W_{(k+1)/2^n} - W_{k/2^n}|, |W_{(k+2)/2^n} - W_{(k+1)/2^n}|\) and \(|W_{(k+3)/2^n} - W_{(k+2)/2^n}|\) are i.i.d. random variables. We have for any \(\epsilon > 0\) and \(k = 0, \ldots, n2^n\) that
\[
P\{X_{n,k} \leq \epsilon\} \leq P\{\max_i \{ |W_{(k+i)/2^n} - W_{(k+i-1)/2^n}| \leq \epsilon, i = 1, 2, 3 \} \}
\leq (P\{W_{(k+1)/2^n} - W_{(k+2)/2^n} \leq \epsilon\})^3 = (P\{W_1/2^n \leq \epsilon\})^3
= (P\{W_1 \leq \epsilon\})^3 \leq (2 \cdot 2^n/\epsilon^3)^3 = 2^{3n/2+1}\epsilon^3.
\]
We have used time-homogeneity in the third step, the time-scaling property in the fourth and the fact that the density of a standard normal random variable is bounded by 1 in the last equality. Next,
\[
P\{Y_n \leq \epsilon\} = P\{\bigcup_{k=1}^{n2^n} \{X_{n,k} \leq \epsilon\} \} \leq \sum_{k=1}^{n2^n} P\{X_{n,k} \leq \epsilon\} \leq n2^n \cdot 2^{3n/2+1}\epsilon^3 = n2^{5n/2+1}\epsilon^3.
\]
Choose \(\epsilon = n/2^n\), we see that \(P\{Y_n \leq n/2^n\} \to 0\), as \(n \to \infty\). This implies that \(P\{B\} = P\{\lim_{n \to \infty} \inf_{Y_n \leq n/2^n}\} \leq \lim_{n \to \infty} P\{Y_n \leq n/2^n\} = 0\). We have used Fatou’s lemma in the last inequality.

QED

1.6 Filtrations and stopping times

If \(W\) is a BM, the increment \(W_{t+h} - W_t\) is independent of ‘what happened up to time \(t\)’. In this section we introduce the concept of a filtration to formalise the notion of ‘information that we have up to time \(t\)’. The probability space \((\Omega, \mathcal{F}, P)\) is fixed again and we suppose that \(T\) is a subinterval of \(\mathbb{Z}_+\) or \(\mathbb{R}_+\).

Definition 1.6.1 A collection \((\mathcal{F}_t)_{t \in T}\) of sub-\(\sigma\)-algebras is called a filtration if \(\mathcal{F}_s \subseteq \mathcal{F}_t\) for all \(s \leq t\). A stochastic process \(X\) defined on \((\Omega, \mathcal{F}, P)\) and indexed by \(T\) is called adapted to the filtration if for every \(t \in T\), the random variable \(X_t\) is \(\mathcal{F}_t\)-measurable. Then \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, P)\) is a filtered probability space.
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We can think of a filtration as a flow of information. The \( \sigma \)-algebra \( \mathcal{F}_t \) contains the events that can happen ‘upto time \( t \)’. An adapted process is a process that ‘does not look into the future’. If \( X \) is a stochastic process, then we can consider the filtration \( \mathcal{F}_t^X \) generated by \( X \):

\[
\mathcal{F}_t^X = \sigma(X_s, s \leq t).
\]

We call this the filtration generated by \( X \), or the natural filtration of \( X \). It is the ‘smallest’ filtration, to which \( X \) is adapted. Intuitively, the natural filtration of a process keeps track of the ‘history’ of the process. A stochastic process is always adapted to its natural filtration. If \( X \) is a canonical process on the subspace \( (\Gamma, \Gamma \cap \mathcal{E}^T) \) of the path space, then \( \mathcal{F}_t^X = \Gamma \cap \mathcal{E}^{[0,t]} \).

Review BN §2, the paragraph on \( \sigma \)-algebra generated by a random variable or a stochastic process.

If \( (\mathcal{F}_t)_{t \in T} \) is a filtration, then for \( t \in T \) we may define the \( \sigma \)-algebra

\[
\mathcal{F}_{t+} = \bigcap_{n=1}^{\infty} \mathcal{F}_{t+1/n}.
\]

This is the \( \sigma \)-algebra \( \mathcal{F}_t \), augmented with the events that ‘happen immediately after time \( t \)’. The collection \( (\mathcal{F}_{t+})_{t \in T} \) is again a filtration (see Exercise 1.16). Cases in which it coincides with the original filtration are of special interest.

**Definition 1.6.2** We call a filtration \( (\mathcal{F}_t)_{t \in T} \) right-continuous if \( \mathcal{F}_{t+} = \mathcal{F}_t \) for all \( t \in T \).

Intuitively, right-continuity of a filtration means that ‘nothing can happen in an infinitesimal small time-interval’ after the observed time instant. Note that for every filtration \( (\mathcal{F}_t) \), the corresponding filtration \( (\mathcal{F}_{t+}) \) is always right-continuous.

In addition to right-continuity it is often assumed that \( \mathcal{F}_0 \) contains all events in \( \mathcal{F}_\infty \) that have probability 0, where

\[
\mathcal{F}_\infty = \sigma(\mathcal{F}_t, t \geq 0).
\]

As a consequence, every \( \mathcal{F}_t \) then also contains these events.

**Definition 1.6.3** A filtration \( (\mathcal{F}_t)_{t \in T} \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) is said to satisfy the usual conditions if it is right-continuous and \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-negligible events of \( \mathcal{F}_\infty \).

We now introduce a very important class of ‘random times’ that can be associated with a filtration.

**Definition 1.6.4** A \([0, \infty)\)-valued random variable \( \tau \) is called a stopping time with respect to the filtration \( (\mathcal{F}_t) \) if for every \( t \in T \) it holds that the event \( \{ \tau \leq t \} \) is \( \mathcal{F}_t \)-measurable. If \( \tau < \infty \) a.s., we call \( \tau \) a finite stopping time.

Loosely speaking, \( \tau \) is a stopping time if for every \( t \in T \) we can determine whether \( \tau \) has occurred before time \( t \) on basis of the information that we have upto time \( t \). Note that \( \tau \) is \( \mathcal{F}/\mathcal{B}([0, \infty]) \)-measurable.

With a stopping time \( \tau \) we can associate the the \( \sigma \)-algebra \( \sigma^\tau \) generated by \( \tau \). However, this \( \sigma \)-algebra only contains the information about when \( \tau \) occurred. If \( \tau \) is associated with an adapted process \( X \), then \( \sigma^\tau \) contains no further information on the history of the process.
upto the stopping time. For this reason we associate with $\tau$ the (generally) larger $\sigma$-algebra $\mathcal{F}_\tau$ defined by

$$\mathcal{F}_\tau = \{ A \in \mathcal{F} : A \cap \{ \tau \leq t \} \in \mathcal{F}_t \text{ for all } t \in T \}.$$  

(see Exercise 1.17). This should be viewed as the collection of all events that happen prior to the stopping time $\tau$. Note that the notation causes no confusion, since a deterministic time $t \in T$ is clearly a stopping time and its associated $\sigma$-algebra is simply the $\sigma$-algebra $\mathcal{F}_t$.

If the filtration $(\mathcal{F}_t)$ is right-continuous, then $\tau$ is a stopping time if and only if $\{ \tau < t \} \in \mathcal{F}_t$ for all $t \in T$ (see Exercise 1.23). For general filtrations we introduce the following class of random times.

**Definition 1.6.5** A $[0, \infty]$-valued random variable $\tau$ is called an *optional time* with respect to the filtration $(\mathcal{F}_t)$ if for every $t \in T$ it holds that $\{ \tau < t \} \in \mathcal{F}_t$. If $\tau < \infty$ almost surely, we call the optional time *finite*.

**Lemma 1.6.6** $\tau$ is an optional time with respect to $(\mathcal{F}_t)$ if and only if it is a stopping time with respect to $(\mathcal{F}_{t+})$. Every stopping time is an optional time.

**Proof.** See Exercise 1.24. QED

**Special stopping times** The so-called *hitting* and *first entrance times* form an important class of stopping times and optional times. They are related to the first time that the process visits a set $B$.

**Lemma 1.6.7** Let $(E, d)$ be a metric space and let $\mathcal{B}(E)$ be the Borel-$\sigma$-algebra of open sets compatible with the metric $d$. Suppose that $X = (X_t)_{t \geq 0}$ is a continuous, $(E, \mathcal{B}(E))$-valued stochastic process and that $B$ is closed in $E$. Then the first entrance time of $B$, defined by

$$\sigma_B = \inf \{ t \geq 0 \mid X_t \in B \},$$

is an $(\mathcal{F}^X_t)$-stopping time.$^1$

**Proof.** Denote the distance of a point $x \in E$ to the set $B$ by $d(x, B)$. In other words

$$d(x, B) = \inf \{ d(x, y) \mid y \in B \}.$$

First note that $x \mapsto d(x, B)$ is a continuous function. Hence it is $\mathcal{B}(E)$-measurable. It follows that $Y_t = d(X_t, B)$ is $(\mathcal{F}^X_t)$-measurable as a composition of measurable maps. Since $X_t$ is continuous, the real-valued process $(Y_t)_t$ is continuous as well. Moreover, since $B$ is closed, it holds that $X_t \in B$ if and only if $Y_t = 0$. By continuity of $Y_t$, it follows that $\sigma_B > t$ if and only if $Y_s > 0$ for all $s \leq t$. This means that

$$\{ \sigma_B > t \} = \{ Y_s > 0, 0 \leq s \leq t \} = \bigcup_{n=1}^{\infty} \bigcap_{q \in \mathbb{Q} \cap [0,t]} \{ Y_q > \frac{1}{n} \} = \bigcup_{n=1}^{\infty} \bigcap_{q \in \mathbb{Q} \cap [0,t]} \{ d(X_q, B) > \frac{1}{n} \} \in \mathcal{F}^X_t.

QED

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$^1$As is usual, we define $\inf \emptyset = \infty$. 

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**Lemma 1.6.8** Let \((E,d)\) be a metric space and let \(B(E)\) be the Borel-\(\sigma\)-algebra of open sets compatible with the metric \(d\). Suppose that \(X = (X_t)_{t \geq 0}\) is a right-continuous, \((E,B(E))\)-valued stochastic process and that \(B\) is an open set in \(E\). Then the hitting time of \(B\), defined by

\[
\tau_B = \inf\{t > 0 \mid X_t \in B\},
\]

is an \((\mathcal{F}_t^X)\)-optional time.

**Proof.** By right-continuity of \(X\) and the fact that \(B\) is open, \(\tau_B(\omega) < t\) if and only if there exists a rational number \(0 < q_x < t\) such that \(X_{q_x}(\omega) \in B\). Hence

\[
\{\tau_B < t\} = \bigcup_{q \in (0,t) \cap \mathbb{Q}} \{X_q \in B\}.
\]

The latter set is \(\mathcal{F}_t^X\)-measurable, and so is the first. \(\Box\)

**Example 1.6.9** Let \(W\) be a BM with continuous paths and, for \(x > 0\), consider the random variable

\[
\tau_x = \inf\{t > 0 \mid W_t = x\}.
\]

Since \(x > 0\), \(W\) is continuous and \(W_0 = 0\) a.s., \(\tau_x\) can a.s. be written as

\[
\tau_x = \inf\{t \geq 0 \mid W_t = x\}.
\]

By Lemma 1.6.7 this is an \((\mathcal{F}_t^W)\)-stopping time. Next we will show that \(P\{\tau_x < \infty\} = 1\).

Note that \(\{\tau_x < \infty\} = \bigcup_{n=1}^\infty \{\tau_x \leq n\}\) is a measurable set. Consider \(A = \{\omega : \sup_{t \geq 0} W_t = \infty, \inf_{t \geq 0} W_t = -\infty\}\). By Corollary 1.4.5 this set has probability 1.

Let \(T > |x|\). For each \(\omega \in A\), there exist \(T_\omega, T'_\omega\), such that \(W_{T_\omega} \geq T, W_{T'_\omega} \leq -T\). By continuity of paths, there exists \(t_\omega \in (T_\omega, T'_\omega, T_\omega \vee T'_\omega)\), such that \(W_{t_\omega} = x\). It follows that \(A \subseteq \{\tau_x < \infty\}\). Hence \(P\{\tau_x < \infty\} = 1\). \(\Box\)

An important question is whether the first entrance time of a closed set is a stopping time for more general stochastic processes than the continuous ones. The answer in general is that this is not true unless the filtration is suitably augmented with null sets (cf. BN §10). Without augmentation we can derive the two following results. Define \(X_{t-} = \liminf_{s \uparrow t} X_s\).

**Lemma 1.6.10** Let \((E,d)\) be a metric space and let \(B(E)\) be the Borel-\(\sigma\)-algebra of open sets compatible with the metric \(d\). Suppose that \(X = (X_t)_{t \geq 0}\) is a (everywhere) cadlag, \((E,B(E))\)-valued stochastic process and that \(B\) is a closed set in \(E\). Then

\[
\gamma_B = \inf\{t > 0 \mid X_t \in B, \text{ or } X_{t-} \in B\},
\]

is an \((\mathcal{F}_t^X)\)-stopping time.

**Lemma 1.6.11** Let \((E,d)\) be a metric space and let \(B(E)\) be the Borel-\(\sigma\)-algebra of open sets compatible with the metric \(d\). Suppose that \(X = (X_t)_{t \geq 0}\) is a right-continuous, \((E,B(E))\)-valued stochastic process and that \(B\) is closed in \(B(E)\). Let \(X\) be defined on the underlying probability space \((\Omega, \mathcal{F}, P)\).

Suppose further that there exist \(\mathcal{F}\)-measurable random times \(0 < \tau_1 < \tau_2 < \cdots\), such that the discontinuities of \(X\) are contained in the set \(\{\tau_1, \tau_2, \ldots\}\). Then
a) $\tau_1, \tau_2, \ldots$ are $(\mathcal{F}_t^X)$-stopping times;

b) $\sigma_B$ is an $(\mathcal{F}_t^X)$-stopping time.

Proof. We prove (a) and check that $\{\tau_1 \leq t\} \in \mathcal{F}_t^X$. Define for $Q_t = \{qt \mid q \in [0,1] \cap \mathbb{Q}\}$

$$G = \bigcup_{m \geq 1} \bigcap_{n \geq 1} \bigcup_{u,s \in Q_t} \{|u-s| \leq \frac{1}{n}, d(X_u, X_s) > \frac{1}{m}\}.$$

Claim: $G = \{\tau_1 \leq t\}$.

Let $\omega \in G$. Then there exists $m_\omega$ such that for each $n$ there exists a pair $(u_{n,\omega}, s_{n,\omega})$ with $|u_{n,\omega} - s_{n,\omega}| < 1/n$ for which $d(X_{u_{n,\omega}}, X_{s_{n,\omega}}) > 1/m_\omega$.

If $\omega \notin \{\tau_1 \leq t\}$, then $\tau_1(\omega) > t$, and $s \rightarrow X_s(\omega)$ would be continuous on $[0,t]$, hence uniformly continuous. As a consequence, for each for $m_\omega$ there exists $n_\omega$ for which $d(X_s, X_u) < 1/m_\omega$ for $|u-s| < 1/n_\omega$. This contradicts the above, and hence $\omega \in \{\tau_1 \leq t\}$.

To prove the converse, assume that $\omega \in \{\tau_1 \leq t\}$, i.o.w., $s = \tau_1(\omega) \leq t$. By right-continuity this implies that there exists a sequence $t_l \uparrow s$, along which $X_{t_l}(\omega) \not\rightarrow X_s(\omega)$. Hence there exists $m$, such that for each $n$ there exists $t_l(n)$ with $|s - t_l(n)| < 1/n$, for which $d(X_{t_l(n)}(\omega), X_s(\omega)) > 1/m$.

By right-continuity, for each $n$ one can find $q_n \geq t_l(n)$ and $q \geq s, q_n, q \in Q_t$, such that $|q - q_n| < 1/n$ and $d(X_{q(n)}(\omega), X_q(\omega)) > 1/2m$. It follows that

$$\omega \in \bigcap_{n \geq 1} \{|q - q(n)| < 1/n, d(X_{q(n)}, X_q) > 1/2m\}.$$  

Hence $\omega \in G$.

To show that $\tau_2$ is a stopping time, we add the requirement $u, s \geq \tau_1$ in the definition of the analogon of the set $G$, etc.

Next we prove (b). We have to cut out small intervals to the left of jumps. On the remainder we can separate the path $X_s(\omega), s \leq t$ and the set $B$, if $\sigma_B(\omega) > t$. To this end define

$$I_{k,n} = \{u \mid \tau_k - \frac{1}{n} \leq u < \tau_k \leq t\}.$$

For each $\omega \in \Omega$ this is a subset of $[0,t]$. Now, given $u \in [0,t]$,

$$\{\omega \mid I_{k,n}(\omega) \ni u\} = \{\omega \mid u < \tau_k(\omega) \leq t \land (u + \frac{1}{n})\} \in \mathcal{F}_t^X.$$

Check that

$$\{\sigma_B > t\} = \bigcap_{n \geq 1} \bigcup_{m \geq 1} \bigcap_{q \in Q_t} \bigcap_{k \neq I_{k,n}} \{q \notin I_{k,n} \cap \{d(X_q, B) > \frac{1}{m}\}\}.$$

QED

**Measurability of $X_\tau$ for $\tau$ an adapted stopping time** We often would like to consider the stochastic process $X$ evaluated at a finite stopping time $\tau$. However, it is not a priori clear that the map $\omega \rightarrow X_\tau(\omega)$ is measurable. In other words, that $X_\tau$ is a random variable. We need measurability of $X$ in both parameters $t$ and $\omega$. This motivates the following definition.
1.6. FILTRATIONS AND STOPPING TIMES

**Definition 1.6.12** An \((E, \mathcal{E})\)-valued stochastic process is called *progressively measurable* with respect to the filtration \((\mathcal{F}_t)\) if for every \(t \in T\) the map \((s, \omega) \rightarrow X_s(\omega)\) is measurable as a map from \([0, t] \times \Omega, \mathcal{B}([0, t]) \times \mathcal{F}_t\) to \((E, \mathcal{E})\).

**Lemma 1.6.13** Let \((E, d)\) be a metric space and \(\mathcal{B}(E)\) the Borel-σ-algebra of open sets compatible with \(d\). Every adapted right-continuous, \((E, \mathcal{B}(E))\)-valued stochastic process \(X\) is progressively measurable.

**Proof.** Fix \(t \geq 0\). We have to check that
\[
\{(s, \omega) \mid X_s(\omega) \in A, s \leq t\} \in \mathcal{B}([0, t]) \times \mathcal{F}_t, \quad \forall A \in \mathcal{B}(E).
\]
For \(n \in \mathbb{Z}_+\) define the process
\[
X^n_s = \sum_{k=0}^{n-1} X_{(k+1)t/n} 1_{\{(kt/n, (k+1)t/n)\}}(s) + X_0 1_{\{0\}}(s).
\]
This is a measurable process, since
\[
\{(s, \omega) \mid X^n_s(\omega) \in A, s \leq t\} = \bigcup_{k=0}^{n-1} \left(\{s \in (kt/n, (k+1)t/n)\} \times \{\omega \mid X_{(k+1)t/n}(\omega) \in A\}\right) \bigcup\{\{0\} \times \{\omega \mid X_0(\omega) \in A\}\}.
\]
Clearly, \(X^n_s(\omega) \rightarrow X_s(\omega), n \rightarrow \infty,\) for all \((s, \omega) \in [0, t] \times \Omega\), pointwise. By BN Lemma 6.1, the limit is measurable. \(\blacksquare\)

Review BN §6 containing an example of a non-progressively measurable stochastic process and a stopping time \(\tau\) with \(X_\tau\) not \(\mathcal{F}_\tau\)-measurable.

**Lemma 1.6.14** Suppose that \(X\) is a progressively measurable process. Let \(\tau\) be a finite stopping time. Then \(X_\tau\) is an \(\mathcal{F}_\tau\)-measurable random variable.

**Proof.** We have to show that \(\{X_\tau \in B\} \cap \{\tau \leq t\} \in \mathcal{F}_t\), for every \(B \in \mathcal{E}\) and every \(t \geq 0\).
Now note that
\[
\{X_{\tau \land t} \in B\} = \{X_\tau \in B, \tau \leq t\} \cup \{X_t \in B, \tau > t\}.
\]
Clearly \(\{X_t \in B, \tau > t\} \in \mathcal{F}_t\). If we can show that \(\{X_{\tau \land t} \in B\} \in \mathcal{F}_t\), it easily follows that \(\{X_\tau \in B, \tau \leq t\} \in \mathcal{F}_t\). Hence, it suffices to show that the map \(\omega \rightarrow X_{\tau(\omega) \land t}(\omega)\) is \(\mathcal{F}_t\)-measurable. This map is the composition of the maps \(\omega \rightarrow (\tau(\omega) \land t, \omega)\) from \(\Omega\) to \([0, t] \times \Omega\), and \((s, \omega) \rightarrow X_s(\omega)\) from \([0, t] \times \Omega\) to \(E\). The first map is measurable as a map from \((\Omega, \mathcal{F}_t)\) to \(([0, t] \times \Omega, \mathcal{B}([0, t]) \times \mathcal{F}_t)\) (this is almost trivial, see Exercise 1.25). Since \(X\) is progressively measurable, the second map is measurable as a map from \(([0, t] \times \Omega, \mathcal{B}([0, t]) \times \mathcal{F}_t)\) to \((E, \mathcal{E})\). This completes the proof, since the composition of measurable maps is measurable. \(\blacksquare\)

Very often problems of interest consider a stochastic process up to a given stopping \(\tau\). To this end we define the *stopped process* \(X^\tau\) by
\[
X^\tau_t = X_{\tau \land t} = \begin{cases} X_t, & t < \tau, \\ X_\tau, & t \geq \tau. \end{cases}
\]
By Lemma 1.6.14 and Exercises 1.18 and 1.20, we have the following result.
Lemma 1.6.15 If \( X \) is progressively measurable with respect to \((\mathcal{F}_t)\) and \( \tau \) an \((\mathcal{F}_t)\)-stopping time, then the stopped process \( X^\tau \) is adapted to the filtrations \((\mathcal{F}_{\tau \wedge t})\) and \((\mathcal{F}_t)\).

In the subsequent chapters we repeatedly need the following technical lemma. It states that every stopping time is the decreasing limit of a sequence of stopping times that take only finitely many values.

Lemma 1.6.16 Let \( \tau \) be a stopping time. Then there exist stopping times \( \tau_n \) that only take finitely many values and such \( \tau_n \downarrow \tau \).

Proof. Define

\[
\tau_n = \sum_{k=1}^{n2^n-1} \frac{k}{2^n} 1\{\tau \in [(k-1)/2^n,k/2^n)\} + \infty 1\{\tau \geq n\}.
\]

Then \( \tau_n \) is a stopping time and \( \tau_n \downarrow \tau \) (see Exercise 1.26). QED

Using the notion of filtrations, we can extend the definition of BM as follows.

Definition 1.6.17 Suppose that on a probability space \((\Omega, \mathcal{F}, P)\) we have a filtration \((\mathcal{F}_t)_{t \geq 0}\) and an adapted stochastic process \( W = (W_t)_{t \geq 0} \). Then \( W \) is called a (standard) Brownian motion (or a Wiener process) with respect to the filtration \((\mathcal{F}_t)\) if

i) \( W_0 = 0 \);

ii) (independence of increments) \( W_t - W_s \) is independent of \( \mathcal{F}_s \) for all \( s \leq t \);

iii) (stationarity of increments) \( W_t - W_s \overset{d}{=} \mathcal{N}(0, t-s) \) distribution;

iv) all sample paths of \( W \) are continuous.

Clearly, process \( W \) that is a BM in the sense of the ‘old’ Definition 1.1.2 is a BM with respect to its natural filtration. If in the sequel we do not mention the filtration of a BM explicitly, we mean the natural filtration. However, we will see that it is sometimes necessary to consider Brownian motions with larger filtrations as well.
1.7 Exercises

Exercise 1.1 Show the claim in the proof of Theorem 1.1.5 that the system $\mathcal{I}$ described there is a $\pi$-system for the $\sigma$-algebra $\sigma(N_u, u \leq s)$.

Exercise 1.2 Complete the proof of Corollary 1.2.4. Give full details.

Exercise 1.3 Give an example of two processes that are versions of each other, but not modifications.

Exercise 1.4 Prove that the process $Y$ defined in the proof of Theorem 1.3.3 is indeed a modification of the process $X$. See remark in Step 4 of the proof of this theorem.

Exercise 1.5 An example of a right-continuous but not continuous stochastic process $X$ is the following. Let $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$ and $P = \lambda$ is the Lebesgue measure on $[0, 1]$. Let $Y$ be the identity map on $\Omega$, i.e. $Y(\omega) = \omega$. Define a stochastic process $X = (X_t)_{t \in [0, 1]}$ by $X_t = 1_{\{Y \leq t\}}$. Hence, $X_t(\omega) = 1_{\{Y(\omega) \leq t\}} = 1_{\{(0, t]\}}(\omega)$.

The process $X$ does not satisfy the conditions of Kolmogorov’s Continuity Criterion, but it does satisfy the condition

$$E|X_t - X_s|^\alpha \leq K|t - s|,$$

for any $\alpha > 0$ and $K = 1$. Show this.

Exercise 1.6 Suppose that $X$ and $Y$ are modifications of each other, and for both $X$ and $Y$ all sample paths are either left or right continuous. Let $T$ be an interval in $\mathbb{R}$. Show that

$$P\{X_t = Y_t, \text{ for all } t \in T\} = 1.$$

Exercise 1.7 Prove that the process $X$ in the proof of Corollary 1.3.4 is a BM process. To this end, you have to show that $X$ has the correct fdd’s, and that $X$ has a.s. continuous sample paths.

Exercise 1.8 Prove Lemma 1.4.2.

Exercise 1.9 Prove parts (i,ii,iii) of Theorem 1.4.4.

Exercise 1.10 Consider the proof of the time-inversion property of Theorem 1.4.4. Prove that

$$P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{q \in (0, 1/n] \cap \mathbb{Q}} \{\omega : |X_q(\omega)| < 1/m\} \cap \Omega^*\right) = P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{q \in (0, 1/n] \cap \mathbb{Q}} \{\omega : |W_q(\omega)| < 1/m\} \cap \Omega^*\right).$$

Exercise 1.11 Let $W$ be a BM and define $X_t = W_{1-t} - W_1$ for $t \in [0, 1]$. Show that $(X_t)_{t \in [0, 1]}$ is a BM as well.
Exercise 1.12 Let $W$ be a BM and fix $t > 0$. Define the process $B$ by

$$B_s = W_{s\wedge t} - (W_s - W_{s\wedge t}) = \begin{cases} W_s, & s \leq t \\ 2W_t - W_s, & s > t. \end{cases}$$

Draw a picture of the processes $W$ and $B$ and show that $B$ is again a BM. We will see another version of this so-called reflection principle in Chapter 3.

Exercise 1.13 i) Let $W$ be a BM and define the process $X_t = W_t - tW_1$, $t \in [0,1]$. Determine the mean and covariance functions of $X$.

ii) The process $X$ of part (i) is called the (standard) Brownian bridge on $[0,1]$, and so is every other continuous Gaussian process indexed by the interval $[0,1]$ that has the same mean and covariance function. Show that the processes $Y$ and $Z$ defined by $Y_t = (1-t)W_{t/(1-t)}$, $t \in [0,1]$, and $Y_1 = 0$, $Z_0 = 0$, $Z_t = tW_{1/(1-t)}$, $t \in (0,1]$ are standard Brownian bridges.

Exercise 1.14 Let $H \in (0,1)$ be given. A continuous, zero-mean Gaussian process $X$ with covariance function $2E X_s X_t = (t^{2H} + s^{2H} - |t-s|^{2H})$ is called a fractional Brownian motion (fBM) with Hurst index $H$. Show that the fBM with Hurst index $1/2$ is simply the BM. Show that if $X$ is a fBM with Hurst index $H$, then for all $a > 0$ the process $a^{-H}X_{at}$ is a fBM with Hurst index $H$ as well.

Exercise 1.15 Let $W$ be a Brownian motion and fix $t > 0$. For $n \in \mathbb{Z}_+$, let $\pi_n$ be a partition of $[0,t]$ given by $0 = t^n_0 < t^n_1 < \cdots < t^n_k = t$ and suppose that the mesh $|\pi_n| = \max_k |t^n_k - t^n_{k-1}|$ tends to zero as $n \to \infty$. Show that

$$\sum_k (W_{t^n_k} - W_{t^n_{k-1}})^2 \xrightarrow{L^2} t,$$

as $n \to \infty$. Hint: show that the expectation of the sum tends to $t$ and the variance to $0$.

Exercise 1.16 Show that if $(\mathcal{F}_t)$ is a filtration, then $(\mathcal{F}_{t+})$ is a filtration as well.

Exercise 1.17 Prove that the collection $\mathcal{F}_\tau$ associated with a stopping time $\tau$ is a $\sigma$-algebra.

Exercise 1.18 Show that if $\sigma, \tau$ are stopping times with $\sigma \leq \tau$, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

Exercise 1.19 Let $\sigma$ and $\tau$ be two $(\mathcal{F}_t)$-stopping times. Show that $\{\sigma \leq \tau\} \subset \mathcal{F}_\tau \cap \mathcal{F}_\sigma$.

Exercise 1.20 If $\sigma$ and $\tau$ are stopping times w.r.t. the filtration $(\mathcal{F}_t)$, show that $\sigma \land \tau$ and $\sigma \lor \tau$ are stopping times as well. Determine the associated $\sigma$-algebras. Hint: show that $A \in \mathcal{F}_{\sigma \lor \tau}$ implies $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_\tau$.

Exercise 1.21 If $\sigma$ and $\tau$ are stopping times w.r.t. the filtration $(\mathcal{F}_t)$, show that $\sigma + \tau$ is a stopping time as well. Hint: for $t > 0$ write

$$\{\sigma + \tau > t\} = \{\tau = 0, \sigma > t\} \cup \{0 < \tau < t, \sigma + \tau > t\} \cup \{\tau > t, \sigma = 0\} \cup \{\tau \geq t, \sigma > 0\}.$$

Only for the second event on the right-hand side it is non-trivial to prove that it belongs to $\mathcal{F}_t$. Now observe that if $\tau > 0$, then $\sigma + \tau > t$ if and only if there exists a positive $q \in \mathbb{Q}$, such that $q < \tau$ and $\sigma + q > t$. 

1.7. EXERCISES

Exercise 1.22 Show that if $\sigma$ and $\tau$ are stopping times w.r.t. the filtration $(\mathcal{F}_t)$ and $X$ is an integrable random variable, then $1_{\{\tau=\sigma\}}E(X \mid \mathcal{F}_\tau) = 1_{\{\tau=\sigma\}}E(X \mid \mathcal{F}_\sigma)$. Hint: show that $1_{\{\tau=\sigma\}}E(X \mid \mathcal{F}_\tau) = 1_{\{\tau=\sigma\}}E(X \mid \mathcal{F}_\tau \cap \mathcal{F}_\sigma)$.

Exercise 1.23 Show that if the filtration $(\mathcal{F}_t)$ is right-continuous, then $\tau$ is an $(\mathcal{F}_t)$-stopping time if and only if $\{\tau < t\} \in \mathcal{F}_t$ for all $t \in T$.

Exercise 1.24 Prove Lemma 1.6.6.

Exercise 1.25 Show that the map $\omega \rightarrow (\tau(\omega) \wedge t, \omega)$ in the proof of Lemma 1.6.14 is measurable as a map from $(\Omega, \mathcal{F}_t)$ to $([0, t] \times \Omega, \mathcal{B}([0, t]) \times \mathcal{F}_t)$.

Exercise 1.26 Show that $\tau_n$ in the proof of Lemma 1.6.16 are indeed stopping times and that they converge to $\tau$.

Exercise 1.27 Translate the definitions of §1.6 to the special case that time is discrete, i.e. $T = \mathbb{Z}_+$.

Exercise 1.28 Let $W$ be a BM and let $Z = \{t \geq 0 \mid W_t = 0\}$ be its zero set. Show that with probability 1 the set $Z$ has Lebesgue measure 0, is closed and unbounded.

Exercise 1.29 We define the last exit time of $x$:

$$L_x = \sup\{t > 0 : W_t = x\},$$

where $\sup\{0\} = 0$.

i) Show that $\tau_0$ is measurable ($\tau_0$ is defined in Lemma 1.6.8).

ii) Show that $L_x$ is measurable for all $x$. Derive first that $\{L_x < t\} = \cap_{n \geq t}\{|W_n - x| > 0, t \leq s \leq n\}$.

iii) Show that $L_x = \infty$ a.s. for all $x$, by considering the set $\{\sup_{t \geq 0} W_t = \infty, \inf_{t \geq 0} W_t = -\infty\}$ as in the proof of Example 1.6.9.

iv) Show that for almost all $\omega \in \Omega$ there exists a decreasing sequence $\{t_n(\omega)\}_n$, $\lim_n t_n(\omega) = 0$, such that $W(t_n(\omega)) = 0$ for all $n$. Hint: time-inversion + (iii). Hence $t = 0$ is a.s. an accumulation point of zeroes of $W$ and so $\tau_0 = 0$ a.s.

Exercise 1.30 Consider Brownian motion $W$ with continuous paths, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $Z(\omega) = \{t \geq 0, W_t(\omega) = 0\}$ be its zero set. In Problem 1.29 you have been asked to show that $t = 0$ is an accumulation point of $Z(\omega)$ for almost all $\omega \in \Omega$.

Let $\lambda$ denote the Lebesgue measure on $[0, \infty)$. Show (by interchanging the order of integration) that

$$\int_{\Omega} \lambda(Z(\omega))d\mathbb{P}(\omega) = 0,$$

and argue from this that $Z$ a.s. has Lebesgue measure 0, i.e. $\lambda(Z(\omega)) = 0$ for a.a. $\omega \in \Omega$.

Exercise 1.31 Let $W$ be a BM with respect to its natural filtration $(\mathcal{F}^W_t)$. Define for $a > 0$

$$S_a = \inf\{t \geq 0 : W_t > a\}.$$
CHAPTER 1. STOCHASTIC PROCESSES

i) Is $S_a$ an optional time? Justify your answer.

Let now $\sigma_a = \inf\{t \geq 0 : W_t = a\}$ be the first entrance time of $a$ and let

$$M_a = \sup\{t \geq 0 : W_t = at\},$$

be the last time that $W_t$ equals $at$.

ii) Is $M_a$ a stopping time? Justify your answer. Show that $M_a < \infty$ with probability 1 (you could use time-inversion for BM).

iii) Show that $M_a$ has the same distribution as $1/\sigma_a$.

**Exercise 1.32** Let $X = (X_t)_{t \geq 0}$ be a Gaussian, zero-mean stochastic process starting from 0, i.e. $X_0 = 0$. Moreover, assume that the process has stationary increments, meaning that for all $t_1 \geq s_1, t_2 \geq s_2, \ldots, t_n \geq s_n$, the distribution of the vector $(X_{t_1} - X_{s_1}, \ldots, X_{t_n} - X_{s_n})$ only depends on the time points through the differences $t_1 - s_1, \ldots, t_n - s_n$.

a) Show that for all $s, t \geq 0$

$$\mathbb{E} X_s X_t = \frac{1}{2} (v(s) + v(t) - v(|t - s|)),$$

where the function $v$ is given by $v(t) = \mathbb{E} X_t^2$.

In addition to stationarity of the increments we now assume that $X$ is $H$-self similar for some parameter $H > 0$. Recall that this means that for every $a > 0$, the process $(X_{at})_t$ has the same finite dimensional distributions as $(a^H X_t)_t$.

b) Show that the variance function $v(t) = \mathbb{E} X_t^2$ must be of the form $v(t) = C t^{2H}$ for some constant $C \geq 0$.

In view of the (a,b) we now assume that $X$ is a zero-mean Gaussian process with covariance function

$$\mathbb{E} X_s X_t = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}),$$

or some $H > 0$.

c) Show that we must have $H \leq 1$. (Hint: you may use that by Cauchy-Schwartz, the (semi-)metric $d(s, t) = \sqrt{\mathbb{E} (X_s - X_t)^2}$ on $[0, \infty)$ satisfies the triangle inequality.)

d) Show that for $H = 1$, we have $X_t = tZ$ a.s., for a standard normal random variable $Z$ not depending on $t$.

e) Show that for every value of the parameter $H \in (0, 1]$, the process $X$ has a continuous modification.