Concurrent lines on Del Pezzo surfaces of degree one

Ronald van Luijk
Rosa Winter

May 29, 2015
Rio de Janeiro

Cubic surfaces

Let $X \subset \mathbb{P}^3$ be a smooth cubic surface over a field $k = \overline{k}$.

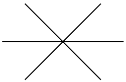
Theorem (Cayley-Salmon).
The surface X contains exactly 27 lines.
Each point lies on at most 3 lines.

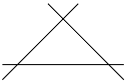
Cubic surfaces

Let $X \subset \mathbb{P}^3$ be a smooth cubic surface over a field $k = \overline{k}$.

Theorem (Cayley-Salmon).
The surface X contains exactly 27 lines.
Each point lies on at most 3 lines.

Proof of second part.
The lines through a point P lie in the tangent plane H,
so in the cubic curve $X \cap H$.



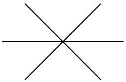


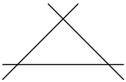
Cubic surfaces

Let $X \subset \mathbb{P}^3$ be a smooth cubic surface over a field $k = \overline{k}$.

Theorem (Cayley-Salmon).
The surface X contains exactly 27 lines.
Each point lies on at most 3 lines.

Proof of second part.
The lines through a point P lie in the tangent plane H,
so in the cubic curve $X \cap H$.





Definition.
An Eckardt point is a point that lies on three lines.

Cubic surfaces

Example.
The Fermat surface $w^3 + x^3 + y^3 + z^3 = 0$ contains the 9 lines

$$w^3 + x^3 = y^3 + z^3 = 0.$$

Three of these go through $[1 : -1 : 0 : 0]$.
In total: 27 lines and 18 Eckardt points.

Cubic surfaces

Example.
The Fermat surface $w^3 + x^3 + y^3 + z^3 = 0$ contains the 9 lines

$$w^3 + x^3 = y^3 + z^3 = 0.$$

Three of these go through $[1 : -1 : 0 : 0]$.
In total: 27 lines and 18 Eckardt points.

Fact. There are 45 tritangent planes.

Fact (Hirschfeld '67).
At most 45 Eckardt points (sharp in characteristic 2).
At most 18 Eckardt points in characteristic not equal to 2.

Cubic surfaces (another point of view)

Fact. The cubic surface $X \subset \mathbb{P}^3$ is isomorphic to the blow up of $\mathbb{P}^2$ in 6 points in general position: no 3 on a line, no 6 on a conic.

Cubic surfaces (another point of view)

Fact. The cubic surface $X \subset \mathbb{P}^3$ is isomorphic to the blow up of $\mathbb{P}^2$ in 6 points in general position: no 3 on a line, no 6 on a conic.

Fact. The canonical divisor K_X is linearly equivalent with $(-n-1+d)H = (-3-1+3)H = -H$ for any hyperplane section H.

Fact (Riemann-Roch and adjunction). The lines on X correspond with the classes E in $\operatorname{Pic} X$ with $E^2 = -1$ and $H \cdot E = 1$.

Cubic surfaces (another point of view)

Fact. The cubic surface $X \subset \mathbb{P}^3$ is isomorphic to the blow up of $\mathbb{P}^2$ in 6 points in general position: no 3 on a line, no 6 on a conic.

Fact. The canonical divisor K_X is linearly equivalent with $(-n-1+d)H = (-3-1+3)H = -H$ for any hyperplane section H.

Fact (Riemann-Roch and adjunction). The lines on X correspond with the classes E in $\operatorname{Pic} X$ with $E^2 = -1$ and $H \cdot E = 1$.

Fact. $\operatorname{Pic} X$ is the orthogonal direct sum $\mathbb{Z}L \oplus \bigoplus_{i=1}^{6} \mathbb{Z}E_i$, where $\pi\colon X \to \mathbb{P}^2$ is the blow up, L is the pull back of a line, and the E_i are the exceptional curves. Moreover, $L^2 = 1$, $E_i^2 = -1$, $L \cdot E_i = 0$, $E_i \cdot E_j = 0$ $(i \neq j)$.

Corollary. We can describe all lines in terms of $\operatorname{Pic} X$.

Cubic surfaces (another point of view)

Corollary. The classes $E \in \operatorname{Pic} X$ with $E^2 = -1$ and $-K_X \cdot E = 1$:

E_i for $1 \leq i \leq 6$,

$L - E_i - E_j$ for $1 \leq i < j \leq 6$,

$2L + E_i - \sum E_j$ for $1 \leq i \leq 6$,

corresponding to the 6 exceptional curves and strict transforms of

the 15 lines through two points and

the 6 conics through five of the six points.

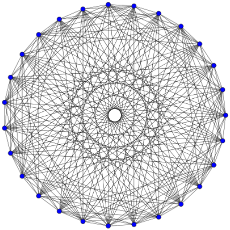
Cubic surfaces (another point of view)

Corollary. The classes $E \in \operatorname{Pic} X$ with $E^2 = -1$ and $-K_X \cdot E = 1$:

- E_i for $1 \leq i \leq 6$,
- $L - E_i - E_j$ for $1 \leq i < j \leq 6$,
- $2L + E_i - \sum E_j$ for $1 \leq i \leq 6$,

corresponding to the 6 exceptional curves and strict transforms of
- the 15 lines through two points and
- the 6 conics through five of the six points.

Corollary. The incidence graph on the 27 lines is independent of X. It is the complement of the Schläfli graph.



Schläfli graph. Each line is connected to 10 other lines, partitioned in five disjoint pairs of intersecting lines.

Cubic surfaces (another point of view)

Corollary. The classes $E \in \operatorname{Pic} X$ with $E^2 = -1$ and $-K_X \cdot E = 1$:

E_i for $1 \leq i \leq 6$,

$L - E_i - E_j$ for $1 \leq i < j \leq 6$,

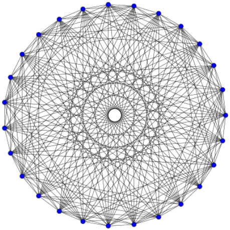
$2L + E_i - \sum E_j$ for $1 \leq i \leq 6$,

corresponding to the 6 exceptional curves and strict transforms of

the 15 lines through two points and

the 6 conics through five of the six points.

Corollary. The incidence graph on the 27 lines is independent of X. It is the complement of the Schläfli graph.



Schläfli graph. Each line is connected to 10 other lines, partitioned in five disjoint pairs of intersecting lines.

Corollary. (Concurrent lines form a complete subgraph, so) each point on at most 3 lines and at most $\frac{27 \cdot 5}{3} = 45$ Eckardt points.

Del Pezzo surfaces

A del Pezzo surface over a field k is a geometrically integral, smooth, projective surface S over k for which there exists an embedding $i\colon S \hookrightarrow \mathbb{P}^n$ and a positive integer a such that the multiple $-aK_S$ of a canonical divisor K_S is linearly equivalent to a hyperplane section. Its degree is $\deg S = K_S^2$.

Examples.

- A smooth cubic surface in $\mathbb{P}^3$, with $a = 1$ and degree 3.
- A double cover of $\mathbb{P}^2$ ramified over a smooth quartic curve, with $a = 2$ (w.r.t. an embedding in $\mathbb{P}^6$) and degree 2.

Fact.
Over an *algebraically closed* field, every del Pezzo surface is isomorphic to

- $\mathbb{P}^1 \times \mathbb{P}^1$, with degree 8, or
- $\mathbb{P}^2$ blown up at $r \leq 8$ points in general position (**!**), with degree $9 - r$ (cubic: $r = 6$).

Del Pezzo surfaces

General position:

- no 3 points on a line,
- no 6 points on a conic,
- no 8 points on a singular cubic with singularity at one of 8.

Del Pezzo surfaces

General position:

- no 3 points on a line,
- no 6 points on a conic,
- no 8 points on a singular cubic with singularity at one of 8.

The (-1)-curves on the blow-up:

- the exceptional curves, and strict transforms of...
- lines through two (of the) points,
- conics through five points,
- cubics through seven points, singular at one (of them),
- quartics through eight points, singular at three,
- quintics through eight points, singular at six,
- sextics through eight singular points, triple point at one.

r	0	1	2	3	4	5	6	7	8
#	0	1	3	6	10	16	27	56	240

Del Pezzo surfaces of degree two

These are double covers of $\mathbb{P}^2$, ramified over smooth quartic curve.
The 28 bitangents pull back to 56 "lines", that is, (-1)-curves.
Each line e intersects its partner e' with multiplicity 2.
Each other line intersects exactly one of e and e' with multiplicity 1.

Del Pezzo surfaces of degree two

These are double covers of $\mathbb{P}^2$, ramified over smooth quartic curve.
The 28 bitangents pull back to 56 "lines", that is, (-1)-curves.
Each line e intersects its partner e' with multiplicity 2.
Each other line intersects exactly one of e and e' with multiplicity 1.

Corollary. Each line intersects 27 lines with multiplicity 1.

Fact. The subgraph on these 27 lines is the graph for cubics **!**
Reason. True for 27 lines intersected with multiplicity 0 and there is an automorphism that sends each line to its partner.

Del Pezzo surfaces of degree two

These are double covers of $\mathbb{P}^2$, ramified over smooth quartic curve.
The 28 bitangents pull back to 56 "lines", that is, (-1)-curves.
Each line e intersects its partner e' with multiplicity 2.
Each other line intersects exactly one of e and e' with multiplicity 1.

Corollary. Each line intersects 27 lines with multiplicity 1.

Fact. The subgraph on these 27 lines is the graph for cubics **!**
Reason. True for 27 lines intersected with multiplicity 0 and there is an automorphism that sends each line to its partner.

Corollary. Each point of a del Pezzo surface X of degree 2 lies on at most 4 lines (generalised Eckardt point: not on branch curve).
There are at most $56 \cdot \frac{27}{4-1} \cdot \frac{1}{4} = 126$ generalised Eckardt points.
This upper bound is sharp: $w^2 = x^4 + y^4 + z^4$ over $\mathbb{F}_9$.

Question. What about characteristics other than 3?

Del Pezzo surfaces: automorphism groups (Manin)

Let X be the blow-up of $\mathbb{P}^2$ in $6 \leq r \leq 8$ points in general position.

Let $\mathcal{E}$ be the set of classes corresponding to lines. Then $\mathcal{E}$ lies in the hyperplane in the lattice $\mathrm{Pic}\, X$ given by $-K_X \cdot e = 1$.

Del Pezzo surfaces: automorphism groups (Manin)

Let X be the blow-up of $\mathbb{P}^2$ in $6 \leq r \leq 8$ points in general position.

Let $\mathcal{E}$ be the set of classes corresponding to lines. Then $\mathcal{E}$ lies in the hyperplane in the lattice $\mathrm{Pic}\, X$ given by $-K_X \cdot e = 1$.

$$\begin{array}{ccc} \mathrm{Aut}\,\mathrm{Pic}\, X \supset \{\sigma \;:\; \sigma K_X = K_X\} \xrightarrow{\cong} \mathrm{Aut}\, K_X^{\perp} \xrightarrow{\cong} W(E_r) \\ \downarrow\cong \\ \mathrm{Sym}(\mathcal{E}) \supset \{\sigma \;:\; \sigma(e)\cdot\sigma(e') = e \cdot e' \text{ for all } e, e' \in \mathcal{E}\} =: G_r \end{array}$$

where E_6, E_7, E_8 are the classical root lattices (of x with $x^2 = -2$).

$$\begin{aligned} \# G_6 &= 2^7 \cdot 3^4 \cdot 5 = 51840, \\ \# G_7 &= 2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2903040, \\ \# G_8 &= 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696729600. \end{aligned}$$

Fact. The group G_r acts transitively on $\mathcal{E}$.

Del Pezzo surfaces of degree one

Every del Pezzo surface X of degree one is the double cover of a cone $C \subset \mathbb{P}^3$, ramified over a curve B of degree 6: the intersection of C with a surface of degree 3.

Del Pezzo surfaces of degree one

Every del Pezzo surface X of degree one is the double cover of a cone $C \subset \mathbb{P}^3$, ramified over a curve B of degree 6: the intersection of C with a surface of degree 3.

Fact. There are 120 planes that are tritangent to B and do not contain the vertex of the cone. They pull back to 240 lines on X. Each intersects its partner with multiplicity 3.

Del Pezzo surfaces of degree one

Every del Pezzo surface X of degree one is the double cover of a cone $C \subset \mathbb{P}^3$, ramified over a curve B of degree 6: the intersection of C with a surface of degree 3.

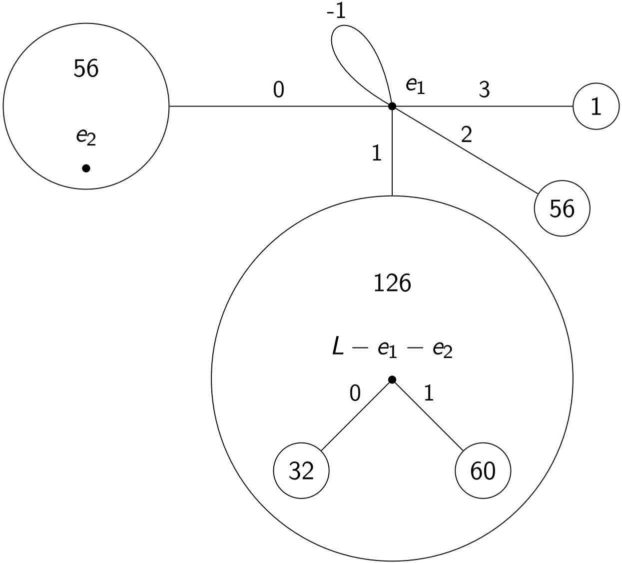
Fact. There are 120 planes that are tritangent to B and do not contain the vertex of the cone. They pull back to 240 lines on X. Each intersects its partner with multiplicity 3.

Fact. If two partnered lines go through a point P, then P lies on the ramification curve.

Fact. If P lies on the ramification curve, then the partner of any line through P also goes through P.

Fact. If e and e' are partners, then $e \cdot f = 2 - e' \cdot f$ for all lines f. ($e + e' \sim -2K_X$)

Del Pezzo surfaces of degree one



Del Pezzo surfaces of degree one

Fact (Manin). The group $G = G_8$ acts transitively on

$$U = \{ (e_1, e_2, \ldots, e_8) \in \mathcal{E}^8 \ : \ e_i \cdot e_j = 0 \text{ for } i \neq j \}.$$

Fact. For every $u = (e_1, \ldots, e_8) \in U$, we can blow down $e_1, \ldots, e_8$ and there is a unique $\ell \in \operatorname{Pic} X$ such that $-K_X = 3\ell - \sum_i e_i$. Since G acts faithfully on $\operatorname{Pic} X$, it acts freely on U, so $|U| = |G|$.

Del Pezzo surfaces of degree one

Fact (Manin). The group $G = G_8$ acts transitively on

$$U = \{\,(e_1, e_2, \ldots, e_8) \in \mathcal{E}^8 \;:\; e_i \cdot e_j = 0 \text{ for } i \neq j\,\}.$$

Fact. For every $u = (e_1, \ldots, e_8) \in U$, we can blow down $e_1, \ldots, e_8$ and there is a unique $\ell \in \operatorname{Pic} X$ such that $-K_X = 3\ell - \sum_i e_i$. Since G acts faithfully on $\operatorname{Pic} X$, it acts freely on U, so $|U| = |G|$.

This action allows us to find the maximal complete subgraphs inside the graph on all 240 lines.

Lemma. The group G acts transitively on the sets

$$V_1 = \{ (e_1, e_2) \ : \ e_1 \cdot e_2 = 0 \}$$

$$V_2 = \{ (e_0, e_1, e_2) \ : \ e_0 \cdot e_1 = e_0 \cdot e_2 = 1 \text{ and } e_1 \cdot e_2 = 0 \}$$

$$V_3 = \{ (e_0, e_1) \ : \ e_0 \cdot e_1 = 1 \}.$$

Lemma. The group G acts transitively on the sets

$$V_1 = \{ (e_1, e_2) \ : \ e_1 \cdot e_2 = 0 \}$$
$$V_2 = \{ (e_0, e_1, e_2) \ : \ e_0 \cdot e_1 = e_0 \cdot e_2 = 1 \text{ and } e_1 \cdot e_2 = 0 \}$$
$$V_3 = \{ (e_0, e_1) \ : \ e_0 \cdot e_1 = 1 \}.$$

Proof. For V_1: the stabiliser G_{e_1} is isomorphic to G_7 and the subgraph on $\{e : e \cdot e_1 = 0\}$ corresponds with $r = 7$. We have $\# V_1 = 240 \cdot 56$.

Lemma. The group G acts transitively on the sets

$$V_1 = \{ (e_1, e_2) \ : \ e_1 \cdot e_2 = 0 \}$$
$$V_2 = \{ (e_0, e_1, e_2) \ : \ e_0 \cdot e_1 = e_0 \cdot e_2 = 1 \text{ and } e_1 \cdot e_2 = 0 \}$$
$$V_3 = \{ (e_0, e_1) \ : \ e_0 \cdot e_1 = 1 \}.$$

Proof. For V_1: the stabiliser G_{e_1} is isomorphic to G_7 and the subgraph on $\{e : e \cdot e_1 = 0\}$ corresponds with $r = 7$. We have $\# V_1 = 240 \cdot 56$.
One fiber of the map $V_2 \to V_1$ has size 72, so all fibers do, so $\# V_2 = 240 \cdot 56 \cdot 72$. For one specific $v = (e_0, e_1, e_2) \in V_2$, the set

$$W_v = \{e : e \cdot e_0 = e \cdot e_1 = e \cdot e_2 = 0\}$$

has 6 elements, and the stabiliser G_v injects into $\mathrm{Sym}(W_v) \cong S_6$, so $\# G_v \leq 720$. Hence, the orbit has size

$$\# Gv = \frac{\# G}{\# G_v} \geq \frac{\# G}{720} = \# V_2,$$

so we have equality, so the action is transitive.

Lemma. The group G acts transitively on the sets

$$V_1 = \{ (e_1, e_2) \ : \ e_1 \cdot e_2 = 0 \}$$
$$V_2 = \{ (e_0, e_1, e_2) \ : \ e_0 \cdot e_1 = e_0 \cdot e_2 = 1 \text{ and } e_1 \cdot e_2 = 0 \}$$
$$V_3 = \{ (e_0, e_1) \ : \ e_0 \cdot e_1 = 1 \}.$$

Proof.
The G-action on the image of the projection $\rho\colon V_2 \to V_3$ is transitive. One fiber has size 32, so all non-empty fibers do. Hence,

$$\# \operatorname{im} \rho = \frac{\# V_2}{32} = \frac{240 \cdot 56 \cdot 72}{32} = 240 \cdot 126 = \# V_3,$$

so ρ is surjective and G acts transitively on V_3.

Lemma. The group G acts transitively on the sets

$$V_1 = \{ (e_1, e_2) \ : \ e_1 \cdot e_2 = 0 \}$$
$$V_2 = \{ (e_0, e_1, e_2) \ : \ e_0 \cdot e_1 = e_0 \cdot e_2 = 1 \text{ and } e_1 \cdot e_2 = 0 \}$$
$$V_3 = \{ (e_0, e_1) \ : \ e_0 \cdot e_1 = 1 \}.$$

Corollary. For any $v = (e_0, e_1, e_2) \in V_2$, there is a blow-down $X \to \mathbb{P}^2$ such that e_1, e_2 are exceptional curves above two of the eight points blown up, and e_0 is the strict transform of the line in $\mathbb{P}^2$ through these two points.

Lemma. The group G acts transitively on the sets

$$V_1 = \{ (e_1, e_2) \ : \ e_1 \cdot e_2 = 0 \}$$
$$V_2 = \{ (e_0, e_1, e_2) \ : \ e_0 \cdot e_1 = e_0 \cdot e_2 = 1 \text{ and } e_1 \cdot e_2 = 0 \}$$
$$V_3 = \{ (e_0, e_1) \ : \ e_0 \cdot e_1 = 1 \}.$$

Corollary. For any $v = (e_0, e_1, e_2) \in V_2$, there is a blow-down $X \to \mathbb{P}^2$ such that e_1, e_2 are exceptional curves above two of the eight points blown up, and e_0 is the strict transform of the line in $\mathbb{P}^2$ through these two points.

Proof. Let $E_1, \dots, E_8$ be the exceptional curves on X above the eight points $P_1, \dots, P_8 \in \mathbb{P}^2$ that were blown up to *define* X. Let E_0 be the strict transform of the line through P_1 and P_2. Let $g \in G \subset \mathrm{Sym}(\mathcal{E})$ be an element that sends

$$(E_0, E_1, E_2) \quad \text{to} \quad (e_0, e_1, e_2).$$

Then g sends $E_1, E_2, \dots, E_8$ to elements $e_1, e_2, \dots, e_8$ that we can blow down.

Del Pezzo surfaces of degree one (example over $\mathbb{F}_{32}$)

The maximal size of a complete subgraph is 16.

There is an example with 16 concurrent lines!

Del Pezzo surfaces of degree one (example over $\mathbb{F}_{32}$)

The maximal size of a complete subgraph is 16.
There is an example with 16 concurrent lines!
Set $F = \mathbb{F}_2[\alpha] = \mathbb{F}_2[x]/(x^5 + x^2 + 1)$ and $P = [0 : 0 : 1] \in \mathbb{P}^2_F$ and:

$$Q_1 = (0 : 1 : 1), \qquad Q_5 = (1 : 1 : 1),$$
$$Q_2 = (0 : 1 : \alpha^{19}), \qquad Q_6 = (\alpha^4 : \alpha^4 : 1),$$
$$Q_3 = (1 : 0 : 1), \qquad Q_7 = (\alpha^{24} : \alpha^{25} : 1),$$
$$Q_4 = (1 : 0 : \alpha^5), \qquad Q_8 = (\alpha^{25} : \alpha^{26} : 1).$$

Then these curves go through P (with $1 \leq i \leq 4$):

- the four lines through Q_{2i-1} and Q_{2i},
- the four cubics through all Q_j with $j \neq 2i - 1$, singular at Q_{2i},
- the four cubics through all Q_j with $j \neq 2i$, singular at Q_{2i-1},
- the four quintics through all Q_j, singular when $j \neq 2i, 2i - 1$.

Only need to check for 9 as these form eight partnered pairs.

Del Pezzo surfaces of degree one

Question. What about characteristic 0 ?

Two cases: the point P on and off the ramification curve.

Del Pezzo surfaces of degree one

Question. What about characteristic 0 ?
Two cases: the point P on and off the ramification curve.

Theorem (Winter, vL).
In both cases the number of concurrent lines is at most 10.

Del Pezzo surfaces of degree one

Question. What about characteristic 0 ?
Two cases: the point P on and off the ramification curve.

Theorem (Winter, vL).
In both cases the number of concurrent lines is at most 10.

Facts (arguments similar to before, to minimise computation).

- Any 6 partnered pairs forming a complete subgraph are contained in a maximal clique of size 16, and G acts transitively on the sets of 6 such pairs.
- Any 11 lines without partners forming a complete subgraph is contained in a clique of size 12 without partners, and G acts transitively on the sets of 11 such lines in any such clique.

Del Pezzo surfaces of degree one

Question. What about characteristic 0 ?
Two cases: the point P on and off the ramification curve.

Theorem (Winter, vL).
In both cases the number of concurrent lines is at most 10.

Facts (arguments similar to before, to minimise computation).

- Any 6 partnered pairs forming a complete subgraph are contained in a maximal clique of size 16, and G acts transitively on the sets of 6 such pairs.
- Any 11 lines without partners forming a complete subgraph is contained in a clique of size 12 without partners, and G acts transitively on the sets of 11 such lines in any such clique.

Corollary. To show that no 6 such pairs (or 11 such lines) are concurrent, it suffices to pick any description in $\mathbb{P}^2$ of 6 such pairs (or of 11 such lines in a clique of 12 such lines).

Del Pezzo surfaces of degree one

Proposition (case P on the ramification curve)
Assume that char $k \neq 2$.
Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position.
Let L_i be the line through Q_{2i} and Q_{2i-1} for $i \in \{1, 2, 3, 4\}$,
and $C_{i,j}$ the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in Q_j.
Assume that the four lines $L_1,\ L_2,\ L_3$ and L_4 all intersect in one point P. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all go through P.

Del Pezzo surfaces of degree one

Proposition (case P on the ramification curve)
Assume that char $k \neq 2$.
Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position.
Let L_i be the line through Q_{2i} and Q_{2i-1} for $i \in \{1, 2, 3, 4\}$,
and $C_{i,j}$ the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in Q_j.
Assume that the four lines $L_1,\ L_2,\ L_3$ and L_4 all intersect in one point P. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all go through P.

Proof. Gröbner bases with a lot of manual help.

Del Pezzo surfaces of degree one

Proposition (case P on the ramification curve)
Assume that char $k \neq 2$.
Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position.
Let L_i be the line through Q_{2i} and Q_{2i-1} for $i \in \{1, 2, 3, 4\}$,
and $C_{i,j}$ the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in Q_j.
Assume that the four lines $L_1,\ L_2,\ L_3$ and L_4 all intersect in one point P. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all go through P.

Proof. Gröbner bases with a lot of manual help.

Corollary. No six partnered pairs through one point, so no point on ramification curve lies on more than 10 lines.

Del Pezzo surfaces of degree one

Proposition (case P on the ramification curve)
Assume that char $k \neq 2$.
Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position.
Let L_i be the line through Q_{2i} and Q_{2i-1} for $i \in \{1, 2, 3, 4\}$,
and $C_{i,j}$ the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in Q_j.
Assume that the four lines $L_1,\ L_2,\ L_3$ and L_4 all intersect in one point P. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all go through P.

Proof. Gröbner bases with a lot of manual help.

Corollary. No six partnered pairs through one point, so no point on ramification curve lies on more than 10 lines.

Fact. In each characteristic there is an example of a del Pezzo surface X with 10 concurrent lines.

Del Pezzo surfaces of degree one

Proposition (case P off the ramification curve)

Assume that char $k = 0$.

Let $Q_1, \dots, Q_8$ be eight points in $\mathbb{P}^2$ in general position. Set

L_1 is the line through Q_1 and Q_2,

L_2 is the line through Q_3 and Q_4,

C_1 is the conic through $Q_1,\ Q_3,\ Q_5,\ Q_6,$ and Q_7,

C_2 is the conic through $Q_1,\ Q_4,\ Q_5,\ Q_6,$ and Q_8,

C_3 is the conic through $Q_2,\ Q_3,\ Q_5,\ Q_7,$ and Q_8,

C_4 is the conic through $Q_2,\ Q_4,\ Q_6,\ Q_7,$ and Q_8,

D_1 is the quartic through all points, singular at $Q_1,\ Q_7,$ and Q_8

D_2 is the quartic through all points, singular at $Q_2,\ Q_5,$ and Q_6

D_3 is the quartic through all points, singular at $Q_3,\ Q_6,$ and Q_8

D_4 is the quartic through all points, singular at $Q_4,\ Q_5,$ and Q_7.

There is no point that lies on all these curves.

Corollary.
No point in X off the ramification curve lies on > 10 lines.

Sketch of proof of Corollary.

- These 10 curves are a subset of a set of 11 (and even 12) curves that form a complete subgraph without any partnered pairs.
- The group G acts transitively on the set of all sets of 11 such curves.
- If there are 11 concurrent lines on X, then (as before), there is a blow down $X \to \mathbb{P}^2$ such that 10 of these curves have images as described in the proposition.
- The proposition gives a contradiction.

Corollary.
No point in X off the ramification curve lies on > 10 lines.

Sketch of proof of Corollary.

- These 10 curves are a subset of a set of 11 (and even 12) curves that form a complete subgraph without any partnered pairs.
- The group G acts transitively on the set of all sets of 11 such curves.
- If there are 11 concurrent lines on X, then (as before), there is a blow down $X \to \mathbb{P}^2$ such that 10 of these curves have images as described in the proposition.
- The proposition gives a contradiction.

Sketch of proof of proposition.
Define

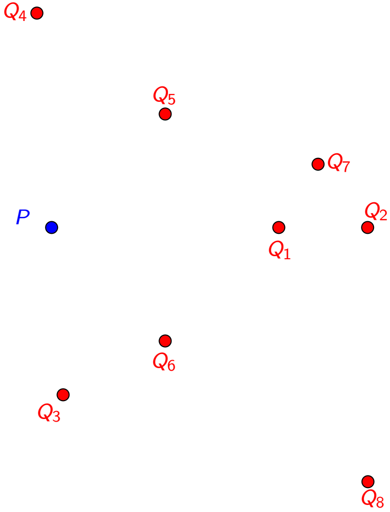
$$(\mathbb{P}^2)^9 \supset \Gamma = \{(P, Q_1, \ldots, Q_8) \ : \ Q_1, Q_2, \ldots, Q_8 \textbf{ not} \text{ in general pos'n}\},$$
$$(\mathbb{P}^2)^9 \supset \Delta = \{(P, Q_1, \ldots, Q_8) \ : \ \text{the curves in the prop'n contain } P\}.$$

We will show $\Delta \subset \Gamma$, or equivalently, $Z := \Delta \cap \left((\mathbb{P}^2)^9 \setminus \Gamma\right) = \emptyset$.

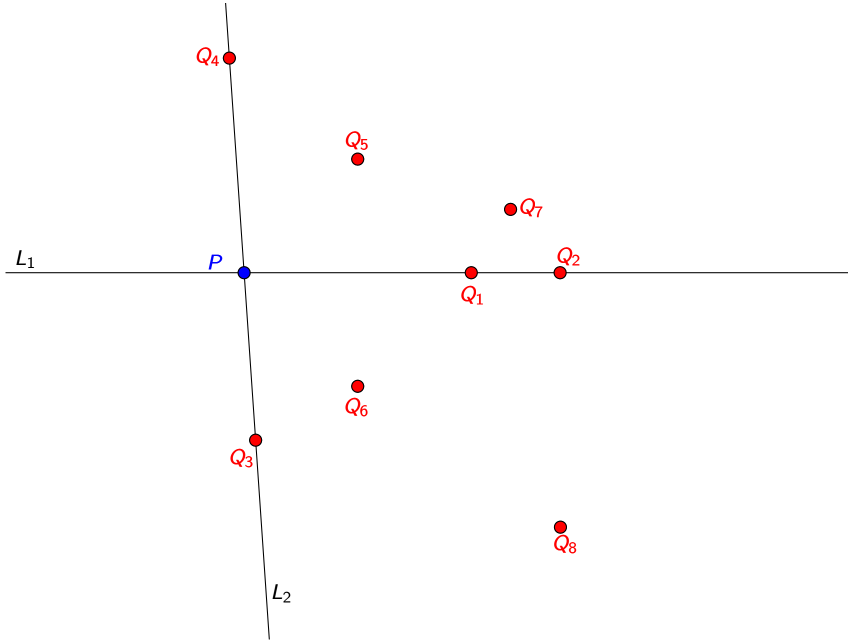
The group $\mathrm{PGL}_3(k)$ acts on everything. After showing that for $(P, Q_1, \ldots, Q_8) \in Z$, no three of P, Q_1, Q_5, Q_6 lie on a line, we may restrict to

$$(\mathbb{P}^2)^9 \supset \mathbb{P} = \left\{(P, Q_1, \ldots, Q_8) \ : \ \begin{matrix} P = [-1 : 0 : 1] & Q_1 = [1 : 0 : 1] \\ Q_5 = [0 : 1 : 1] & Q_6 = [0 : -1 : 1] \end{matrix} \right\}$$

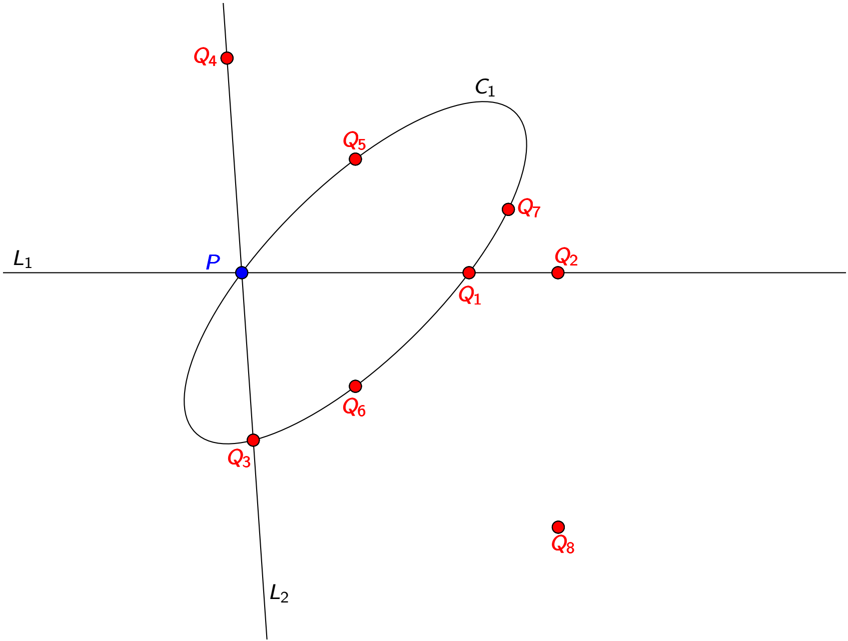
$$\Gamma' = \Gamma \cap \mathbb{P}$$
$$\Delta' = \Delta \cap \mathbb{P}$$
$$Z' = Z \cap \mathbb{P} = \Delta' \cap (\mathbb{P} \setminus \Gamma')$$



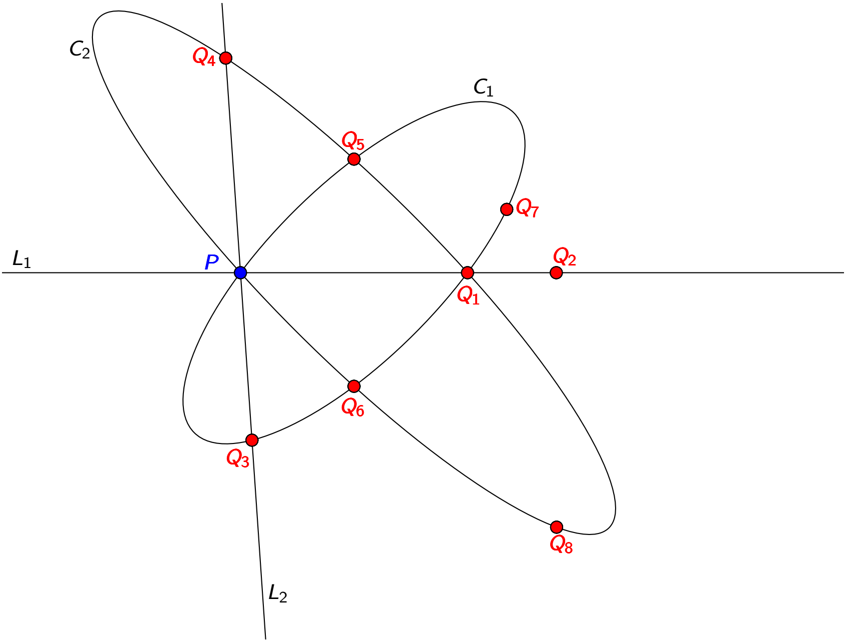
Q_4
Q_5
Q_7
P
Q_2
Q_1
Q_6
Q_3
Q_8

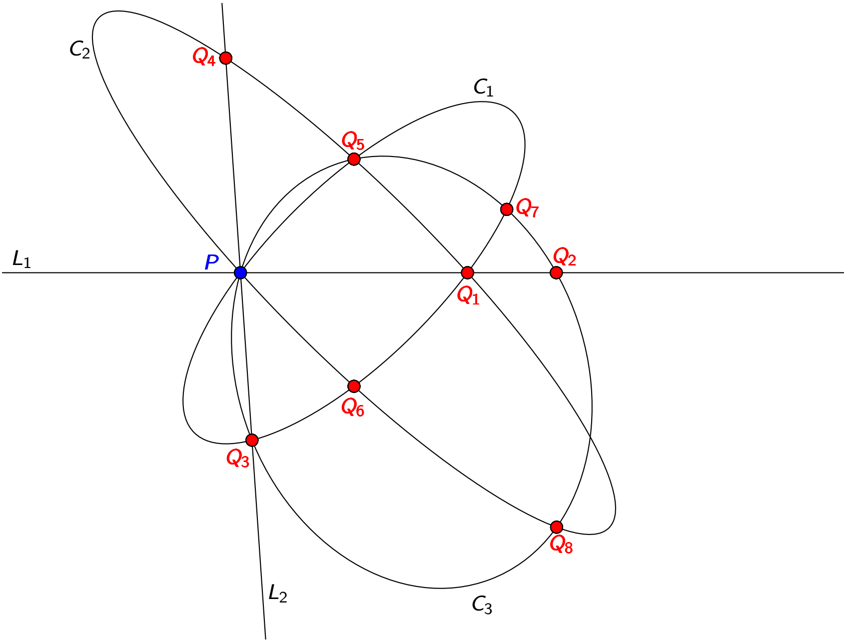


Q_4
Q_5
Q_7
L_1
P
Q_2
Q_1
Q_6
Q_3
Q_8
L_2

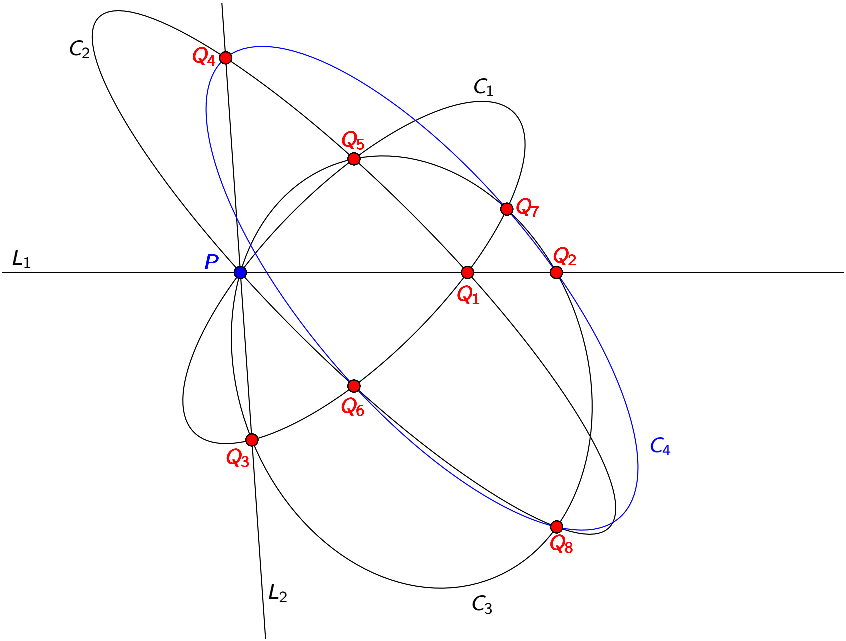


Q_4
C_1
Q_5
Q_7
L_1
P
Q_2
Q_1
Q_6
Q_3
Q_8
L_2

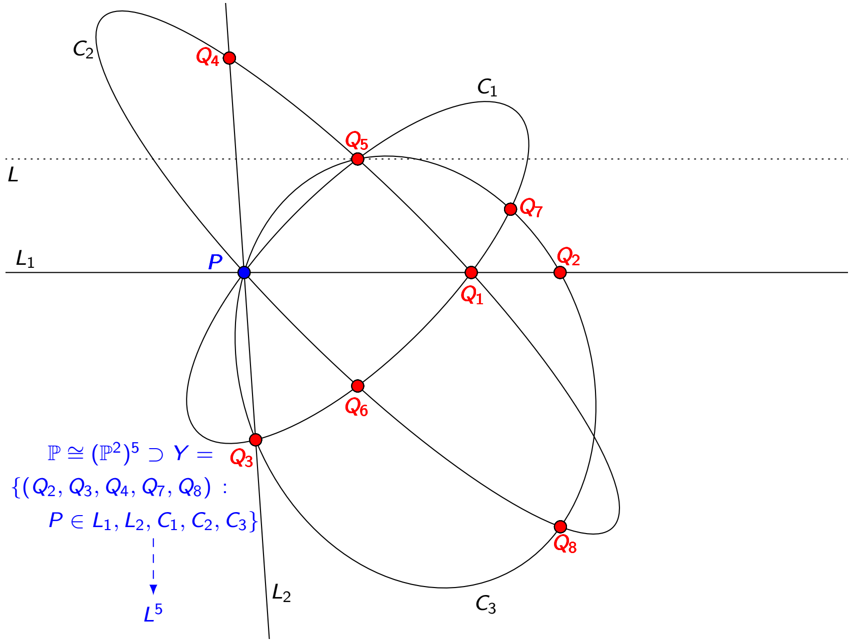

C2
Q4
C1
Q5
Q7
L1
P
Q2
Q1
Q6
Q3
Q8
L2

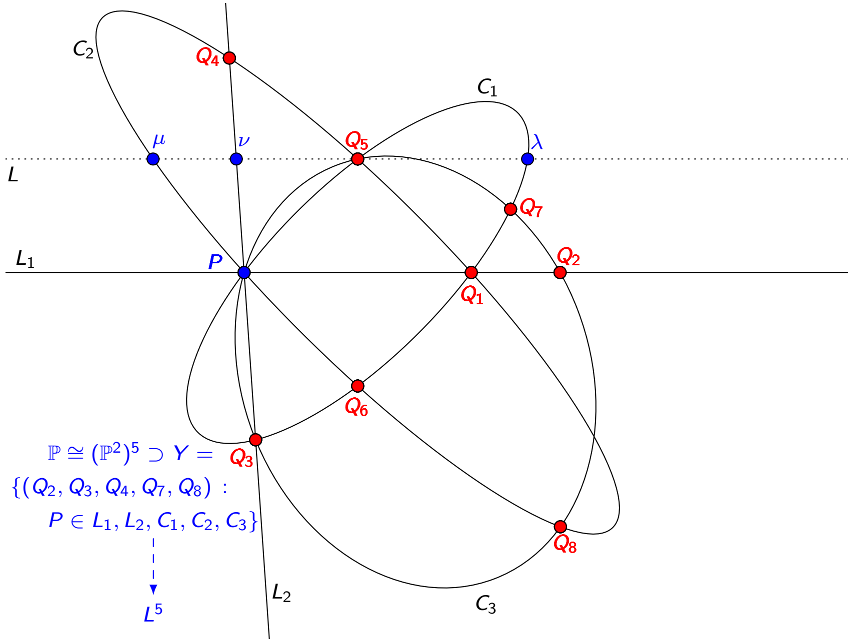


C_2
Q_4
C_1
Q_5
Q_7
L_1
P
Q_2
Q_1
Q_6
Q_3
Q_8
L_2
C_3

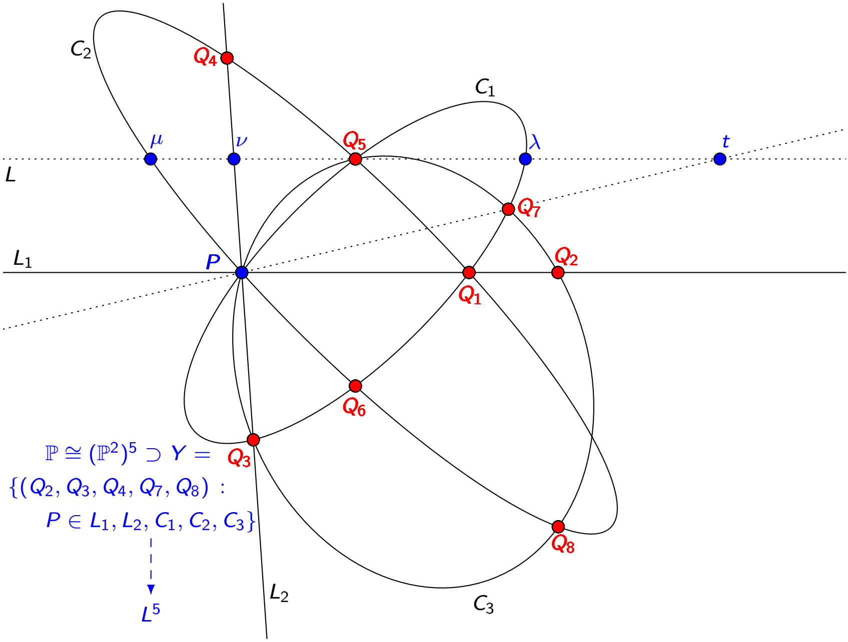


C_2
Q_4
C_1
Q_5
Q_7
L_1
P
Q_2
Q_1
Q_6
C_4
Q_3
Q_8
L_2
C_3

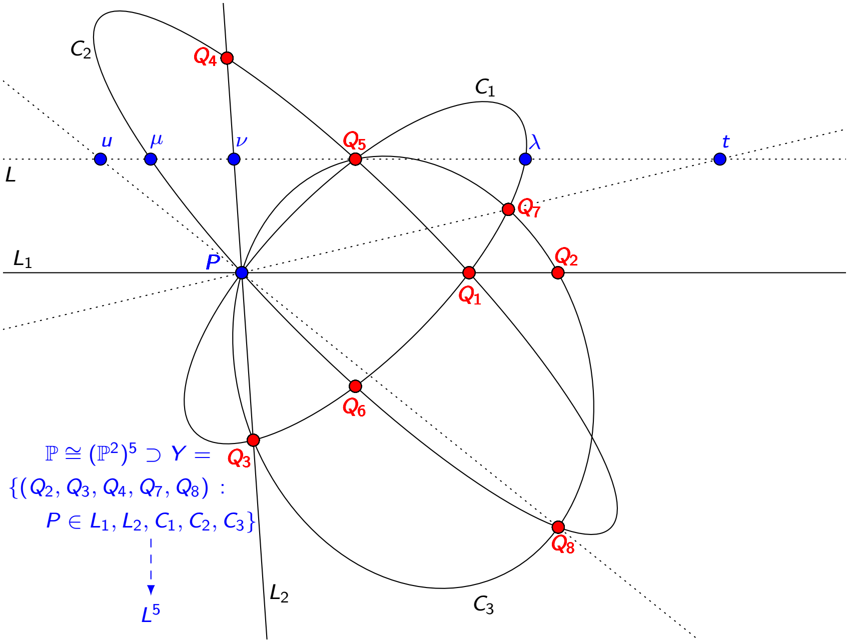

C_2
Q_4
C_1
Q_5
L
Q_7
L_1
P
Q_2
Q_1
Q_6
$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y =$
$\{(Q_2, Q_3, Q_4, Q_7, Q_8)$:
$P \in L_1, L_2, C_1, C_2, C_3\}$
Q_3
Q_8
L_2
C_3
L^5



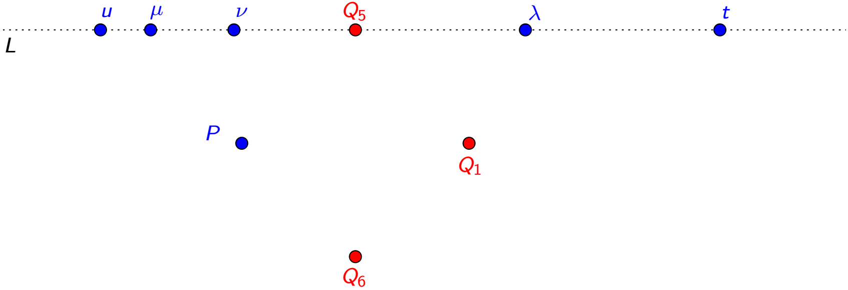
C_2
Q_4
C_1
μ
ν
Q_5
λ
L
Q_7
L_1
P
Q_2
Q_1
Q_6
$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y =$
$\{(Q_2, Q_3, Q_4, Q_7, Q_8) :$
$P \in L_1, L_2, C_1, C_2, C_3\}$
Q_3
Q_8
L_2
C_3
L^5



C_2
Q_4
C_1
μ
ν
Q_5
λ
t
L
Q_7
L_1
P
Q_2
Q_1
Q_6
$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y =$
$\{(Q_2, Q_3, Q_4, Q_7, Q_8)\ :$
$P \in L_1, L_2, C_1, C_2, C_3\}$
Q_3
Q_8
L_2
C_3
L^5



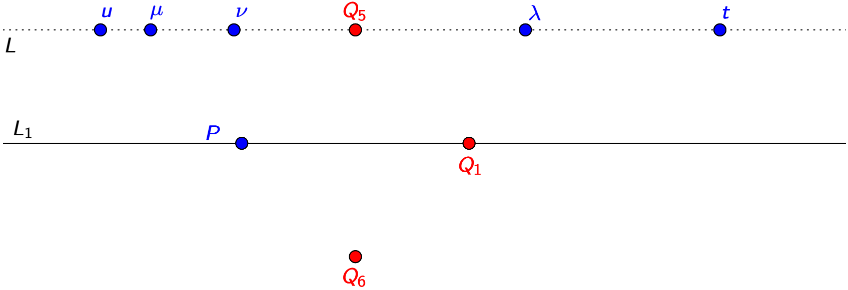
C_2
Q_4
C_1
u
μ
ν
Q_5
λ
t
L
Q_7
L_1
P
Q_2
Q_1
Q_6
$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y =$
$\{(Q_2, Q_3, Q_4, Q_7, Q_8)\ :$
$P \in L_1, L_2, C_1, C_2, C_3\}$
Q_3
Q_8
L_2
C_3
L^5



$$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3\}$$

$$\updownarrow$$

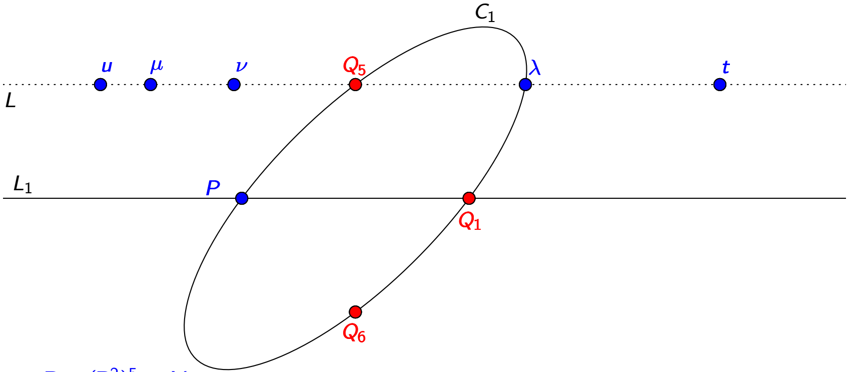
$$L^5$$



$$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3\}$$

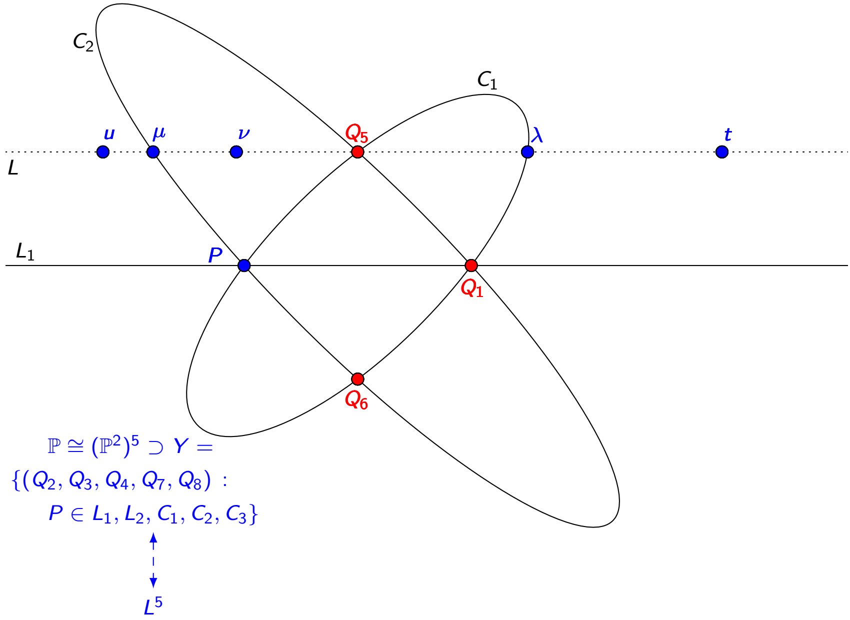
$\updownarrow$

L^5

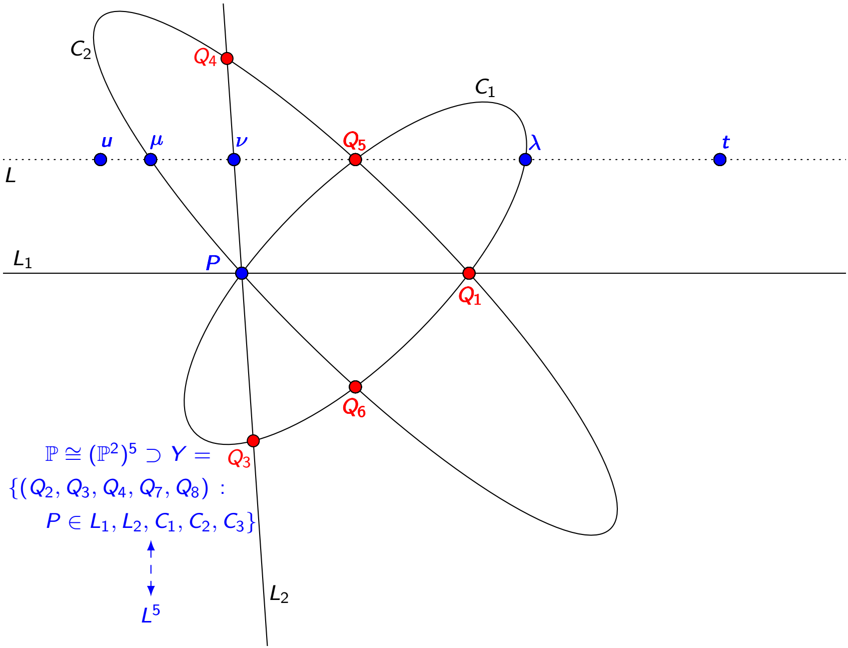


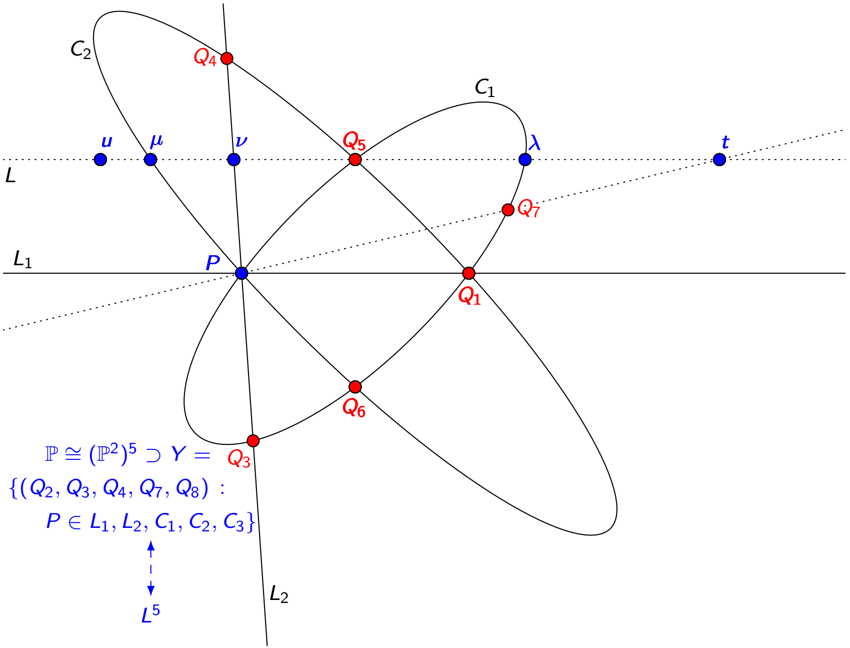
$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y =$
$\{(Q_2, Q_3, Q_4, Q_7, Q_8) :$
$P \in L_1, L_2, C_1, C_2, C_3\}$

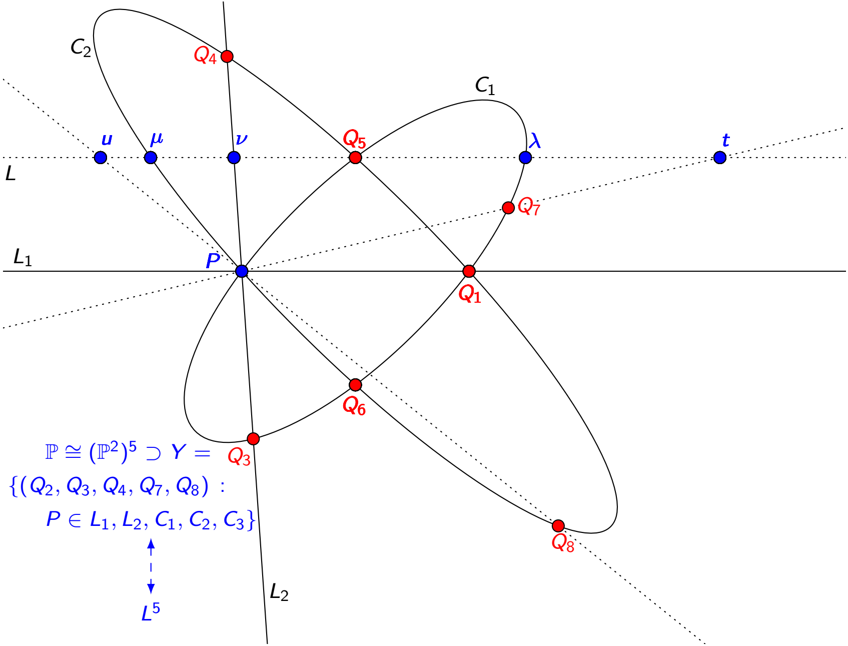
L^5

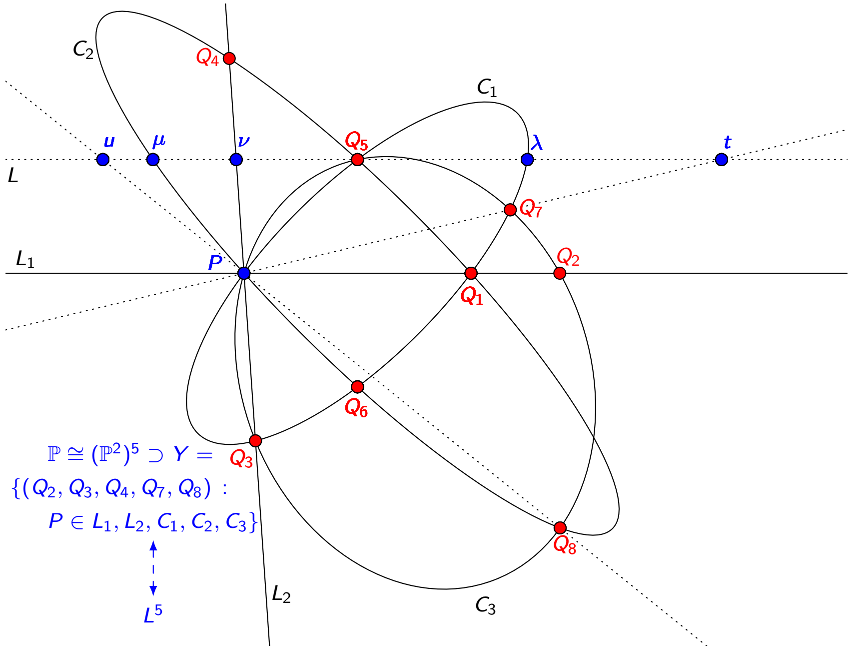


C_2
C_1
u
μ
ν
Q_5
λ
t
L
L_1
P
Q_1
Q_6
$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y =$
$\{(Q_2, Q_3, Q_4, Q_7, Q_8) :$
$P \in L_1, L_2, C_1, C_2, C_3\}$
L^5


C_2
Q_4
C_1
u
μ
ν
Q_5
λ
t
L
L_1
P
Q_1
Q_6
Q_3
ℙ ≅ (ℙ^2)^5 ⊃ Y =
{(Q_2, Q_3, Q_4, Q_7, Q_8) :
P ∈ L_1, L_2, C_1, C_2, C_3}
L^5
L_2


C_2
Q_4
C_1
u
μ
ν
Q_5
λ
t
L
Q_7
L_1
P
Q_1
Q_6
Q_3
$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y =$
$\{(Q_2, Q_3, Q_4, Q_7, Q_8)$:
$P \in L_1, L_2, C_1, C_2, C_3\}$
L^5
L_2


C_2
Q_4
C_1
u
μ
ν
Q_5
λ
t
L
Q_7
L_1
P
Q_1
Q_6
$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y =$
Q_3
$\{(Q_2, Q_3, Q_4, Q_7, Q_8)\ :$
$P \in L_1, L_2, C_1, C_2, C_3\}$
Q_8
L_2
L^5


C_2
Q_4
C_1
u
μ
ν
Q_5
λ
t
L
Q_7
L_1
P
Q_2
Q_1
Q_6
$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y =$
$\{(Q_2, Q_3, Q_4, Q_7, Q_8)$:
$P \in L_1, L_2, C_1, C_2, C_3\}$
Q_3
Q_8
L^5
L_2
C_3

$$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3\} \dashrightarrow L^5$$

The extra requirement $P \in C_4$ yields a hypersurface in $L^5(\lambda, \mu, \nu, t, u)$ that is a conic bundle over $L^3(\lambda, \mu, \nu)$ with a section. Hence, it is birational to $\mathbb{A}^4$.

$$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) \, : P \in L_1, L_2, C_1, C_2, C_3\} \dashleftarrow\dashrightarrow L^5$$

The extra requirement $P \in C_4$ yields a hypersurface in $L^5(\lambda, \mu, \nu, t, u)$ that is a conic bundle over $L^3(\lambda, \mu, \nu)$ with a section. Hence, it is birational to $\mathbb{A}^4$.

In $\mathbb{A}^4$, the four conditions $P \in D_i$ for $1 \leq i \leq 4$ define a set that is contained in the contained in the set that describes **not** being in general position (at this point MAGMA is able to help out).

This proves the proposition.

$$\mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) \, : P \in L_1, L_2, C_1, C_2, C_3\} \dashrightarrow L^5$$

The extra requirement $P \in C_4$ yields a hypersurface in $L^5(\lambda, \mu, \nu, t, u)$ that is a conic bundle over $L^3(\lambda, \mu, \nu)$ with a section. Hence, it is birational to $\mathbb{A}^4$.

In $\mathbb{A}^4$, the four conditions $P \in D_i$ for $1 \leq i \leq 4$ define a set that is contained in the contained in the set that describes **not** being in general position (at this point MAGMA is able to help out).

This proves the proposition.

Proof of the theorem (case P off the ramification curve).
The corollary already said that > 10 concurrent lines is impossible. There is a 2-dimensional family of examples with 10 concurrent lines.

Thank you