

# An Elliptic K3 surface associated to Heron triangles

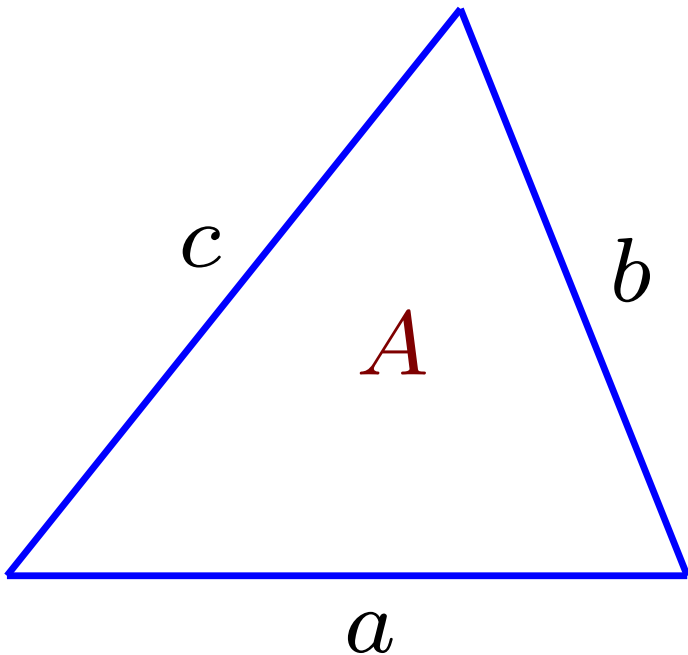
Ronald van Luijk  
CRM, Montreal  
MSRI, Berkeley

October 7, 2005  
Queen's University

Goals:

- (1) Solve a Diophantine problem
- (2) Sketch an algebraic geometric point of view
- (3) Open problems

## Heron triangles

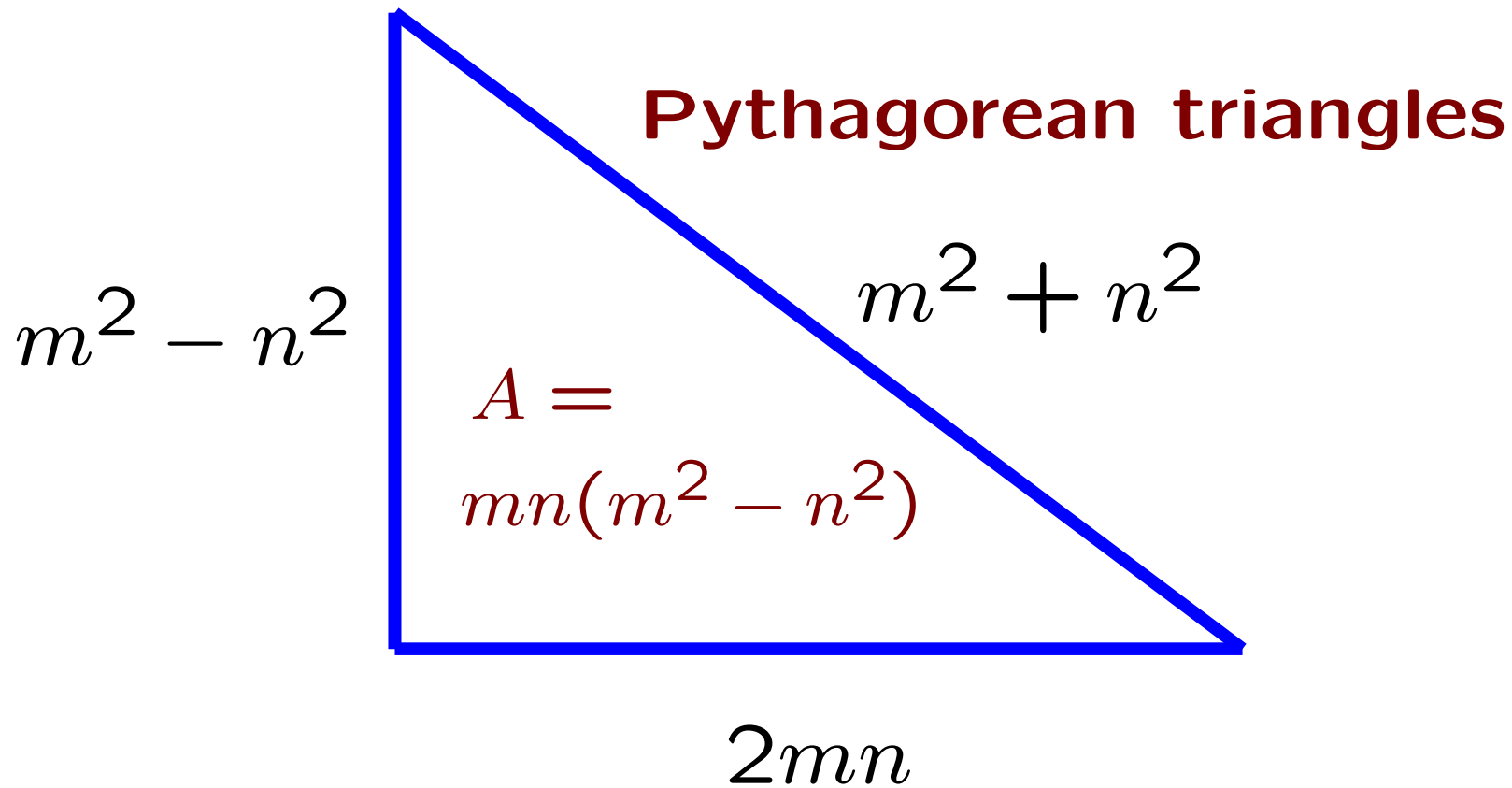


A **Heron triangle** is a triangle with integral sides and integral area.

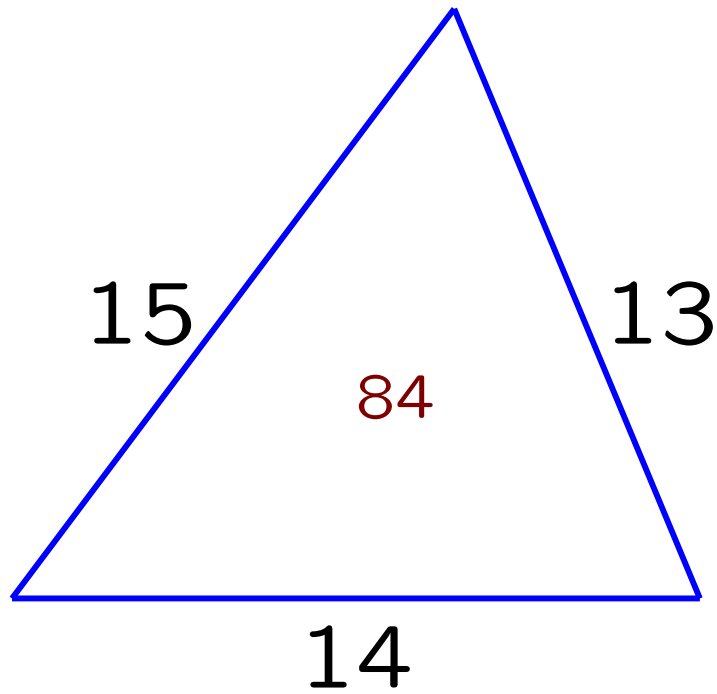
$$16A^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c)$$

(Heron's formula)

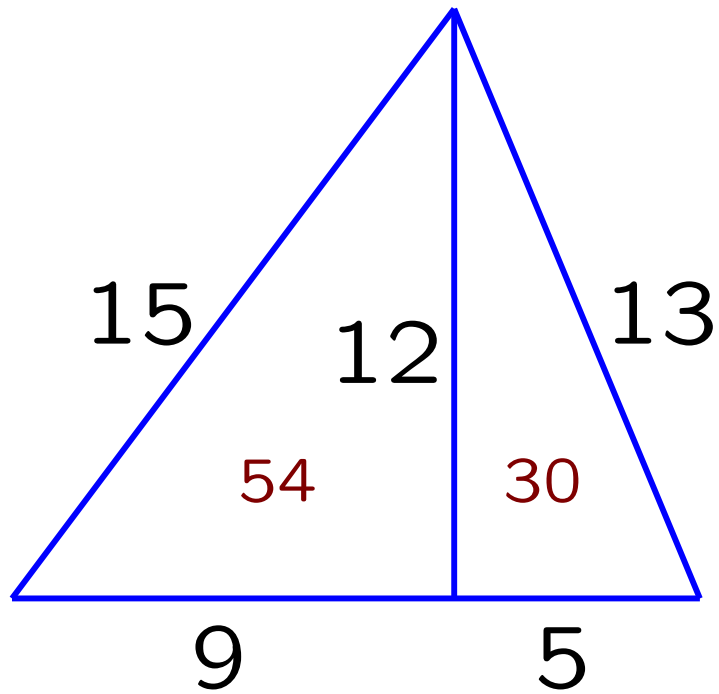
## Examples of Heron triangles



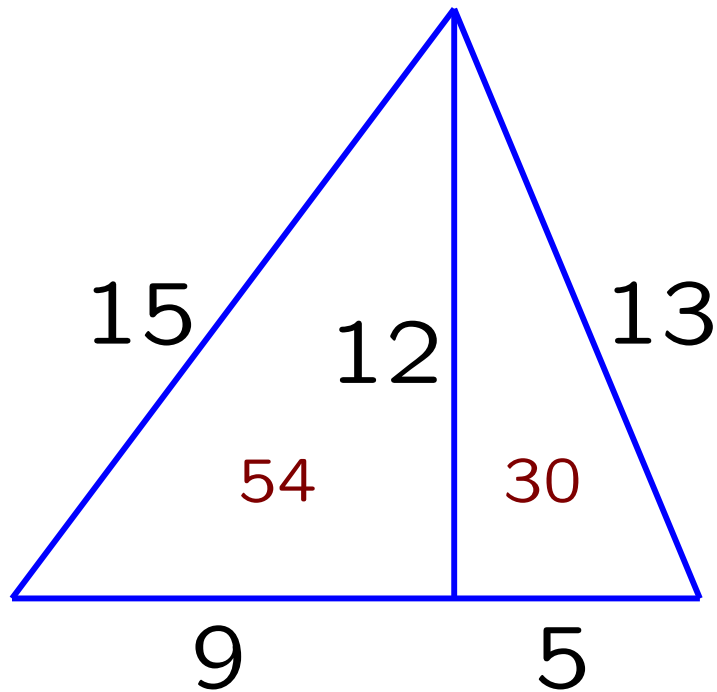
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**Fact** All Heron triangles can be obtained by glueing together Pythagorean triangles.

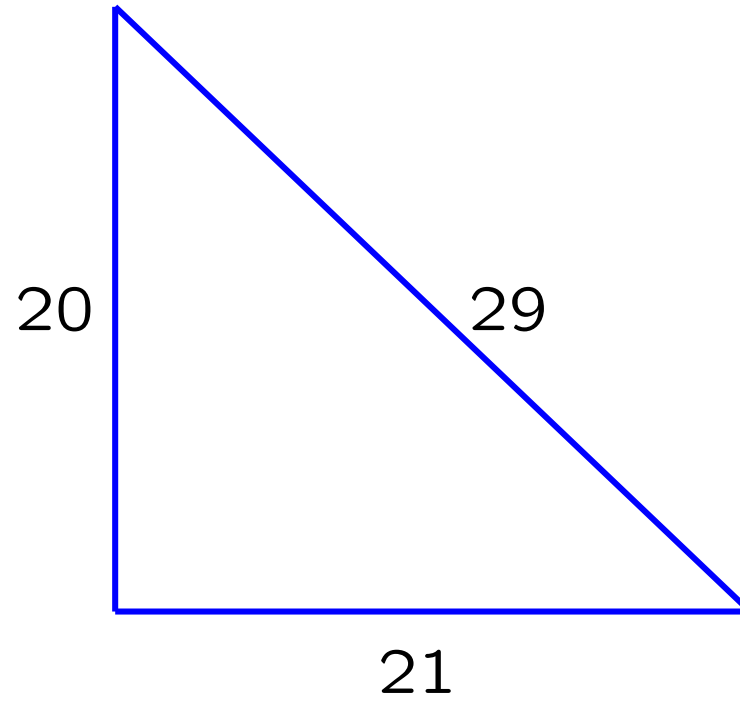
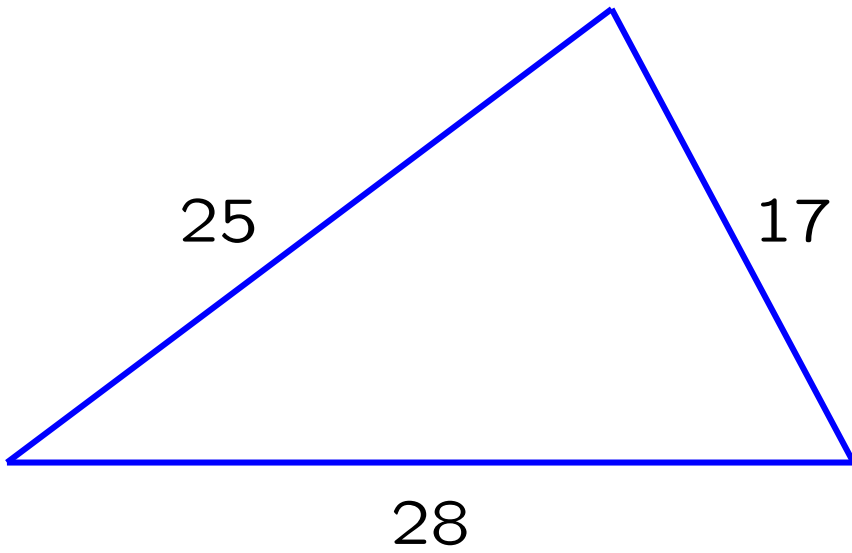
## Examples of Heron triangles

$$\begin{aligned}a &= u(v^2 + w^2) & 0 < v < u \\b &= v(u^2 + w^2) & 0 < w \\c &= (u - v)(uv + w^2) \\A &= uvw(u - v)(uv + w^2)\end{aligned}$$

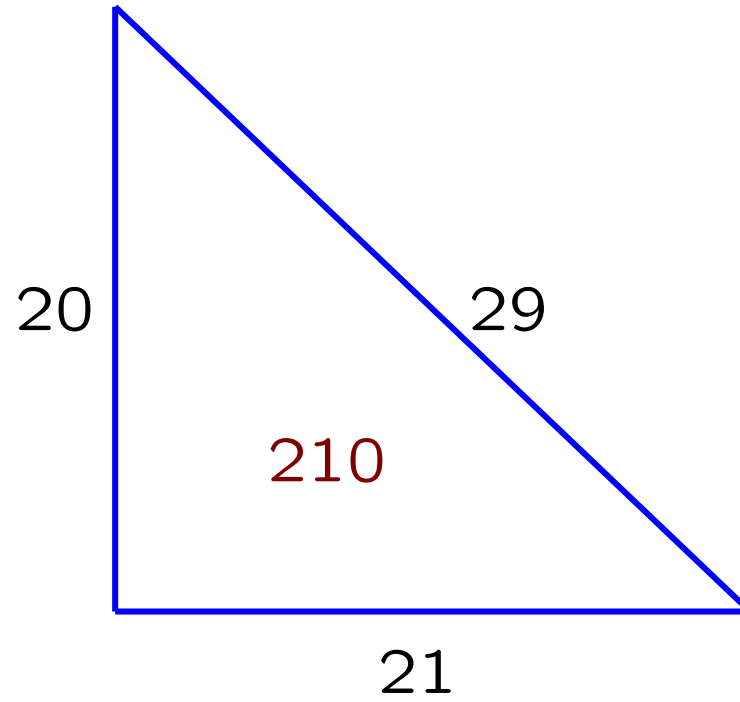
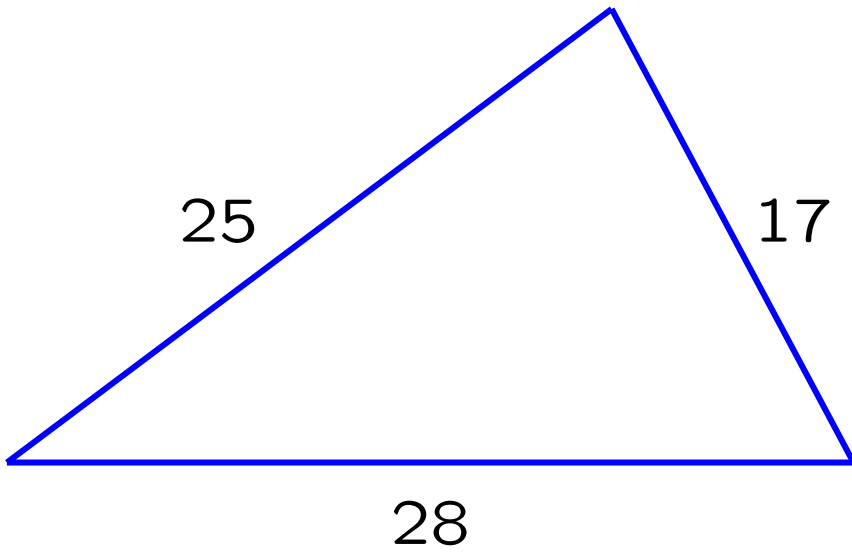
What more could we want than a parametrization?



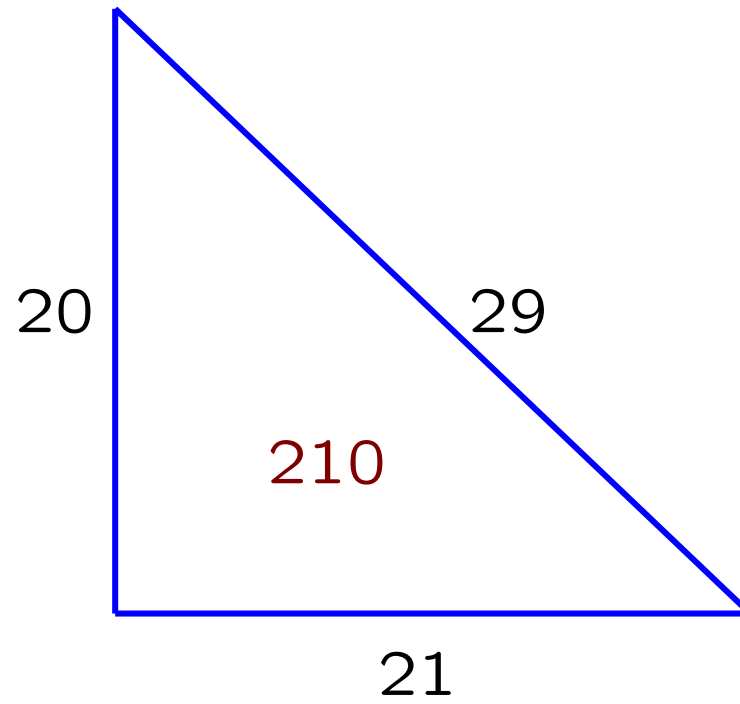
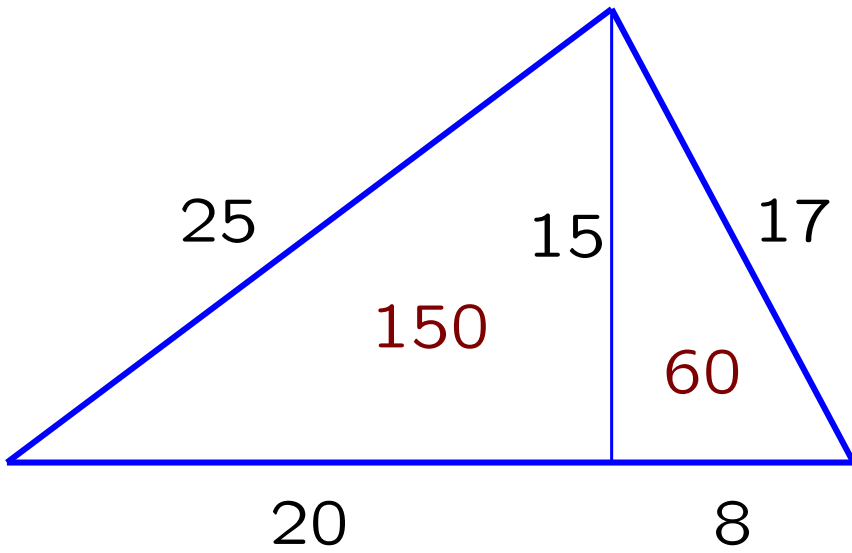
# Pairs of Heron triangles



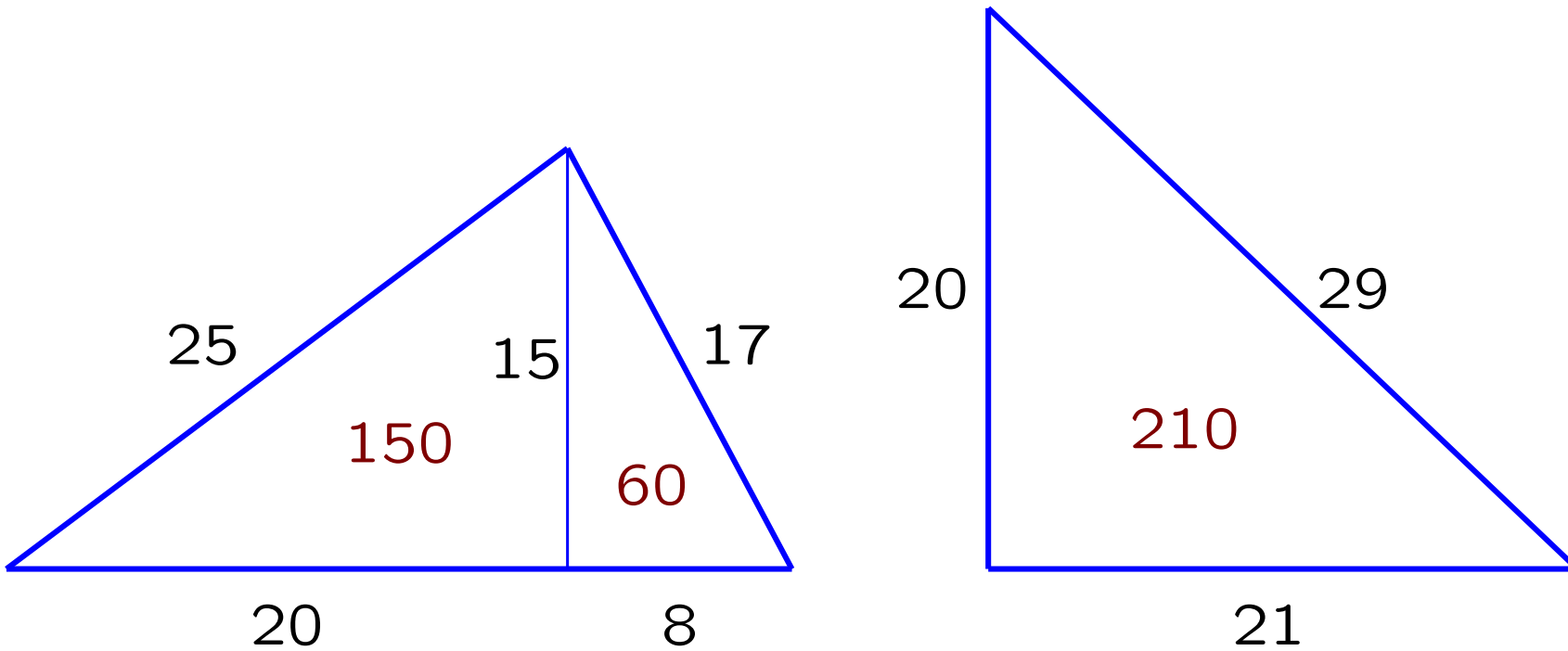
# Pairs of Heron triangles



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## Pairs of Heron triangles



Same area and same perimeter!

Infinitely many pairs (Aassila, Kramer& Luca):

$$a_1 = t^{10} + 6t^8 + 15t^6 + 19t^4 + 11t^2 + 1$$

$$b_1 = t^{10} + 5t^8 + 10t^6 + 10t^4 + 6t^2 + 3$$

$$c_1 = t^8 + 5t^6 + 9t^4 + 7t^2 + 2$$

$$a_2 = t^{10} + 6t^8 + 15t^6 + 18t^4 + 9t^2 + 1$$

$$b_2 = t^{10} + 6t^8 + 14t^6 + 16t^4 + 9t^2 + 2$$

$$c_2 = t^6 + 4t^4 + 6t^2 + 3$$

$$p = 2t^{10} + 12t^8 + 30t^6 + 38t^4 + 24t^2 + 6$$

$$A = t(t^2 + 1)^4(t^2 + 2)(t^4 + 3t^3 + 3)$$

## First Question:

Are there  $n$ -tuples of Heron triangles with the same area and the same perimeter for  $n \geq 3$ ?

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## Answer:

Yes, in fact there are for any  $n \geq 1$ !

<i>a</i>	<i>b</i>	<i>c</i>
1154397878350700583600	2324466316136026062000	2632653985016982326400
1096939160423742636000	2485350726331508315280	2529228292748458020720
1353301222256224441200	2044007602377661720800	2714209354869822810000
1326882629217053462400	2076293397636039582000	2708342152650615927600
1175291957596867110000	2287901677455234640800	2648324544451607221200
1392068029775844821400	1997996327914674087000	2721453821813190063600
1664717974861560418800	1703885276761144351875	2742914927881004201325
1159621398162242215200	2314969007387768550000	2636927773953698206800
1582886815525601586000	1787918651729320350240	2740712712248787035760
1363338670812365847600	2031949206689694692400	2716230302001648432000
1629738181200989059200	1739432097243363322800	2742347901059356590000
1958819929328111850000	1426020908550865426800	2726677341624731695200
2256059203526140412400	1195069414854334519500	2660389561123234040100
2227944754401017652000	1213597769548172408400	2669975655554518911600
2005582596002614412784	1385590865209533198216	2720344718291561361000
2462169105650632177800	1100472310428896790000	2548876763424180004200
2198208931289532607600	1234160196742812482000	2679149051471363882400
2440795514101169425200	1105486738297174396800	2565235927105365150000
2469616851505228370400	1099107024377149242000	2542794303621331359600
2623055767363274578335	1143817472264343917040	2344644939876090476625
$p = a + b + c = 6111518179503708972000$		
$A = 1340792724147847711994993266314426038400000$		



## Main Question:

Are there parametrizations of  $n$ -tuples of Heron triangles with the same area and the same perimeter for  $n \geq 3$ ?

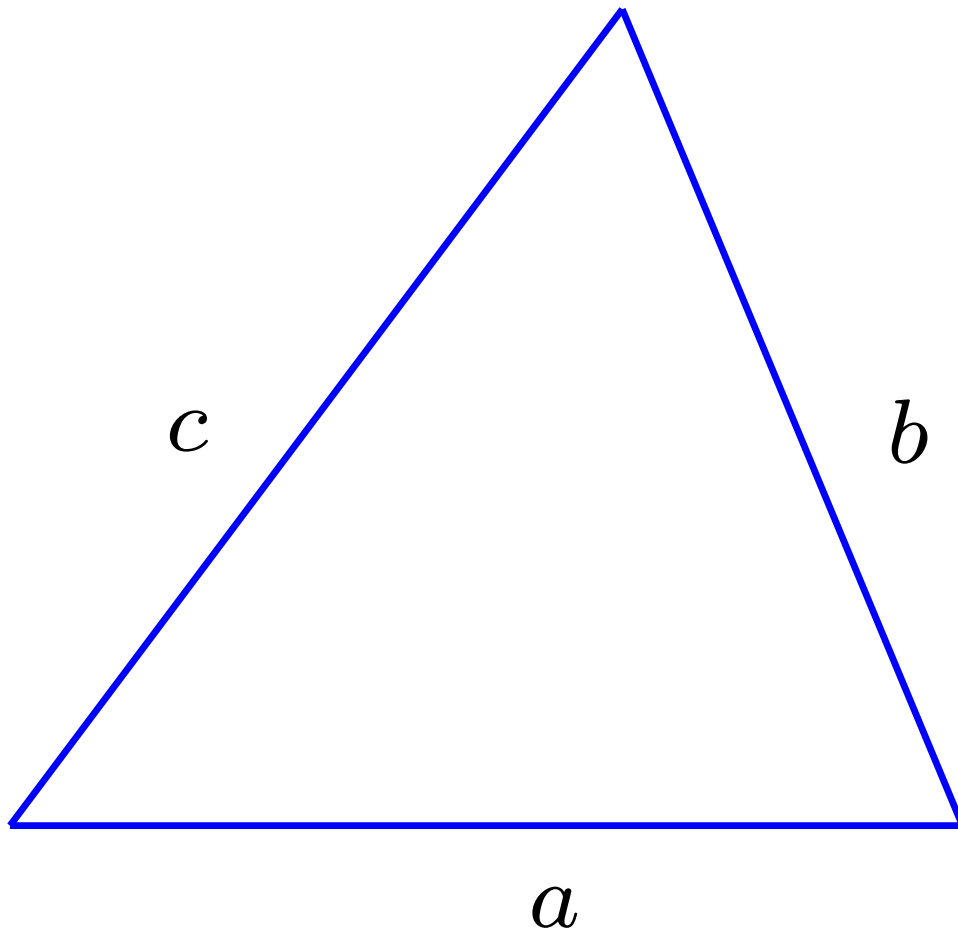
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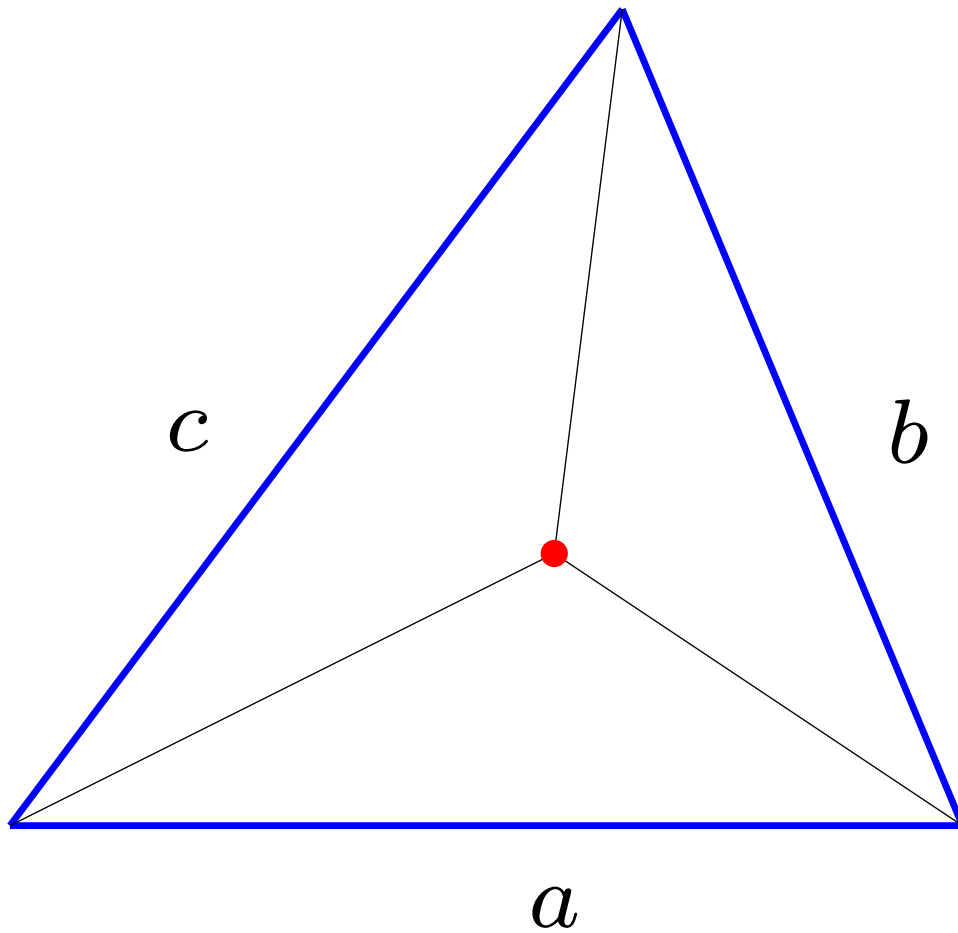
### **Answer:**

Yes, such parametrizations exist for any  $n$ .

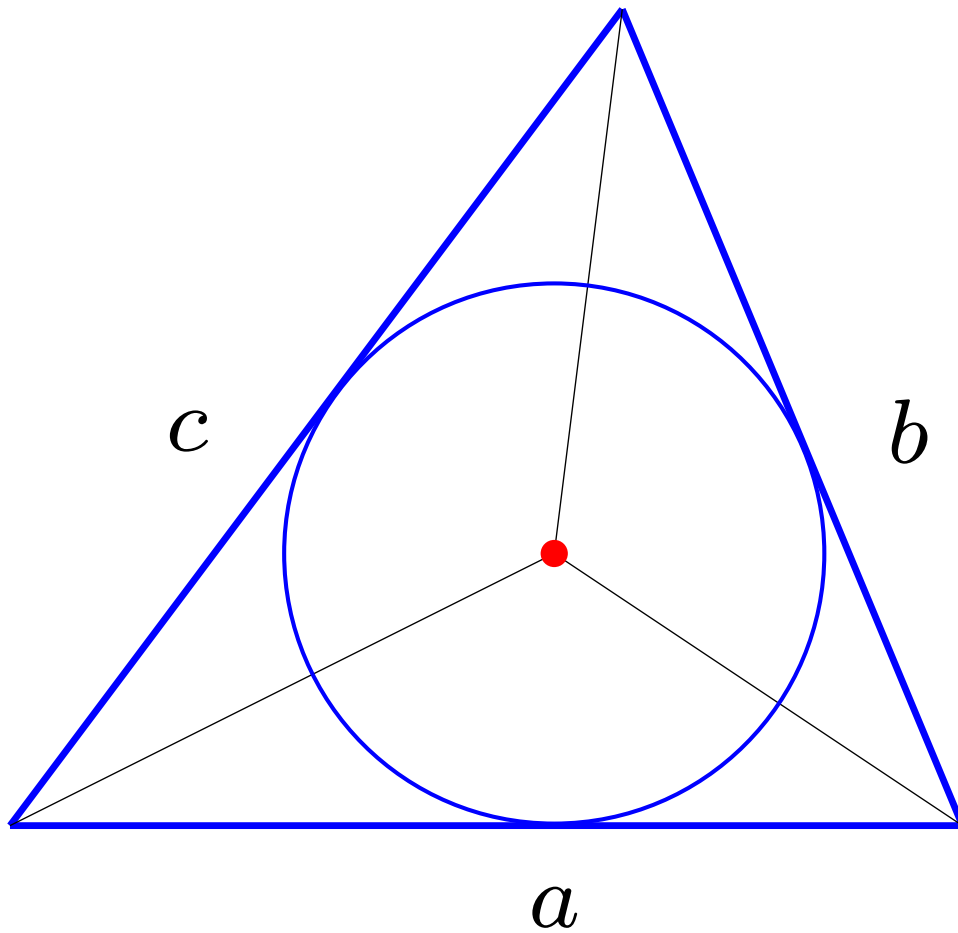
## New variables



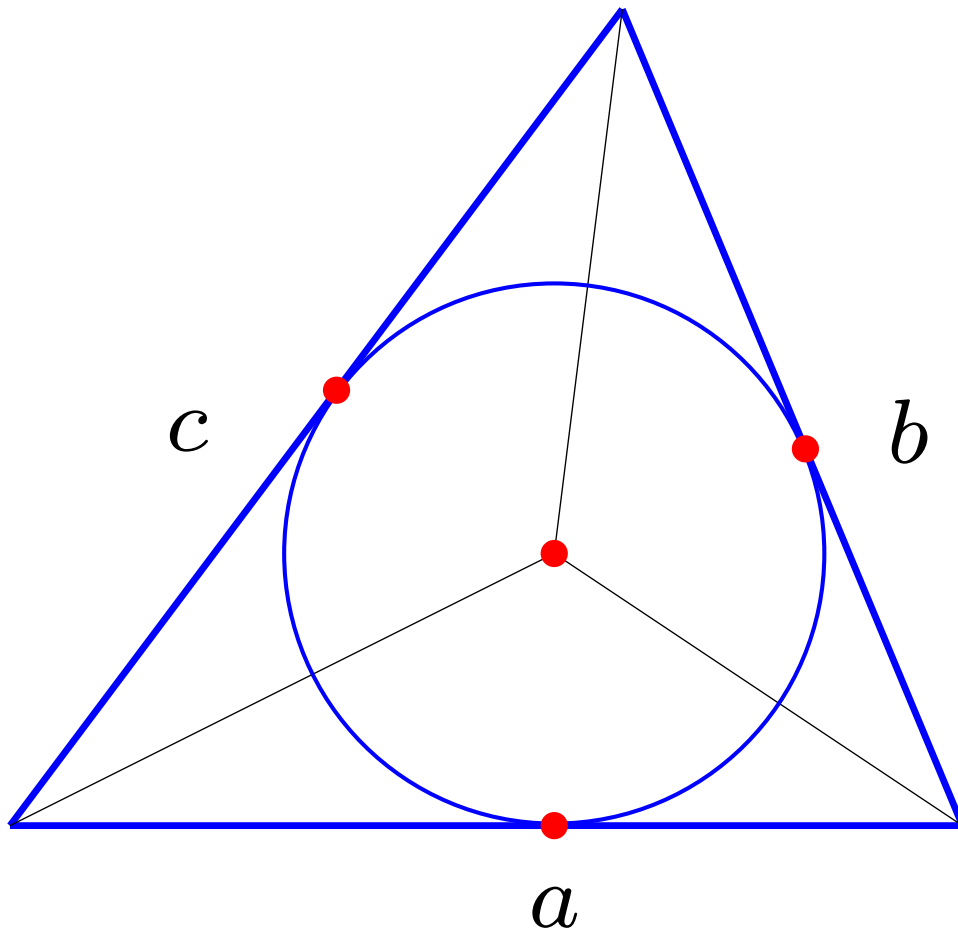
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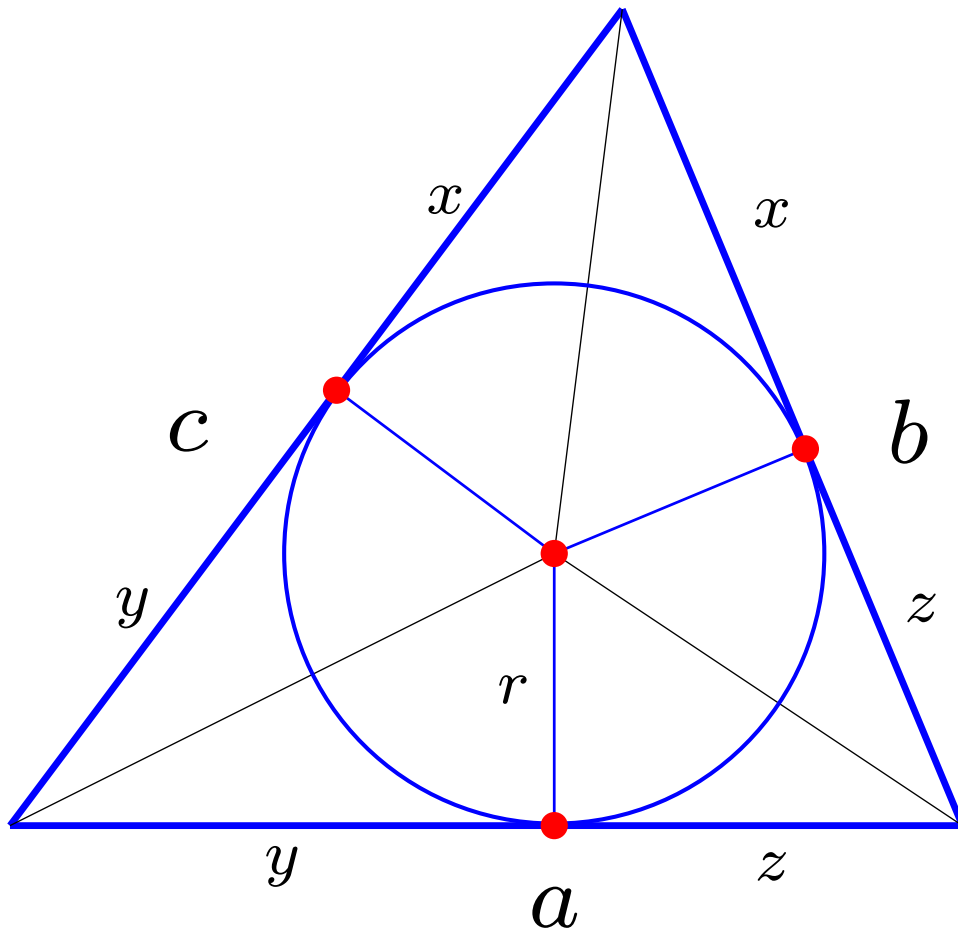
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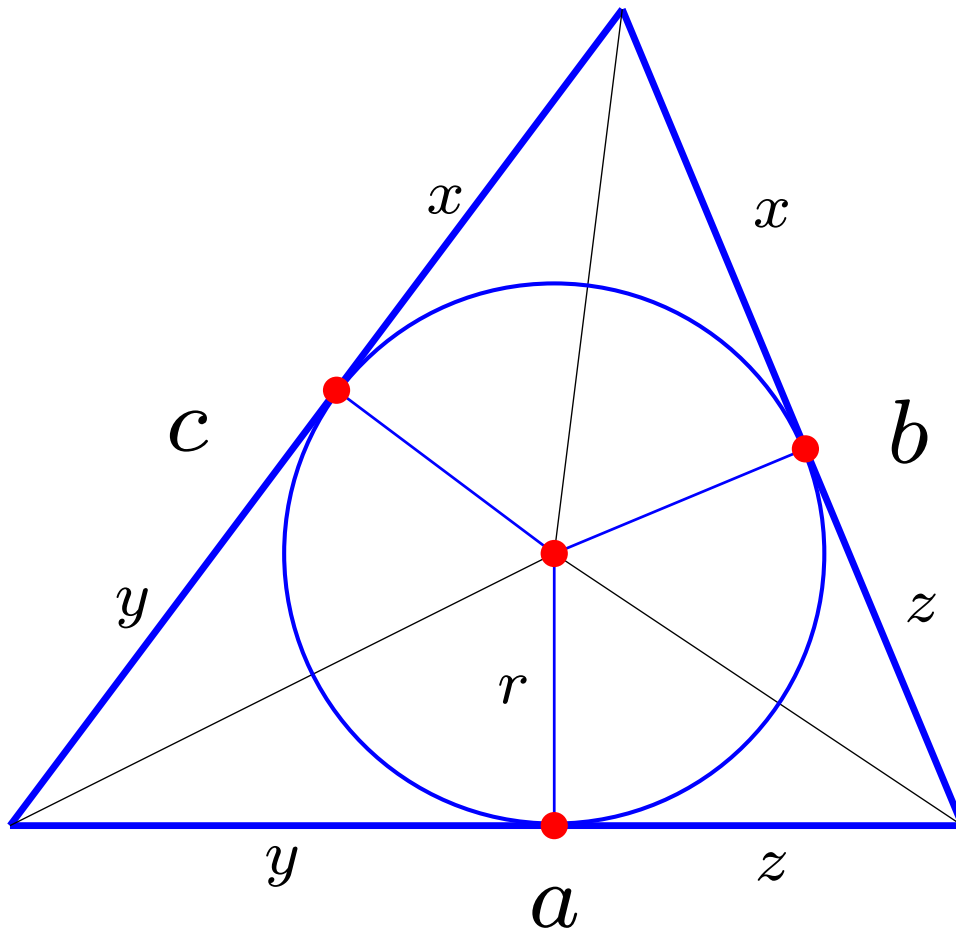
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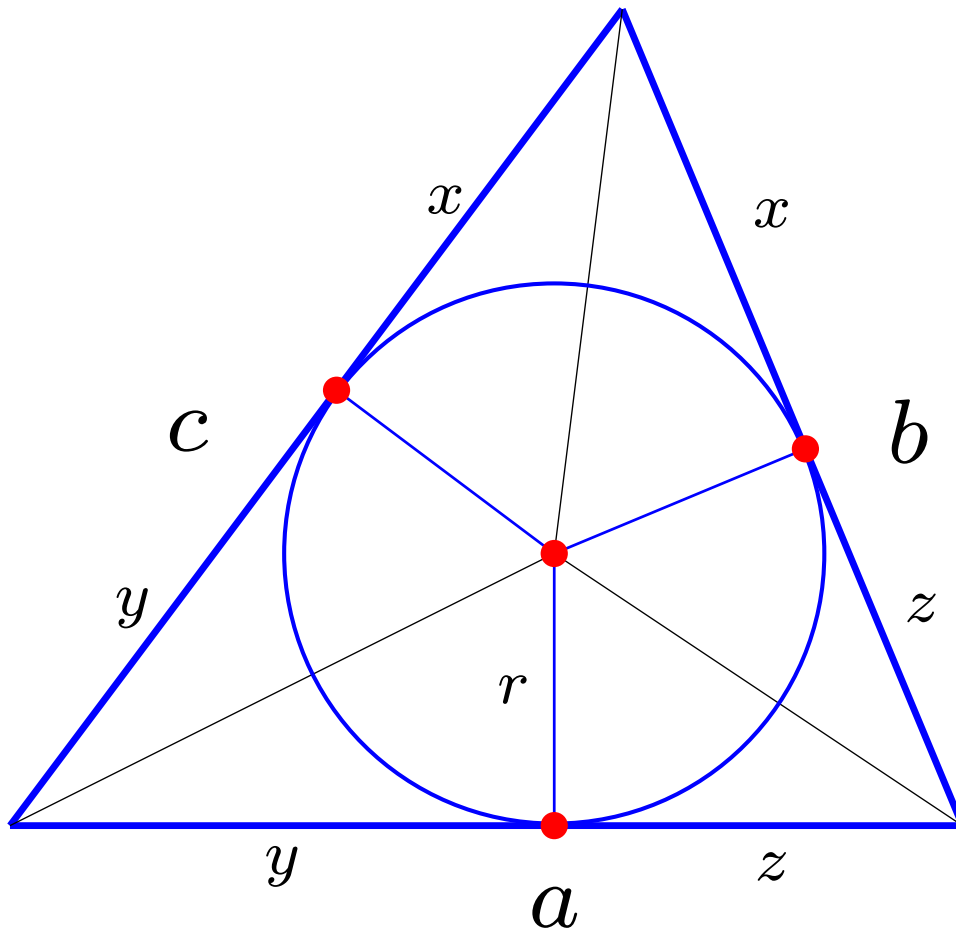
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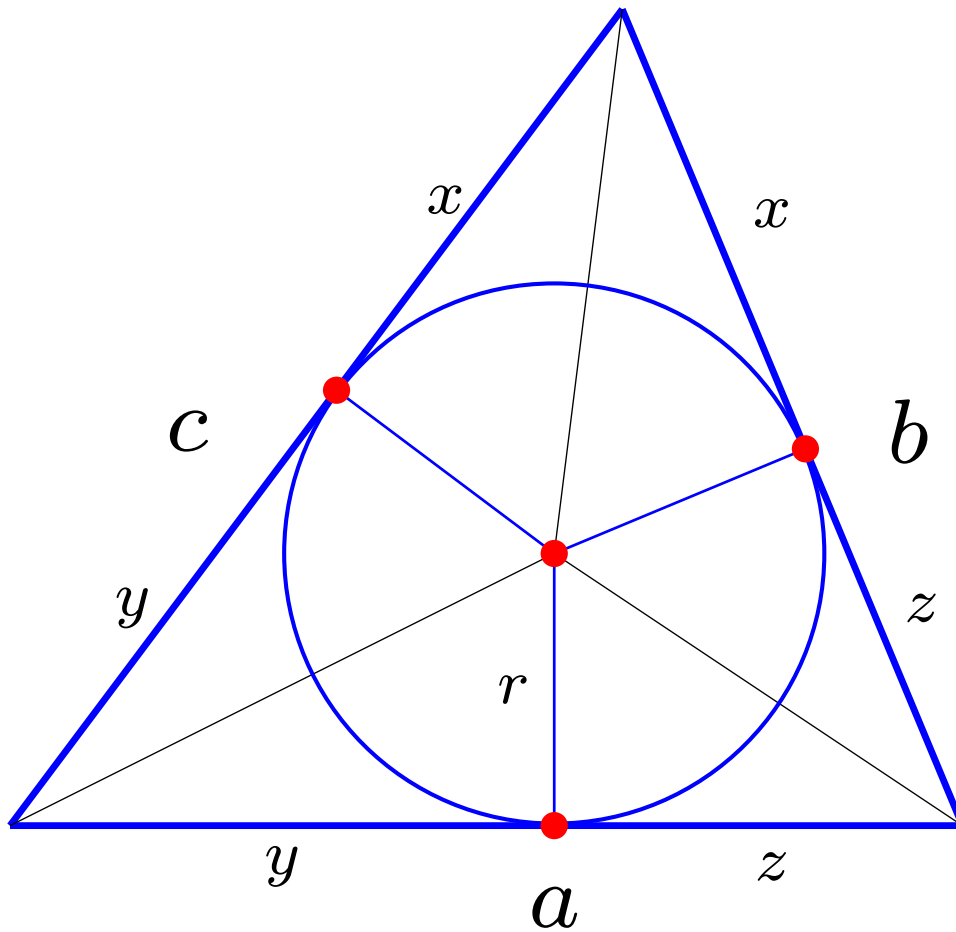
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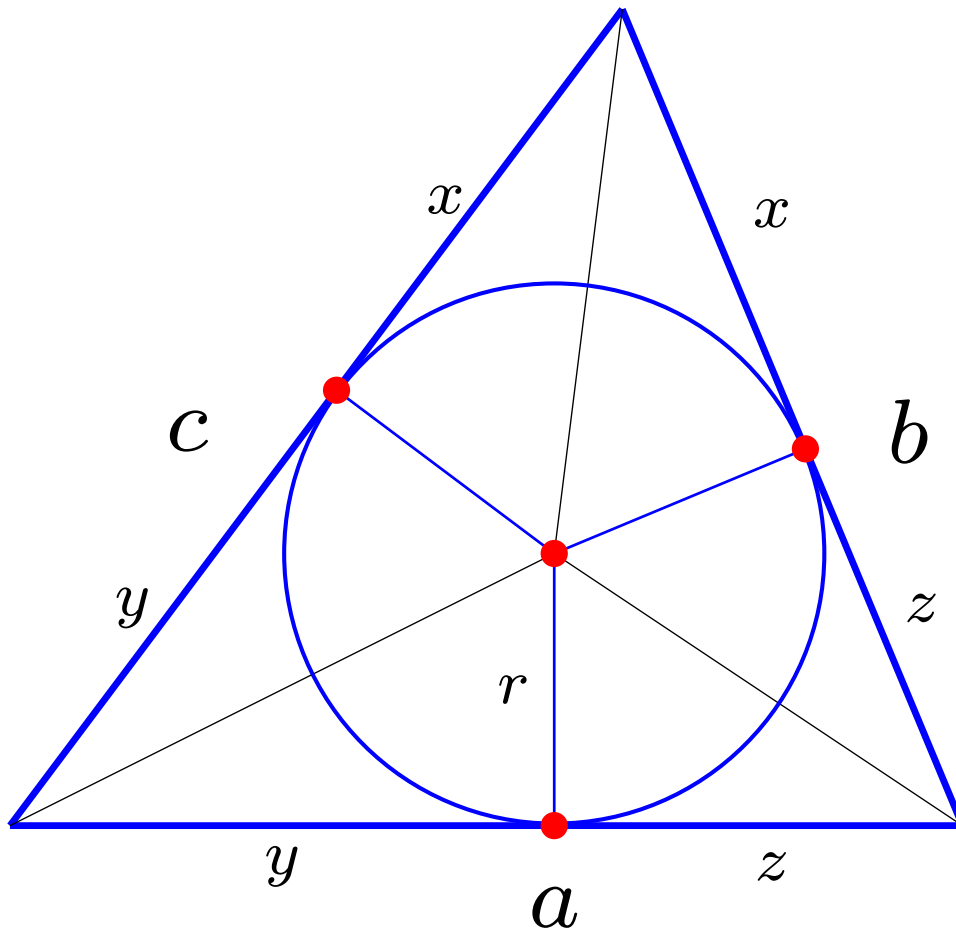
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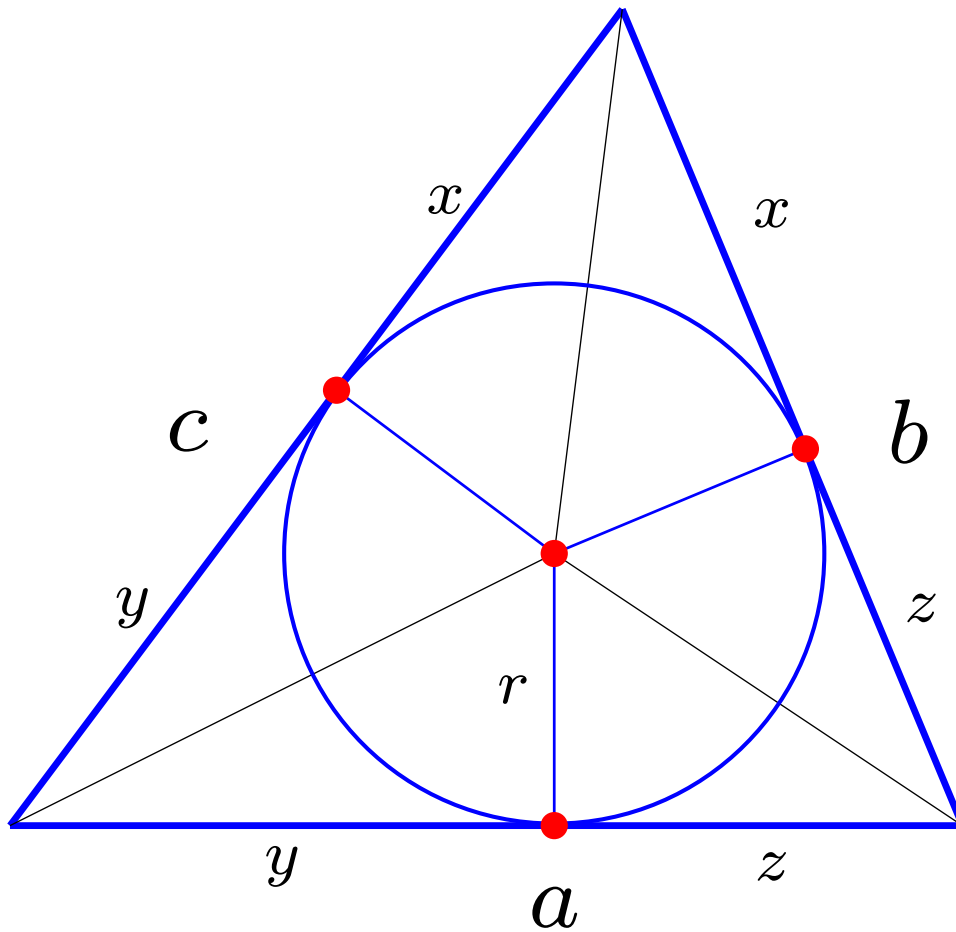
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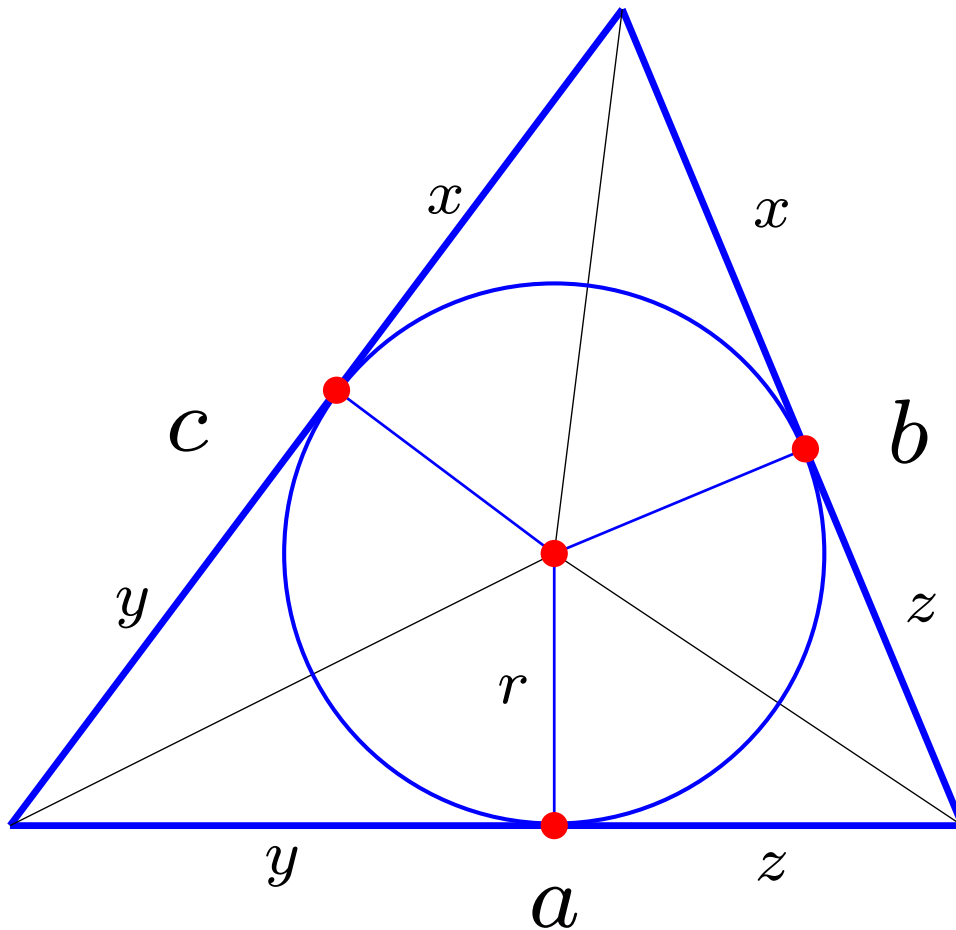
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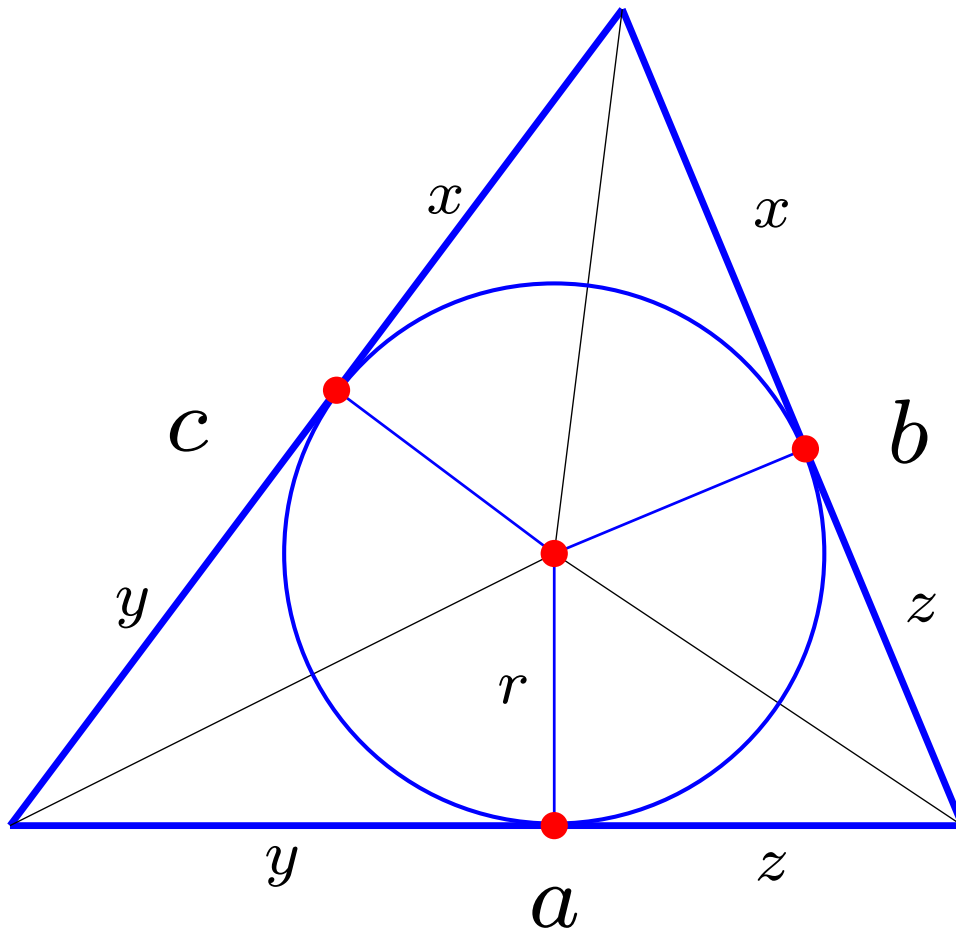
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We want constant  $\frac{r}{x+y+z}$

Every triangle gives  $x, y, z, r > 0$  satisfying

$$r^2(x + y + z) = xyz.$$

Triangles with the same area and perimeter yield the same ratio

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We can find solutions for  $t$  of the form

$$t = \frac{s - 1}{s(s + 1)}.$$

**Theorem 1.** *There exists a sequence  $\{(x_n, y_n, z_n)\}_{n \geq 1}$  of triples of elements in  $\mathbb{Q}(s)$  such that*

1. *for all  $n \geq 1$  and all  $\sigma \in \mathbb{R}$  with  $\sigma > 1$ , there exists a triangle  $\Delta_n(\sigma)$  with sides  $y_n(\sigma) + z_n(\sigma)$ ,  $x_n(\sigma) + z_n(\sigma)$ , and  $x_n(\sigma) + y_n(\sigma)$ , and inradius  $(\sigma - 1)\sigma^{-1}(\sigma + 1)^{-1}(x_n(\sigma) + y_n(\sigma) + z_n(\sigma))$ , and*
2. *for all  $m, n \geq 1$  and  $\sigma_0, \sigma_1 \in \mathbb{Q}$  with  $\sigma_0, \sigma_1 > 1$ , the rational triangles  $\Delta_m(\sigma_0)$  and  $\Delta_n(\sigma_1)$  are similar if and only if  $m = n$  and  $\sigma_0 = \sigma_1$ .*

First triples in the sequence:

$$(x_1, y_1, z_1) = (1 + s, -1 + s, (-1 + s)s),$$

$$x_2 = (-1 + s)(1 + 6s - 2s^2 - 2s^3 + s^4)^3,$$

$$y_2 = (-1 + s)(-1 + 4s + 4s^2 - 4s^3 + s^4)^3,$$

$$z_2 = s(1 + s)(3 + 4s^2 - 4s^3 + s^4)^3,$$

$$x_3 = (-1 + s)(1 + 2s + 2s^2 - 2s^3 + s^4)^3$$

$$(-1 - 22s + 66s^2 + 14s^3 - 72s^4 + 30s^5 + 6s^6 - 6s^7 + s^8)^3,$$

$$y_3 = (1 + s)(-1 + 20s + 68s^2 - 84s^3 + 139s^4 + 32s^5 - 224s^6 + 64s^7 + 149s^8 - 148s^9 + 60s^{10} - 12s^{11} + s^{12})^3,$$

$$z_3 = (-1 + s)s(5 + 10s + 126s^2 + 62s^3 - 225s^4 + 52s^5 + 28s^6 + 12s^7 + 27s^8 - 62s^9 + 38s^{10} - 10s^{11} + s^{12})^3,$$

$$x_4 = (1 + s)(-1 - 62s + 198s^2 + 1698s^3 + 7764s^4 - 8298s^5 - 10830s^6 + 43622s^7 - 15685s^8 - 45356s^9 - 1348s^{10} + 75284s^{11} - 13088s^{12} - 93076s^{13} + 85220s^{14} + 12s^{15} - 49467s^{16} + 40842s^{17} - 16034s^{18} + 2282s^{19} + 844s^{20} - 546s^{21} + 138s^{22} - 18s^{23} + s^{24})^3,$$

$$y_4 = (-1 + s)(-1 + 54s + 550s^2 - 10s^3 + 5092s^4 + 16674s^5 + 98s^6 - 51662s^7 + 22875s^8 + 41916s^9 - 63076s^{10} + 45628s^{11} + 13088s^{12} - 63644s^{13} + 38884s^{14} + 17668s^{15} - 31195s^{16} + 8302s^{17} + 8990s^{18} - 9554s^{19} + 4476s^{20} - 1254s^{21} + 218s^{22} - 22s^{23} + s^{24})^3,$$

$$z_4 = (-1 + s)s(-7 - 28s - 1168s^2 - 2588s^3 + 5170s^4 + 6940s^5 + 20176s^6 - 10628s^7 - 70305s^8 + 46664s^9 + 85440s^{10} - 107832s^{11} + 380s^{12} + 66840s^{13} - 46848s^{14} + 13656s^{15} - 1465s^{16} - 2796s^{17} + 5712s^{18} - 5228s^{19} + 2738s^{20} - 884s^{21} + 176s^{22} - 20s^{23} + s^{24})^3.$$

Sketch of a very unenlightening proof:

**Step 1. Show there is one solution to**

$$\frac{(s-1)^2}{s^2(s+1)^2}(x+y+z)^3 = xyz.$$

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A solution is  $(x, y, z) = (s+1, s-1, s(s-1))$  (positive for  $s > 1$ ).

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**Step 2. Show that if there is a solution, then there are infinitely many solutions.**

If  $(x, y, z)$  is a solution, then so is  $(x', y', z')$  with ...

$$\begin{aligned}
x' &= -3(s+1)(2s^2-s+1)(s-1)^3z^3 - 2(s+1)(4s^3-10s^2+7s-3)(s-1)^2xz^2 \\
&\quad + 3(2s^6-s^4+3s^3+5s^2-3s+2)(s-1)^2yz^2 - 3(s+1)(-1+s)^5x^2z \\
&\quad + 3s(s^4+s^2+2)(s-1)^3y^3 + s(s^4-s^3+s^2-3s+6)(s-1)^3x^2y \\
&\quad - (s-1)(s^9+s^8-6s^7+15s^6-6s^5+6s^4-14s^3+31s^2-15s+3)y^2z \\
&\quad + 3s(s^4-s^3+s^2-s+4)(s-1)^3xy^2 + 3(s^2+2s+2)(s-1)^6xyz, \\
y' &= -3(s+1)(s^4-s^3+4s^2-s+1)(s-1)^2z^3 + 3s(s^2+s+2)(s-1)^4y^3 \\
&\quad + (s^9-s^8+4s^7+6s^6-3s^5+9s^4+46s^3-40s^2+16s-6)yz^2 + 6s(s-1)^4x^2y \\
&\quad - (s+1)(s^4-4s^3+10s^2-6s+3)(s-1)^2x^2z - 6(5s^2-2s+1)(s-1)^2xyz \\
&\quad - 3(s^6+4s^4+3s^3+10s^2-3s+1)(s-1)^2y^2z - s(s^3-2s^2-3s-12)(s-1)^4xy^2 \\
&\quad - 3(s+1)(s^4-3s^3+7s^2-3s+2)(s-1)^2xz^2, \\
z' &= -s(-1+s)(s+1)^4z^3 + 3s^2(-1+s)^2(s+1)^3yz^2 + s^4(s+1)(-1+s)^4y^3 \\
&\quad - 3s^3(s+1)^2(-1+s)^3y^2z.
\end{aligned}$$

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**Problem 3:** Are the triangles nonsimilar?

**Solution:** We need an interpretation of these formulas.

The change of variables

$$x = -s(s + 1)p + q$$

$$y = -s(s + 1)p - q$$

$$z = 8s(s + 1)(s - 1)^2$$

shows that our curve in  $\mathbb{P}^2$  with parameter  $s$  given by

$$(s - 1)^2(x + y + z)^3 = s^2(s + 1)^2xyz$$

is isomorphic to the curve given by

$$q^2 = (p - 4(s - 1)^2)^3 + s^2(s + 1)^2p^2$$

together with the point at infinity.

# Interlude on elliptic curves

## Definition:

An **elliptic curve** is a curve given by an equation of the form

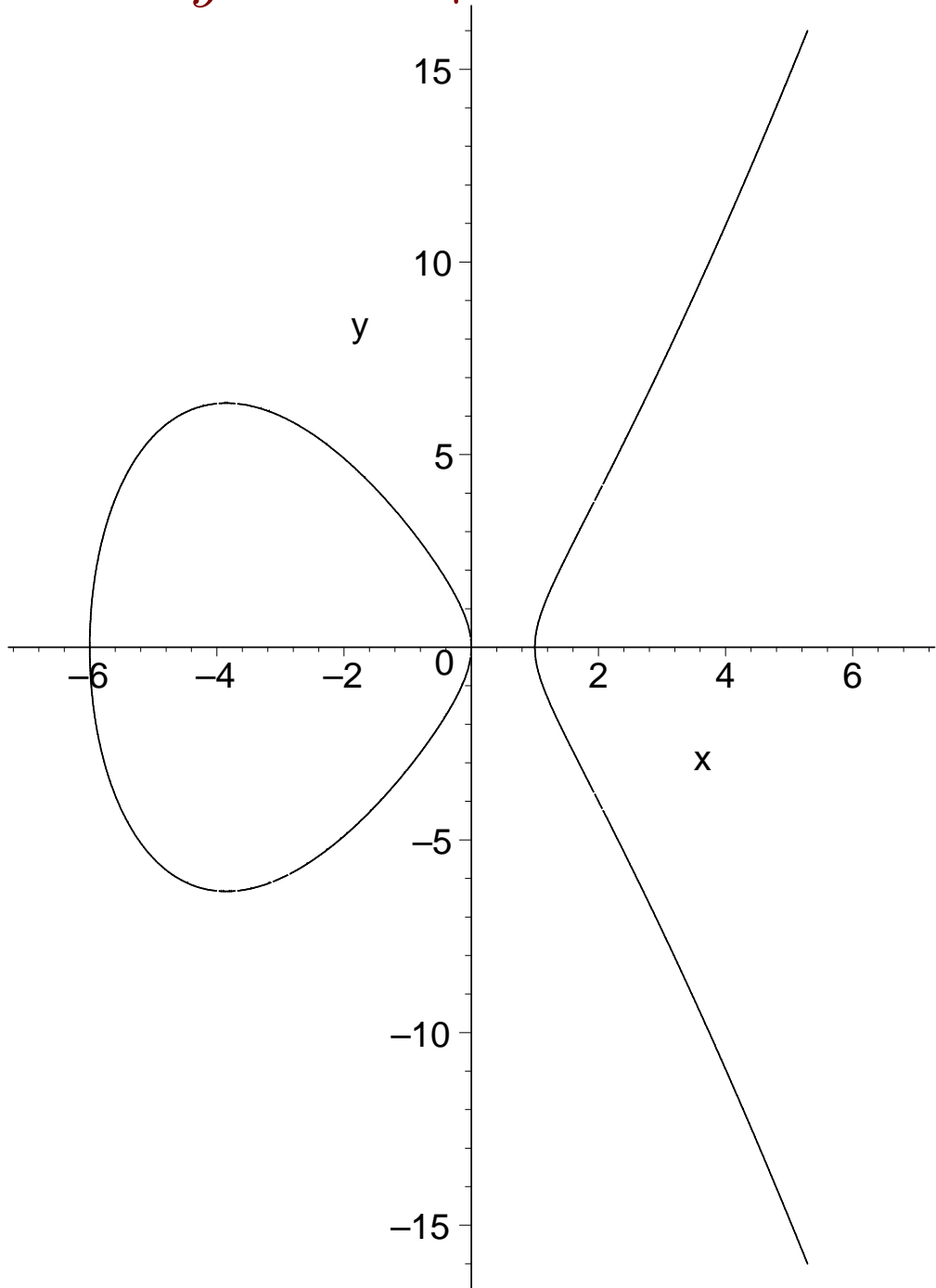
$$y^2 = f(x)$$

with  $f$  a separable polynomial of degree 3, together with the point at infinity.

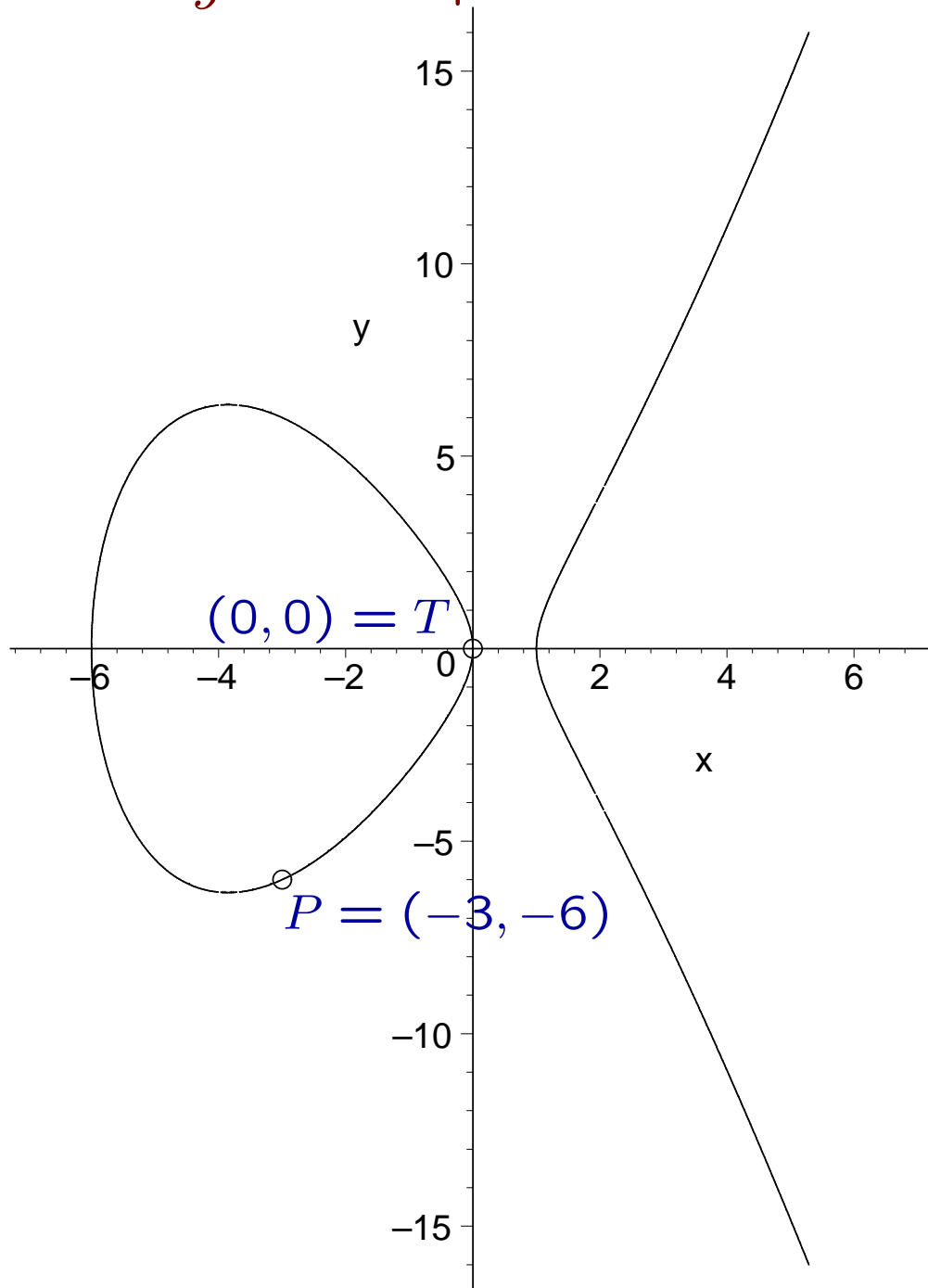
Elliptic curves are

- 1-dimensional lie-groups,
- Calabi-Yau 1-manifolds,
- 1-dimensional abelian varieties.

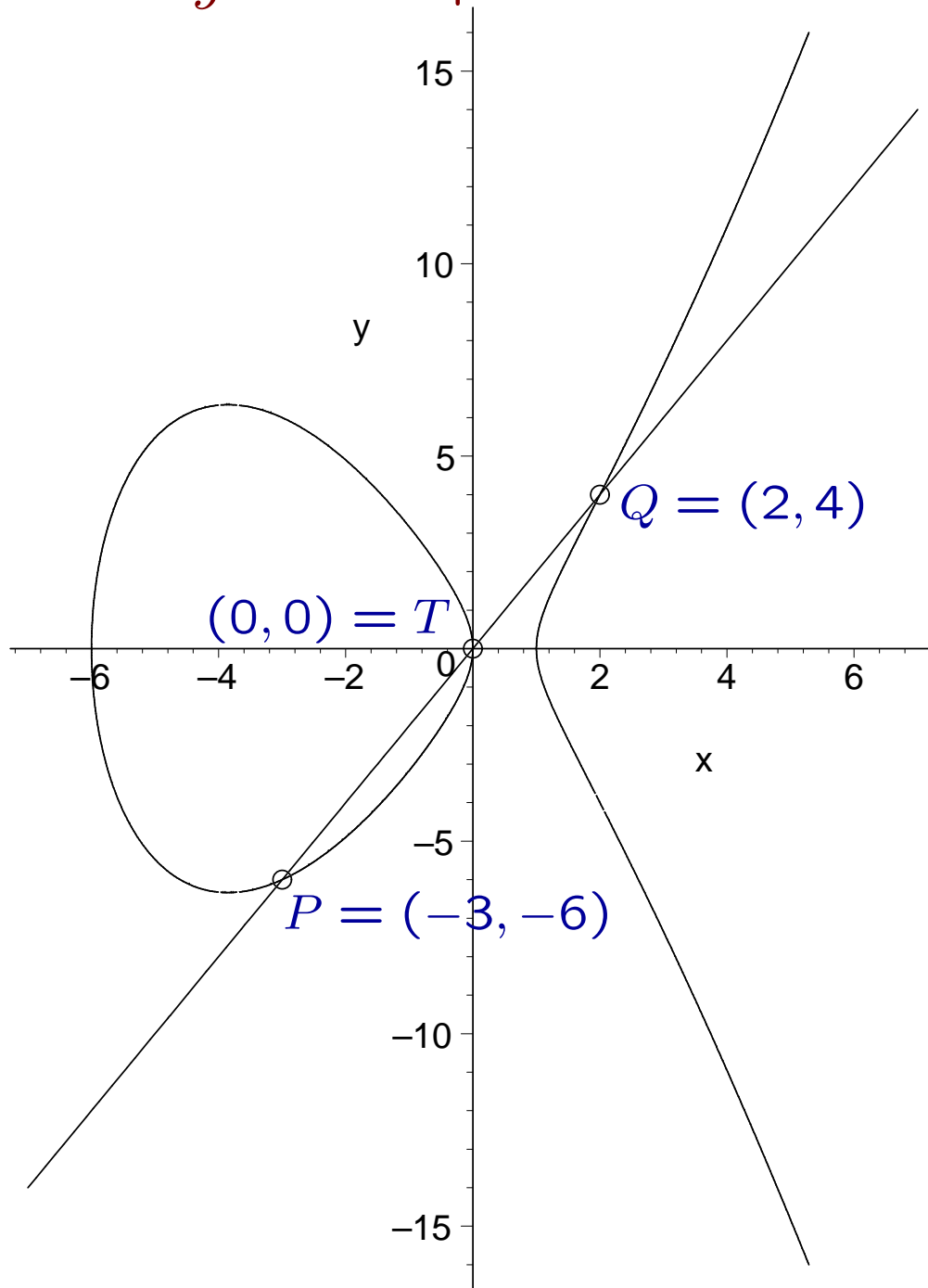
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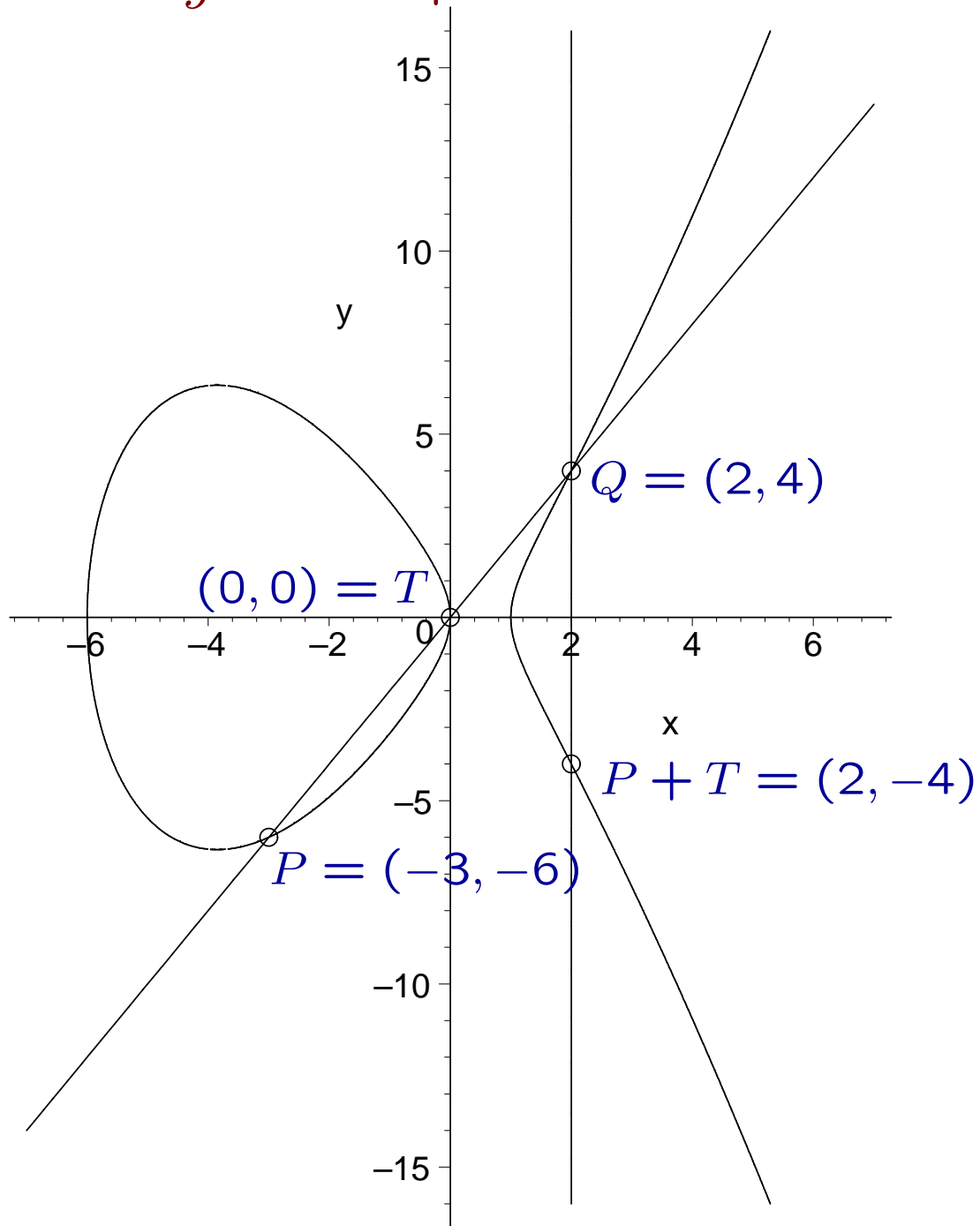


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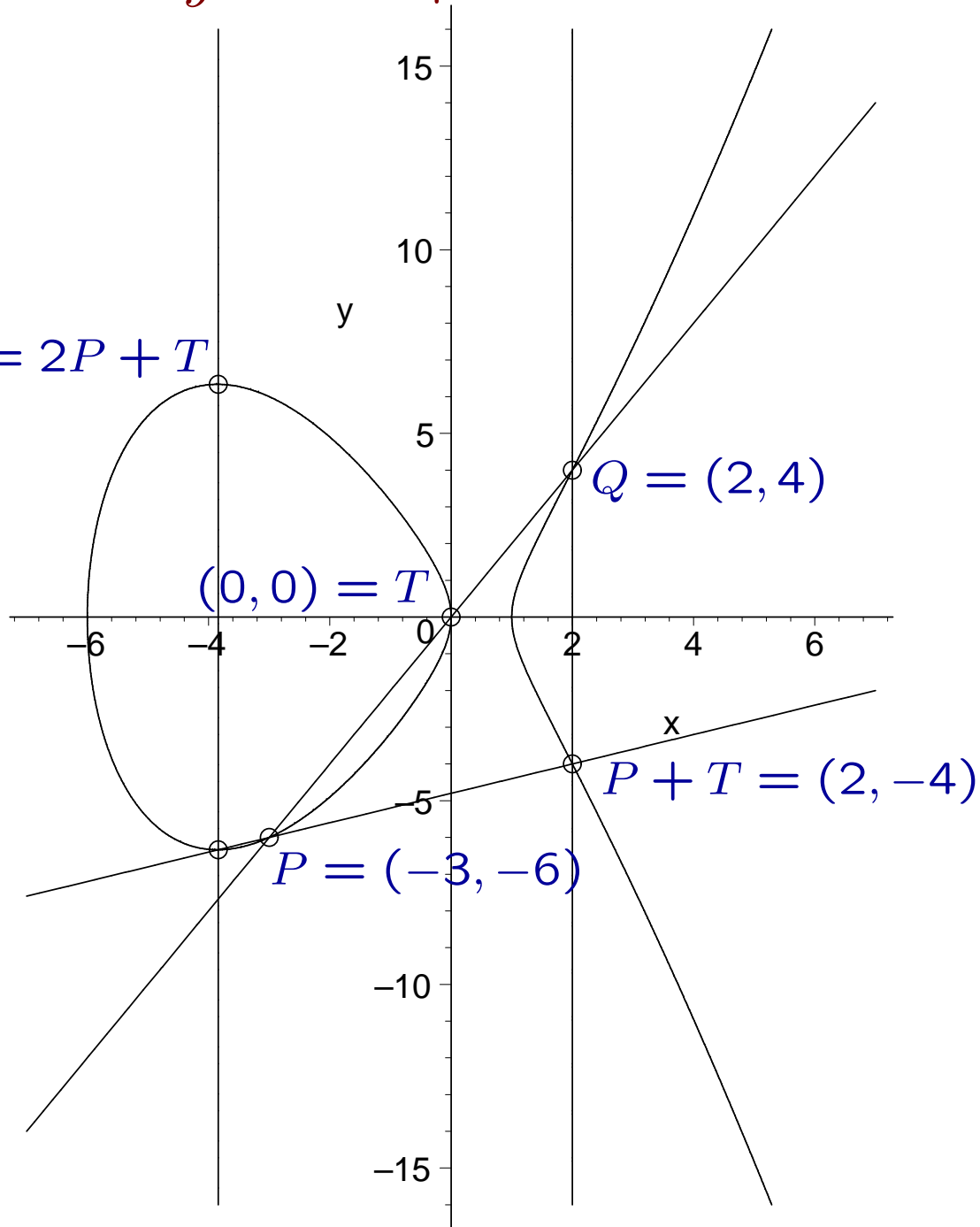


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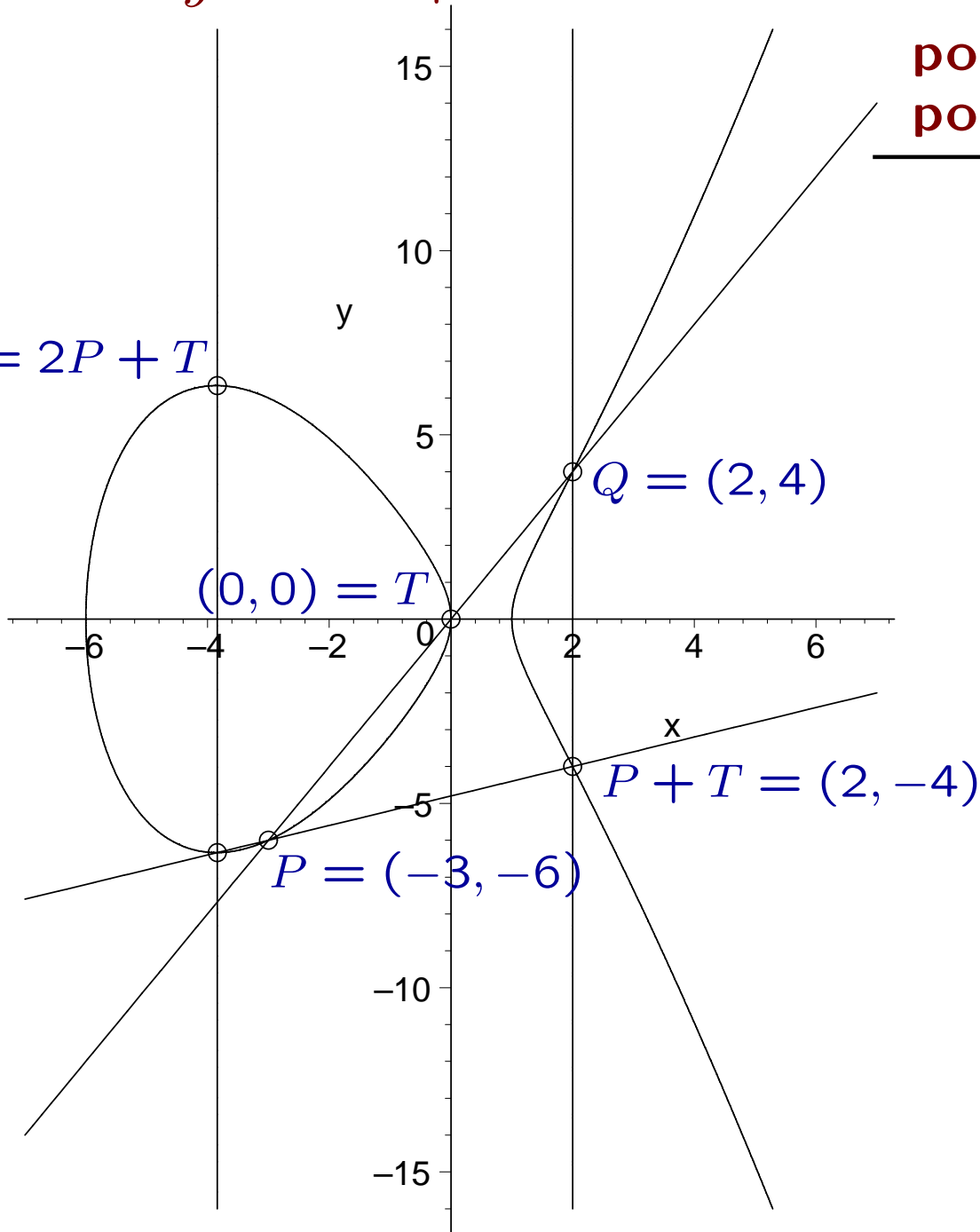
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point ON egg  
point ON egg +

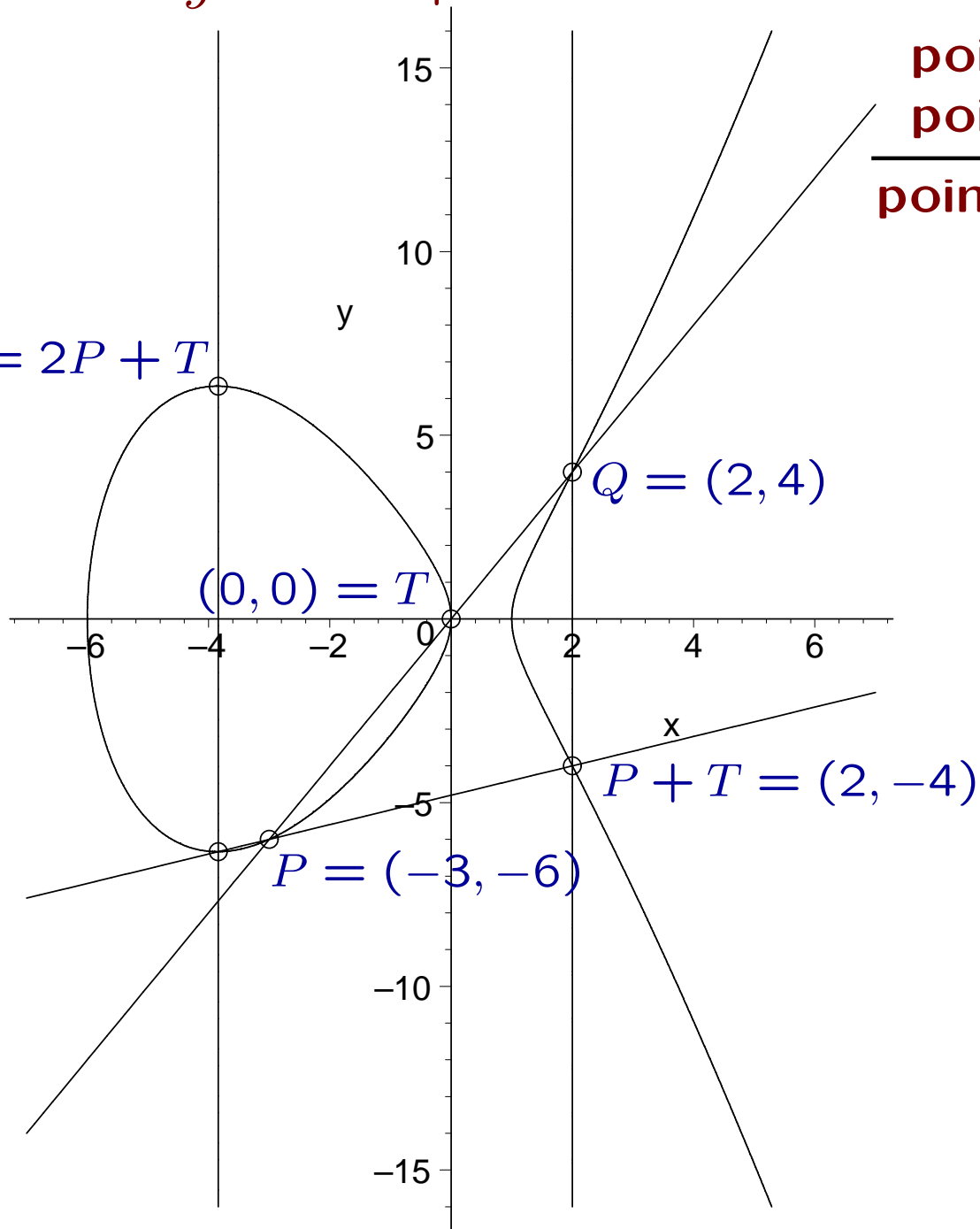
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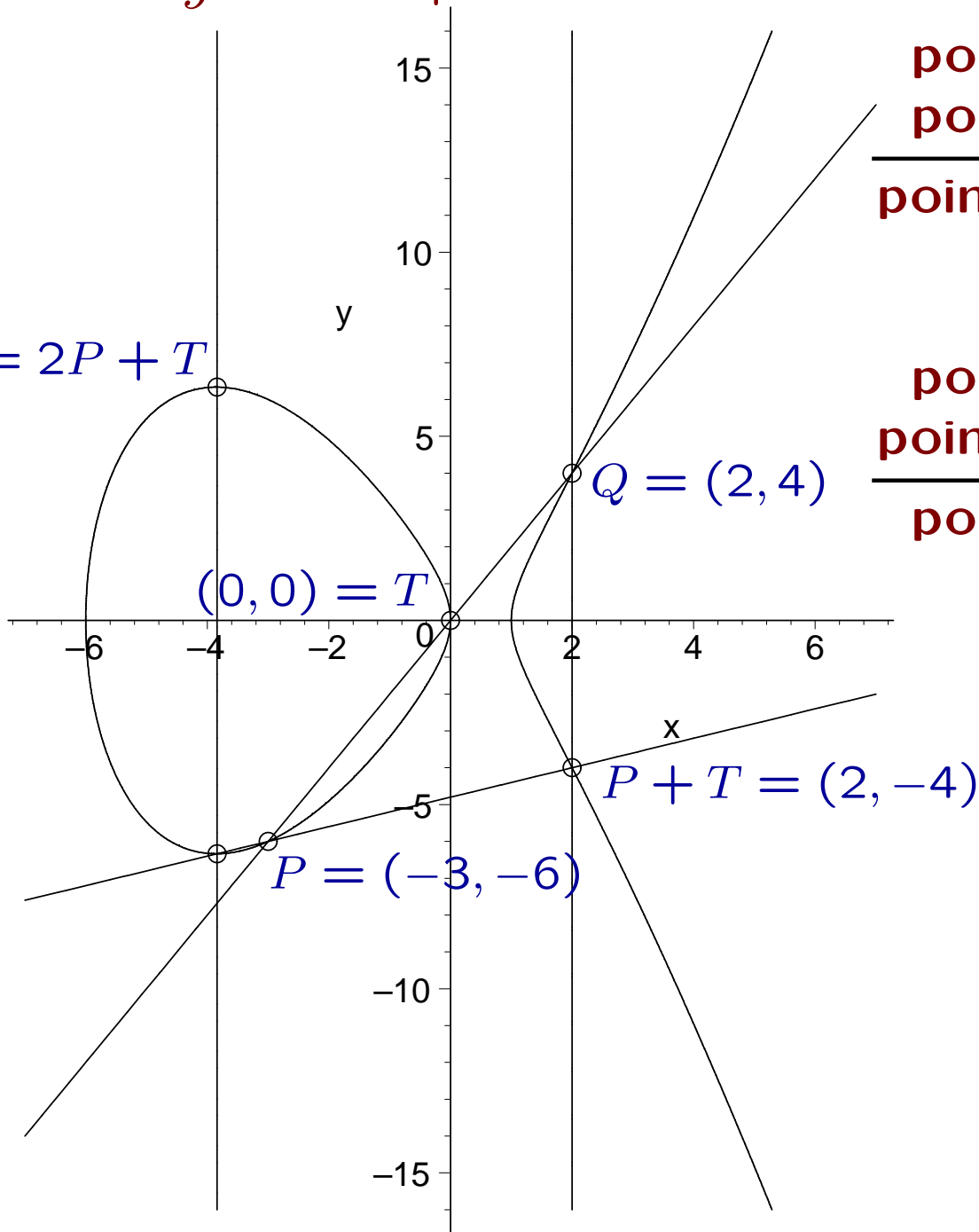
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point ON egg  
 point ON egg +  


---

 point OFF egg

point ON egg  
 point OFF egg +  


---

 point ON egg

Back to our curve

$$(s - 1)^2(x + y + z)^3 = s^2(s + 1)^2xyz$$

and the isomorphism

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to the curve given by

$$q^2 = (p - 4(s - 1)^2)^3 + s^2(s + 1)^2p^2.$$

Back to our curve

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For  $s > 1$  the inequalities  $x, y, z > 0$  are equivalent with

$$p < 0, \quad \text{and} \quad s^2(s + 1)^2p^2 > q^2 = (p - 4(s - 1)^2)^3 + s^2(s + 1)^2p^2,$$

so just to  $p < 0$ . This turns out to be equivalent to

“lying on the egg-part of the elliptic curve.”

Our simple solution  $(x, y, z) = (s + 1, s - 1, s(s - 1))$  corresponds to the point  $R = (8 - 8s, 8s^2 - 8)$  on the egg-part of the elliptic curve for  $s > 1$ .

The point  $P = (4(s - 1)^2, 4s(s + 1)(s - 1)^2)$  has order 3.



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- (2) All odd multiples of  $R$  lie on the egg-part,
- (3)  $mR$  and  $nR$  give similar triangles iff  $mR = \pm nR + kP$  for some  $k$ ,

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- (4) big formula  $(x', y', z')$  corresponds to  $Q \mapsto Q + R$ .

**Unsatisfying:**

## Unsatisfying:

- Where did the substitution

$$t = \frac{s - 1}{s(s + 1)}$$

come from?

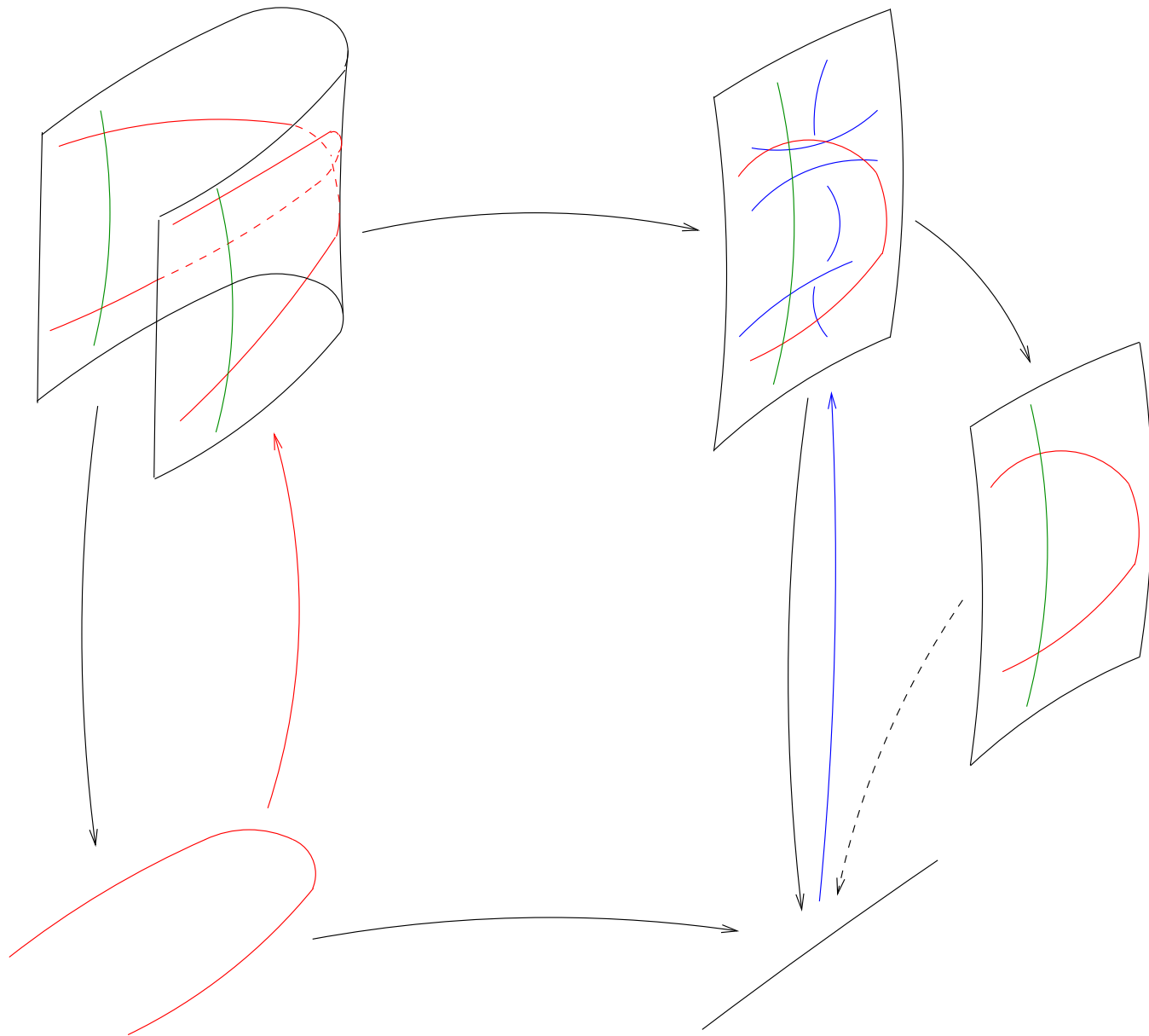
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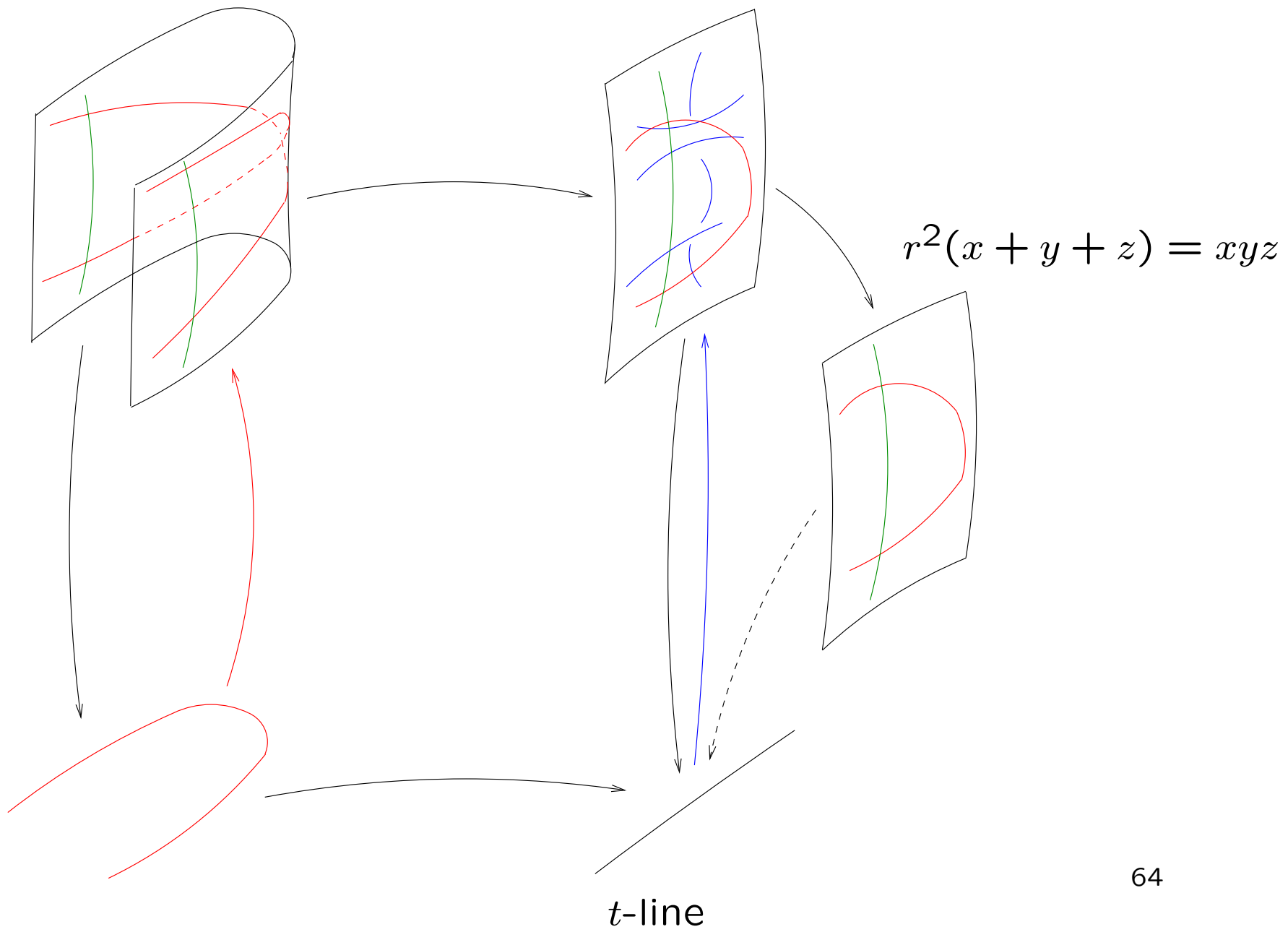
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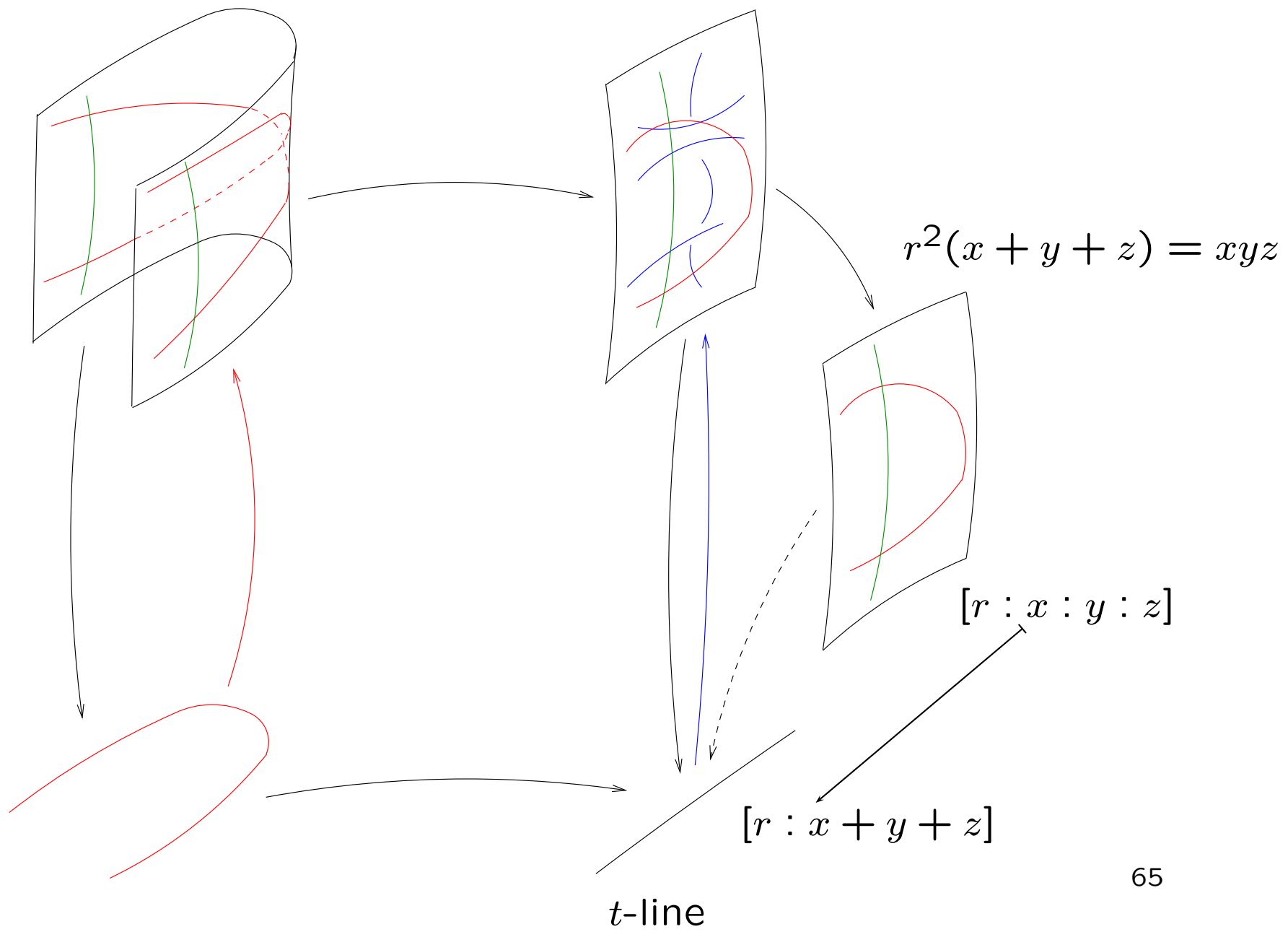
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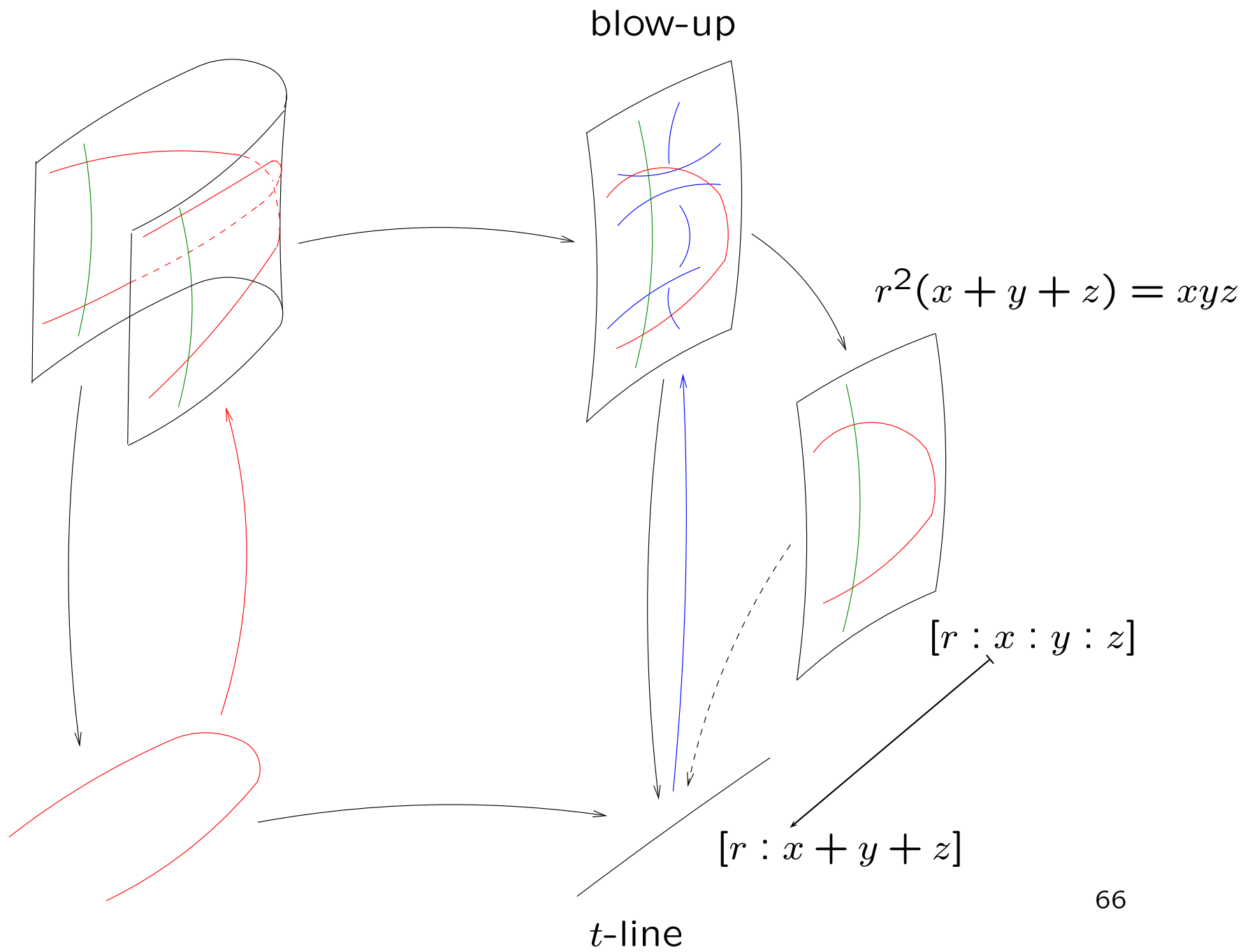
- Where are the surfaces from the title of this talk?

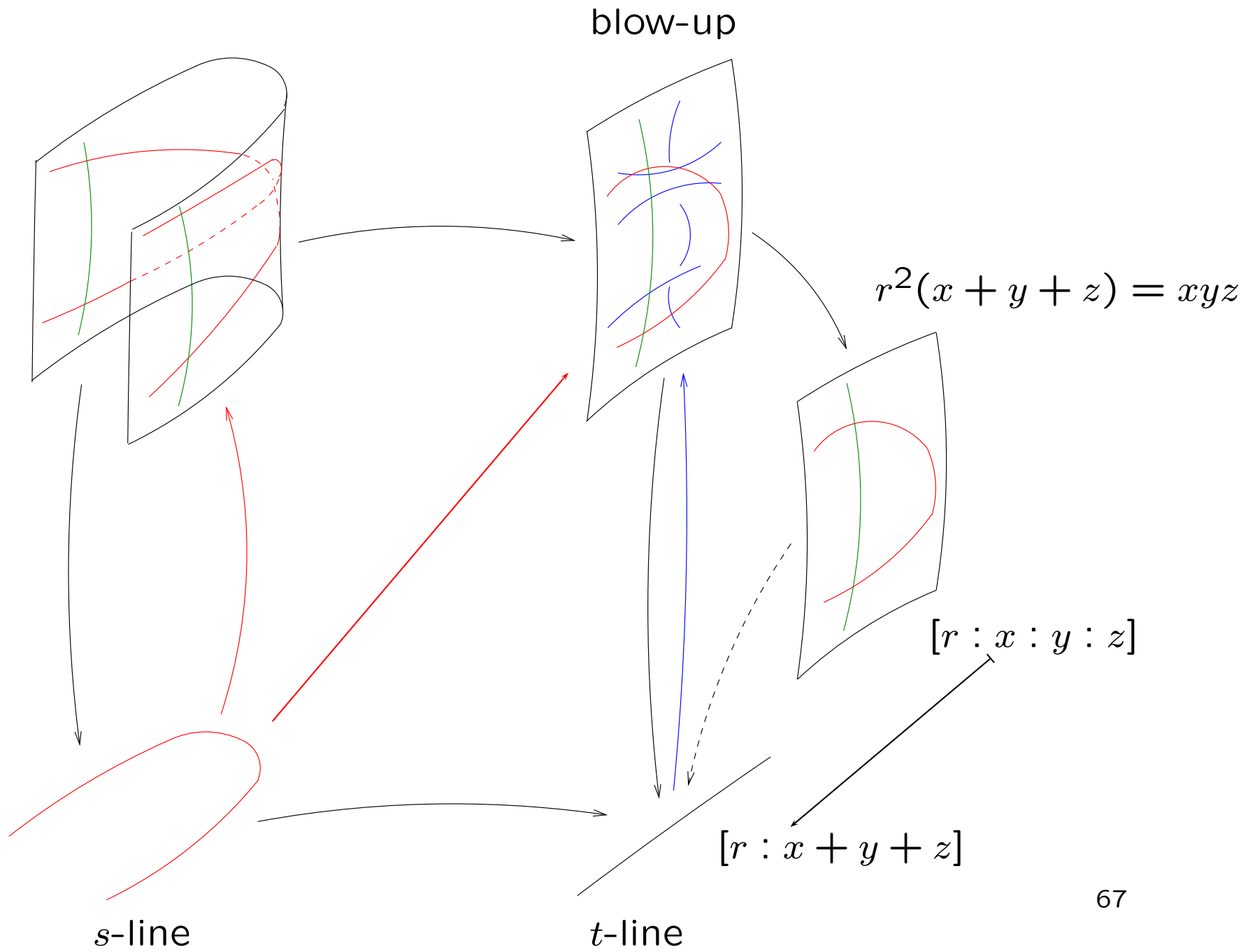


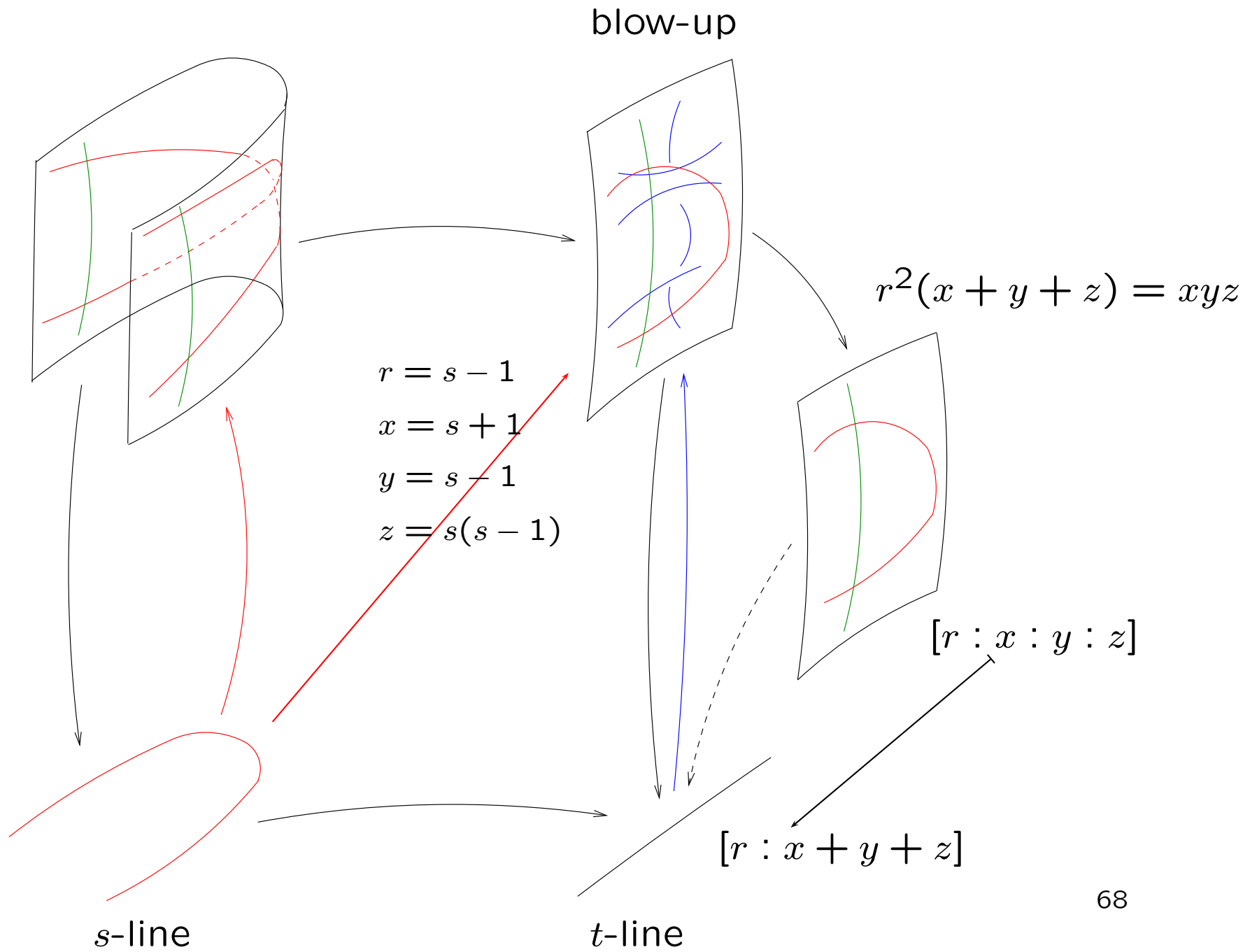


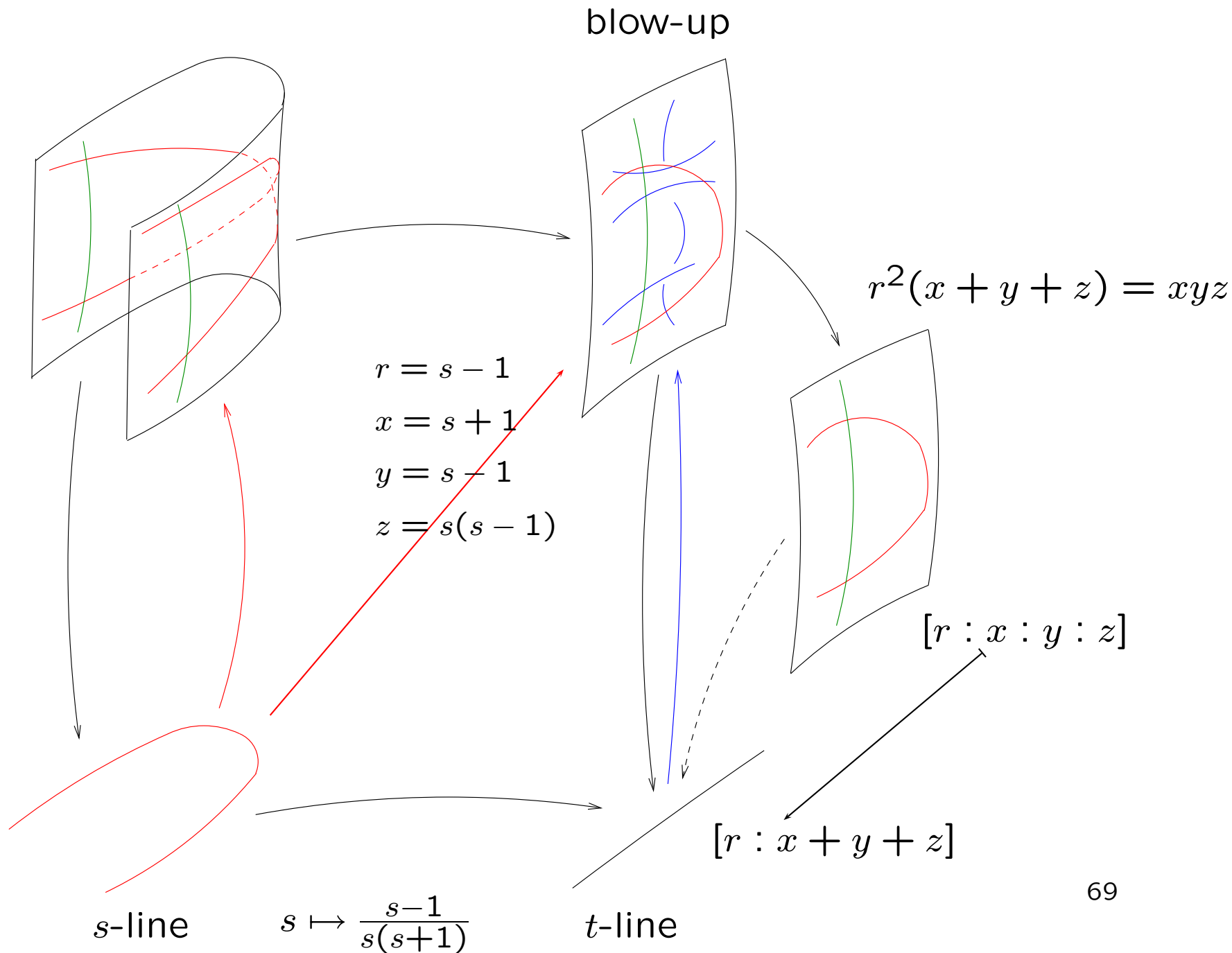








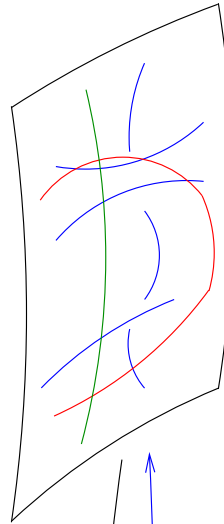
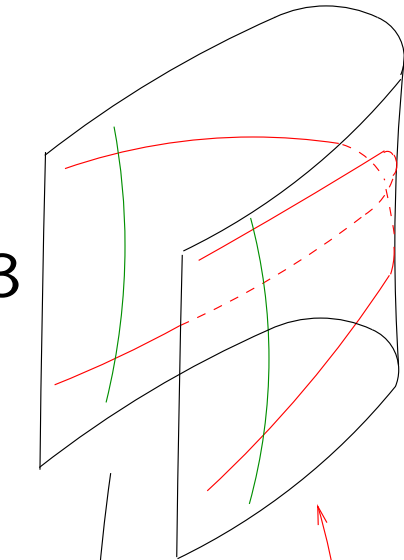




fibered product

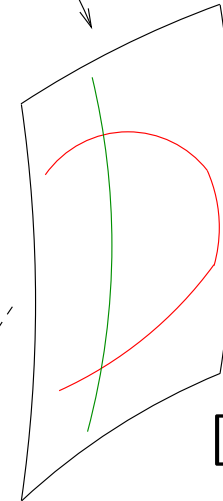
blow-up

K3

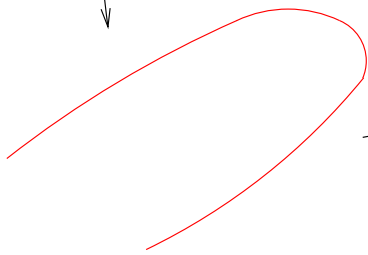


$$r^2(x + y + z) = xyz$$

$$\begin{aligned} r &= s - 1 \\ x &= s + 1 \\ y &= s - 1 \\ z &= s(s - 1) \end{aligned}$$

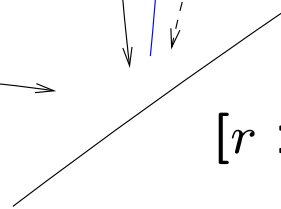


$$[r : x : y : z]$$



s-line

$$s \mapsto \frac{s-1}{s(s+1)}$$



t-line

$$[r : x + y + z]$$

K3 surfaces are 2-dimensional Calabi-Yau manifolds.

Examples of K3 surfaces are smooth quartic surfaces in  $\mathbb{P}^3$ .

We showed that the K3 surface has infinitely many fibers with infinitely many rational points.

The set of rational points is dense on the K3 surface.

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**Open Problem:**

Is there a K3 surface on which the set of rational points is neither empty, nor dense?



Swinnerton-Dyer suspected perhaps the quartic surface

$$x^4 + 2y^4 = z^4 + 4w^4$$

has only 2 rational points.

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However, in 2004 Elsenhans and Jahnel found

$$1484801^4 + 2 \cdot 1203120^4 = 1169407^4 + 4 \cdot 1157520^4$$