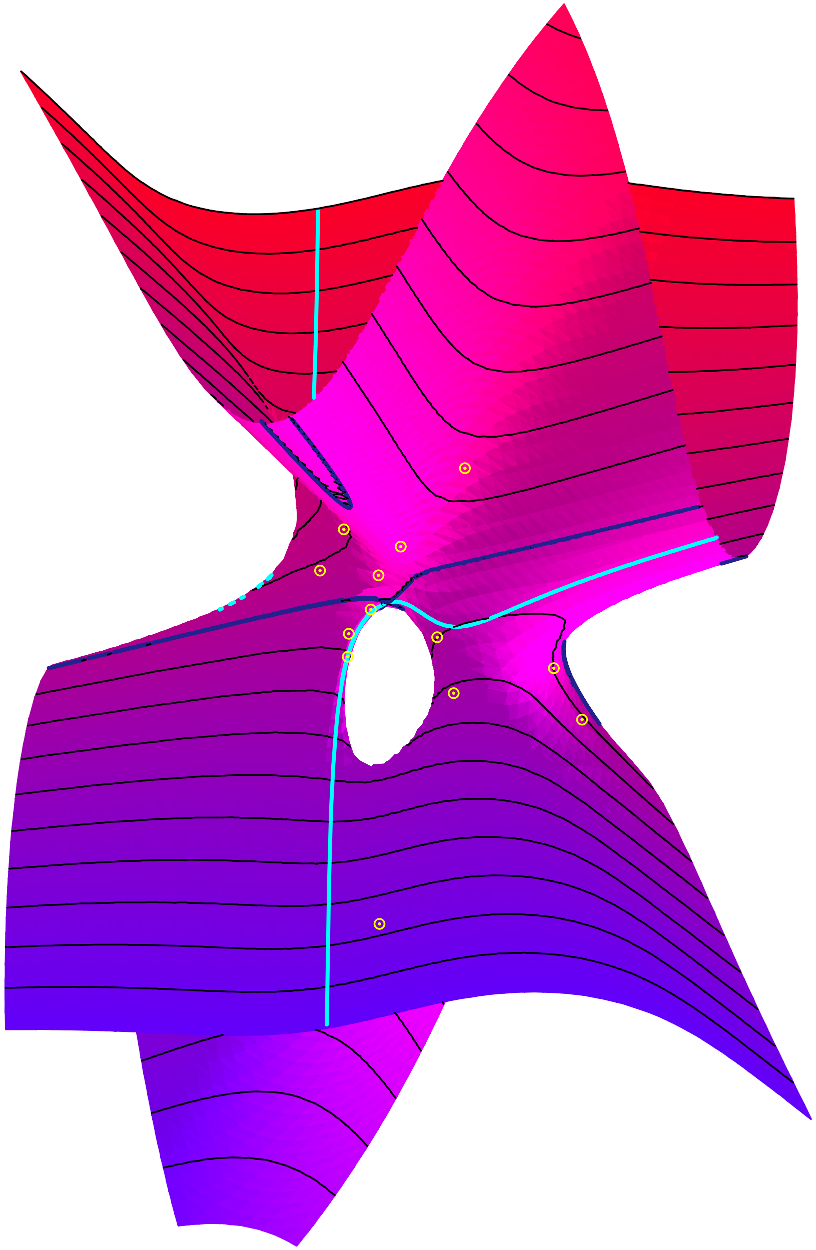
Manin Conjectures for K3 surfaces

Ronald van Luijk
CRM, Montreal
MSRI, Berkeley
PIMS, UBC/SFU

May 17, 2006
Banff



A K3 surface

with 2 singular curves
of genus 0 and
several rational points

Question 1. *Is there a K3 surface, defined over a number field, such that its set of rational points is neither empty, nor dense?*

Question 1. *Is there a K3 surface, defined over a number field, such that its set of rational points is neither empty, nor dense?*

Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3$.

Question 2 (Swinnerton-Dyer).
Does $X \subset \mathbb{P}^3$ *given by*

$$x^4 + 2y^4 = z^4 + 4w^4$$

have more than 2 rational points?

Question 1. *Is there a K3 surface, defined over a number field, such that its set of rational points is neither empty, nor dense?*

Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3$.

Question 2 (Swinnerton-Dyer).
Does $X \subset \mathbb{P}^3$ given by

$$x^4 + 2y^4 = z^4 + 4w^4$$

have more than 2 rational points?

Answer (Elsenhans, Jahnel, 2004):

$$1484801^4 + 2 \cdot 1203120^4 \ = \ 1169407^4 + 4 \cdot 1157520^4$$

We will look at the growth of the number of rational points of bounded height. Consider a surface X/K, choose a height H, and set

$$N_U(B) = \#\{x \in U(K) \ : \ H(x) \leq B\}.$$

Conjecture 1 (Batyrev, Manin). *Let X be a smooth, geometrically integral, projective variety over a number field K, and let D be a hyperplane section. Assume that the canonical sheaf K_X satisfies $-K_X = aD$ for some $a > 0$. Then there exists a finite field extension L, a constant C, and an open subset $U \subset X$, such that with $b = \mathrm{rk}\,\mathrm{Pic}(X_L)$ we have*

$$N_{U_L}(B) \ \approx \ CB^a(\log B)^{b-1}.$$

We will look at the growth of the number of rational points of bounded height. Consider a surface X/K, choose a height H, and set

$$N_U(B) = \#\{x \in U(K) \ : \ H(x) \leq B\}.$$

Conjecture 1 (Batyrev, Manin). *Let X be a smooth, geometrically integral, projective variety over a number field K, and let D be a hyperplane section. Assume that the canonical sheaf K_X satisfies $-K_X = aD$ for some $a > 0$. Then there exists a finite field extension L, a constant C, and an open subset $U \subset X$, such that with $b = \mathrm{rk}\,\mathrm{Pic}(X_L)$ we have*

$$N_{U_L}(B) \ \approx \ CB^a(\log B)^{b-1}.$$

What if $a = 0$, in particular, if X is K3?

Define the height-zeta function

$$Z(U,s) = \sum_{x \in U(K)} H(x)^{-s}.$$

From

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s \, \frac{ds}{s} = \begin{cases} 1 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases} \qquad (c > 0)$$

we get

$$N(U,x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(U,s) \, x^s \, \frac{ds}{s} \qquad (c >> 0)$$

Assuming $Z(U,s)$ is analytic on $\Re(s) > a - \epsilon$, except for a pole of order b at a, we can write

$$N(U,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z(U,s)\, x^s \frac{ds}{s} \qquad (c >> 0)$$

$$= \mathrm{res}_{s=a}\left[Z(U,s)s^{-1}\exp(s\log x)\right] + \frac{1}{2\pi i}\int_{a-\epsilon-i\infty}^{a-\epsilon+i\infty} Z(U,s)\, x^s \frac{ds}{s}.$$

Assuming $Z(U,s)$ is analytic on $\Re(s) > a - \epsilon$, except for a pole of order b at a, we can write

$$N(U,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z(U,s)\, x^s \,\frac{ds}{s} \qquad (c >> 0)$$
$$= \mathrm{res}_{s=a}\left[Z(U,s)s^{-1}\exp(s\log x)\right] + \frac{1}{2\pi i}\int_{a-\epsilon-i\infty}^{a-\epsilon+i\infty} Z(U,s)\, x^s \,\frac{ds}{s}.$$

The main term is

$$x^a p(\log x)$$

for some polynomial p of degree $\begin{cases} b-1 & \text{if } a \neq 0, \\ b & \text{if } a = 0. \end{cases}$

Assuming $Z(U,s)$ is analytic on $\Re(s) > a - \epsilon$, except for a pole of order b at a, we can write

$$\begin{aligned} N(U,x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z(U,s)\, x^s \,\frac{ds}{s} \qquad (c >> 0) \\ &= \mathrm{res}_{s=a}\left[Z(U,s)s^{-1}\exp(s\log x)\right] + \frac{1}{2\pi i}\int_{a-\epsilon-i\infty}^{a-\epsilon+i\infty} Z(U,s)\, x^s\,\frac{ds}{s}. \end{aligned}$$

The main term is

$$x^a p(\log x)$$

for some polynomial p of degree $\begin{cases} b-1 & \text{if } a \neq 0, \\ b & \text{if } a = 0. \end{cases}$

For X a K3 surface: $N(U,B) \sim C(\log B)^{\mathrm{rk}\,\mathrm{Pic}\,X}$?

For X a K3 surface: $N(U,B) \sim C(\log B)^{\operatorname{rk}\operatorname{Pic}X}$?

Not if X admits an elliptic fibration (in particular, if $\operatorname{rk}\operatorname{Pic}X \geq 5$).

For X a K3 surface: $N(U,B) \sim C(\log B)^{\operatorname{rk}\operatorname{Pic} X}$?

Not if X admits an elliptic fibration (in particular, if $\operatorname{rk}\operatorname{Pic} X \geq 5$).

More problems:

- infinitely many curves of genus 0 or 1,
- infinitely many automorphisms (cf. A. Baragar),
- Swinnerton-Dyer's surface has very slow growth,
- which height to use?

For X a K3 surface: $N(U,B) \sim C(\log B)^{\operatorname{rk}\operatorname{Pic} X}$?

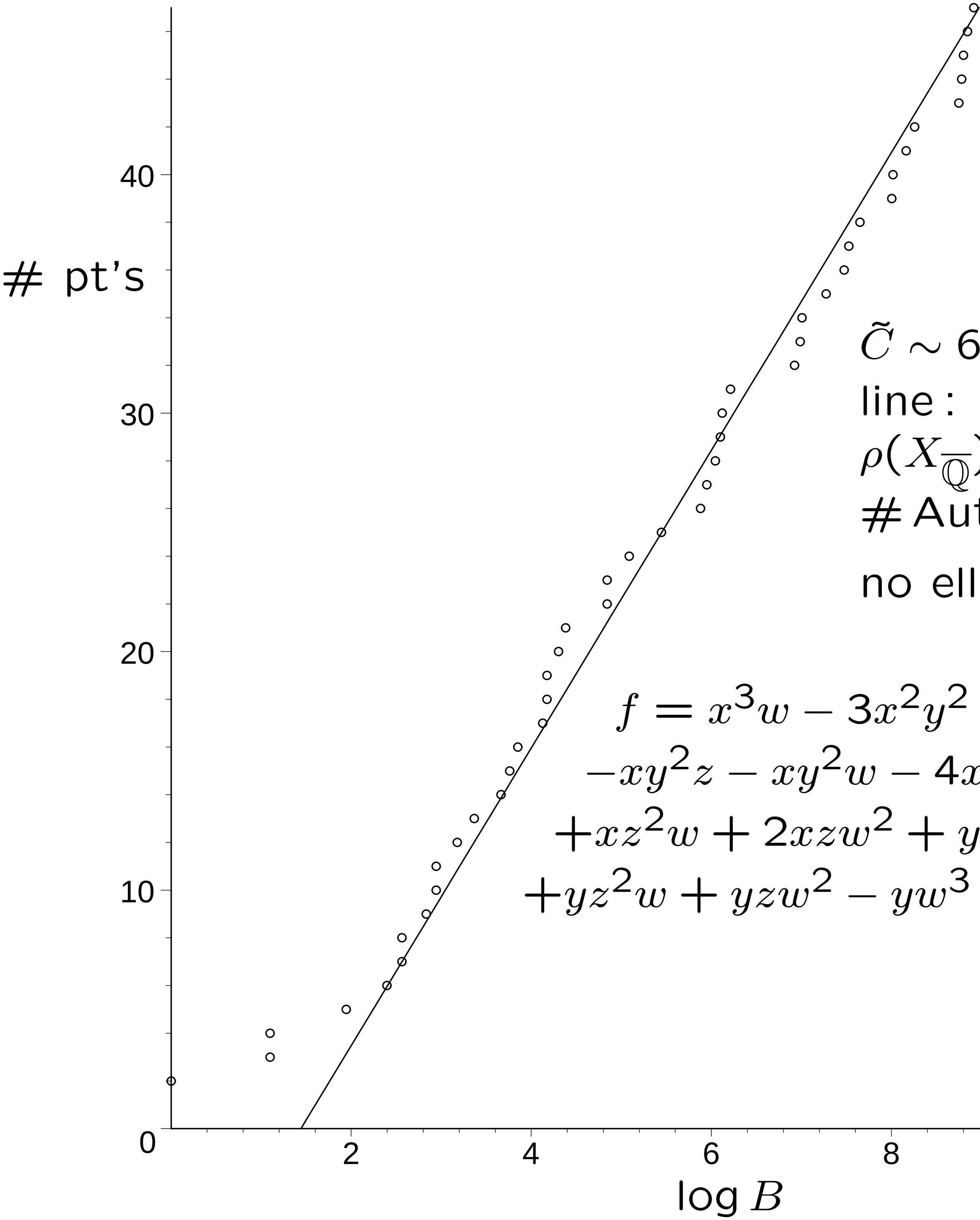
Not if X admits an elliptic fibration (in particular, if $\operatorname{rk}\operatorname{Pic} X \geq 5$).

More problems:

- infinitely many curves of genus 0 or 1,
- infinitely many automorphisms (cf. A. Baragar),
- Swinnerton-Dyer's surface has very slow growth,
- which height to use?

We will run experiments on quartic surfaces, comparing exponents, and the main coefficient C to a relatively naive heuristic coefficient $\tilde{C}$ (part of Peyre's constant for del Pezzo).

There are some subtle rational numbers that C and $\tilde{C}$ may be off by.



$\tilde{C} \sim 6.24$

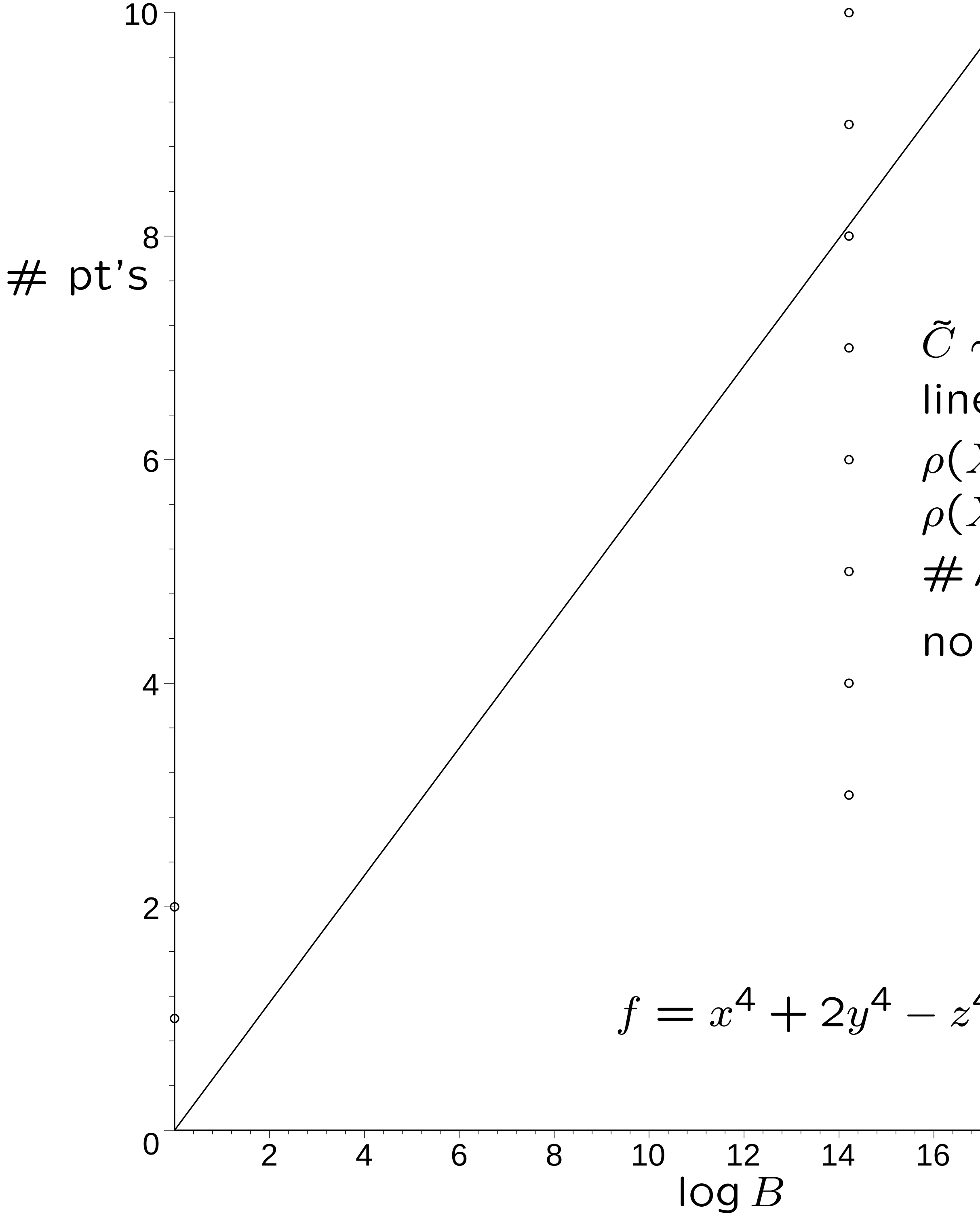
line: $\quad \tilde{C}(\log B) - 9$

$\rho(X_{\overline{\mathbb{Q}}}) = 1$

$\# \operatorname{Aut} X_{\overline{\mathbb{Q}}} < \infty$

no elliptic fibration

$$\begin{aligned} f = {} & x^3w - 3x^2y^2 - x^2yw - x^2zw + x^2w^2 \\ & -xy^2z - xy^2w - 4xyz^2 - xyzw + 2xyw^2 - 2xz^3 \\ & +xz^2w + 2xzw^2 + y^3w + y^2zw - y^2w^2 - 5yz^3 \\ & +yz^2w + yzw^2 - yw^3 - z^4 + z^2w^2 + zw^3 + 2w^4 \end{aligned}$$



$\tilde{C} \sim 0.57$

line: $\tilde{C}(\log B)$

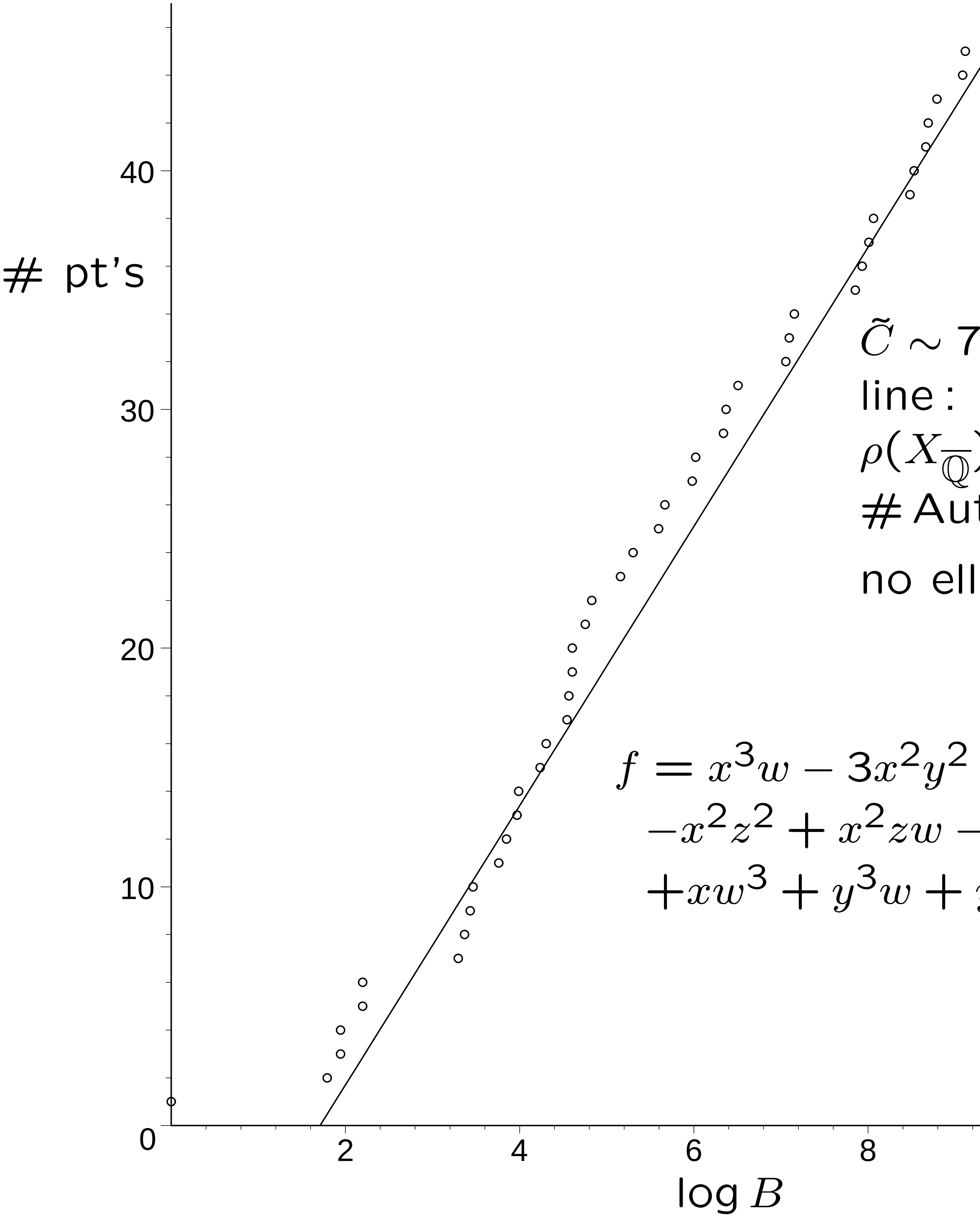
$\rho(X) = 1$

$\rho(X_{\overline{\mathbb{Q}}}) = 20$

$\# \operatorname{Aut} X_{\overline{\mathbb{Q}}} = \infty$

no elliptic fibration over $\mathbb{Q}$

$$f = x^4 + 2y^4 - z^4 - 4w^4$$



$\tilde{C} \sim 7.8$

line: $\frac{3}{4}\tilde{C}(\log B) - 10$

$\rho(X_{\overline{\mathbb{Q}}}) = 1$

$\# \operatorname{Aut} X_{\overline{\mathbb{Q}}} < \infty$

no elliptic fibration

$$f = x^3w - 3x^2y^2 + 4x^2yz - x^2z^2$$
$$-x^2z^2 + x^2zw - xy^2z - xyz^2$$
$$+xw^3 + y^3w + y^2z^2 + z^3w$$

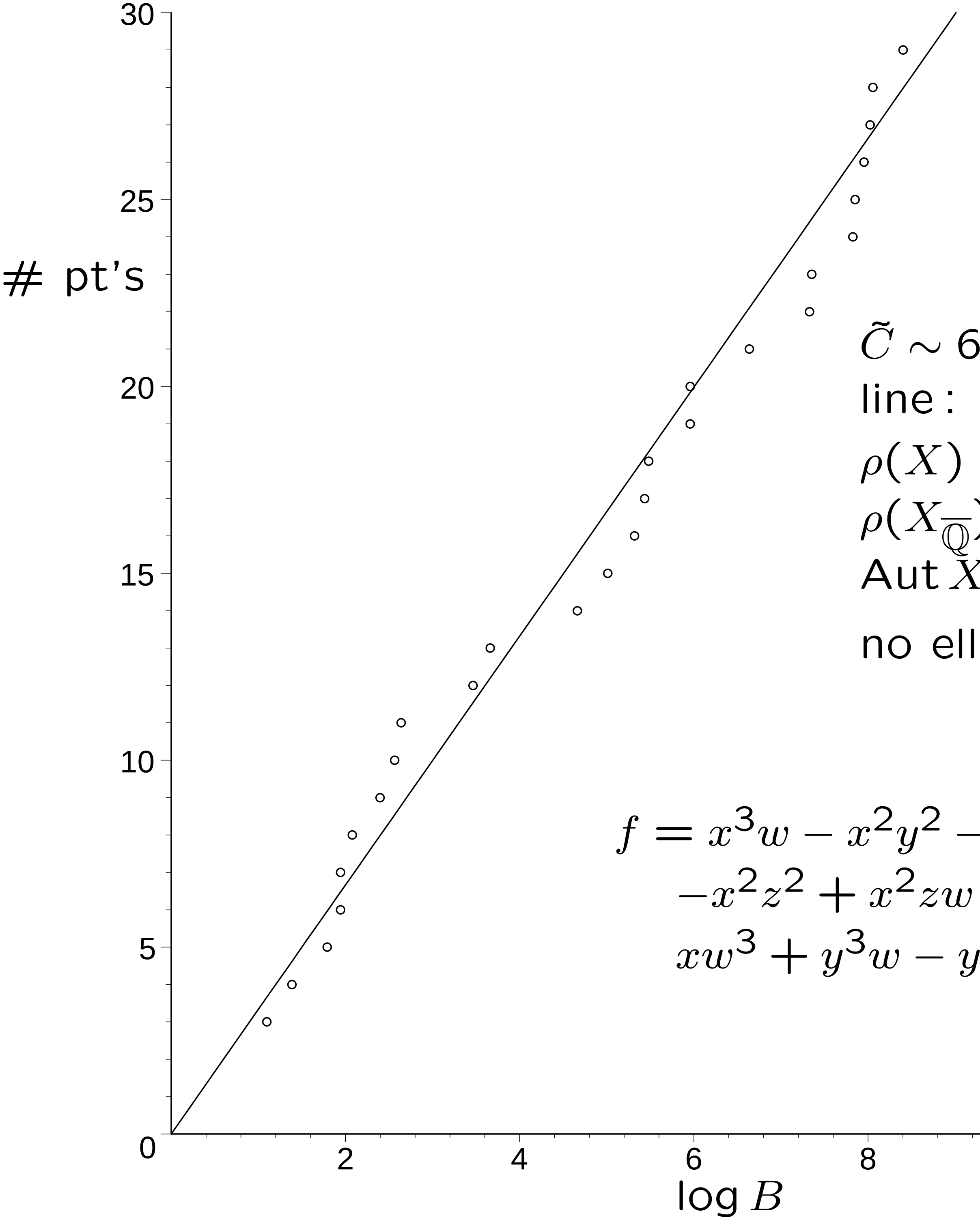
Consider X given by

$$f + 6m(w^4 + 2x^4 - (y+w)^4 - 4z^4) \quad \text{for} \quad m \in \{\pm 1, \pm 2, \pm 3\}$$

with f as on the previous page.

Then X has geometric Picard number 1.

m	$\tilde{C}$	$\tilde{C} \cdot \log(12000)$	$\#\{x : H(x) \leq 12000\}$
0	7.8	73	46
1	0.30	2.8	5
-1	0.58	5.4	9
2	0.39	3.7	2
-2	0.35	3.3	6
3	0.18	1.7	4
-3	0.26	2.4	4



$\tilde{C} \sim 6.66$

line: $\frac{1}{2}\tilde{C}(\log B)$

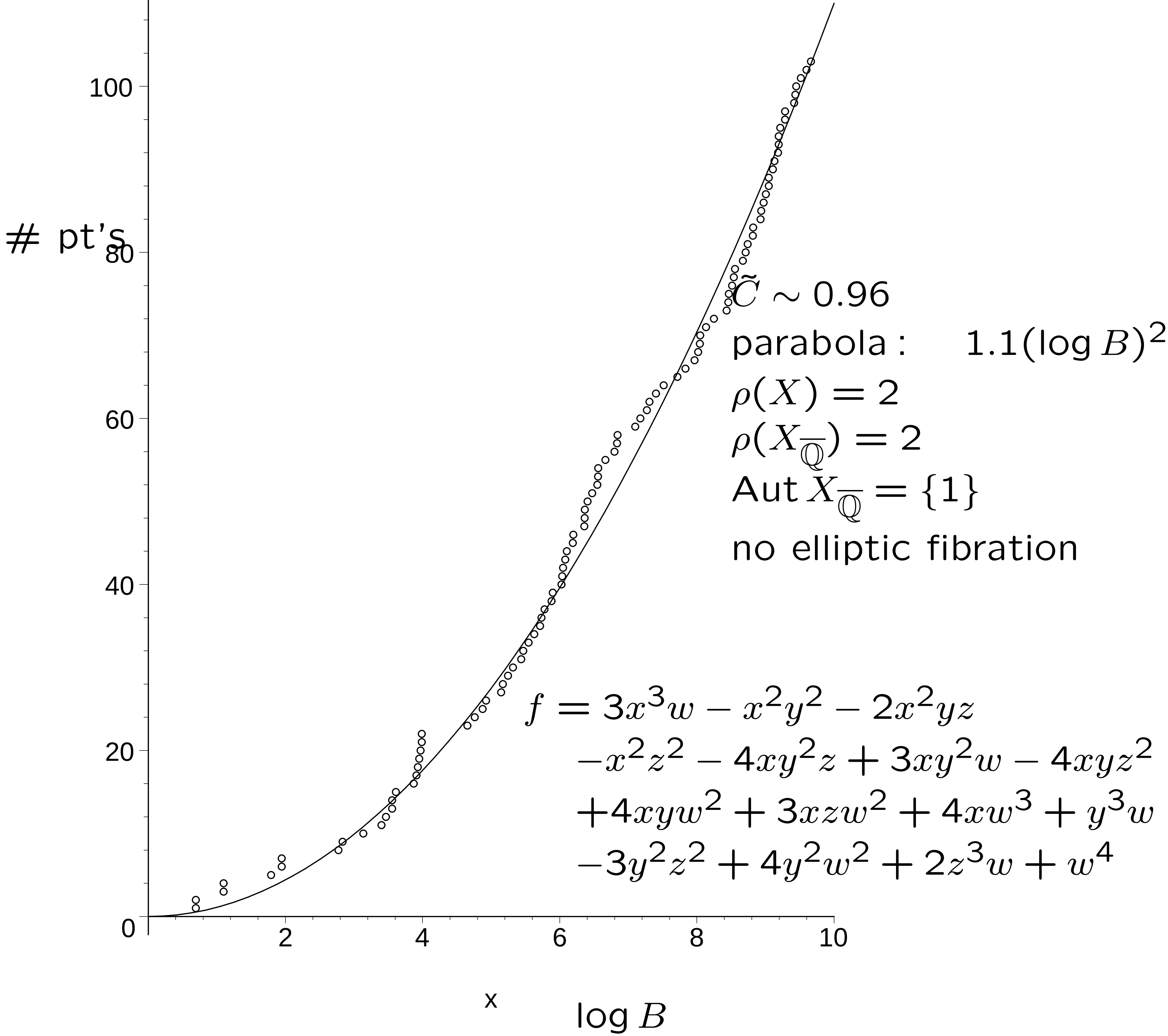
$\rho(X) = 1$

$\rho(X_{\overline{\mathbb{Q}}}) = 2$ (over $\mathbb{Q}(\sqrt{3})$)

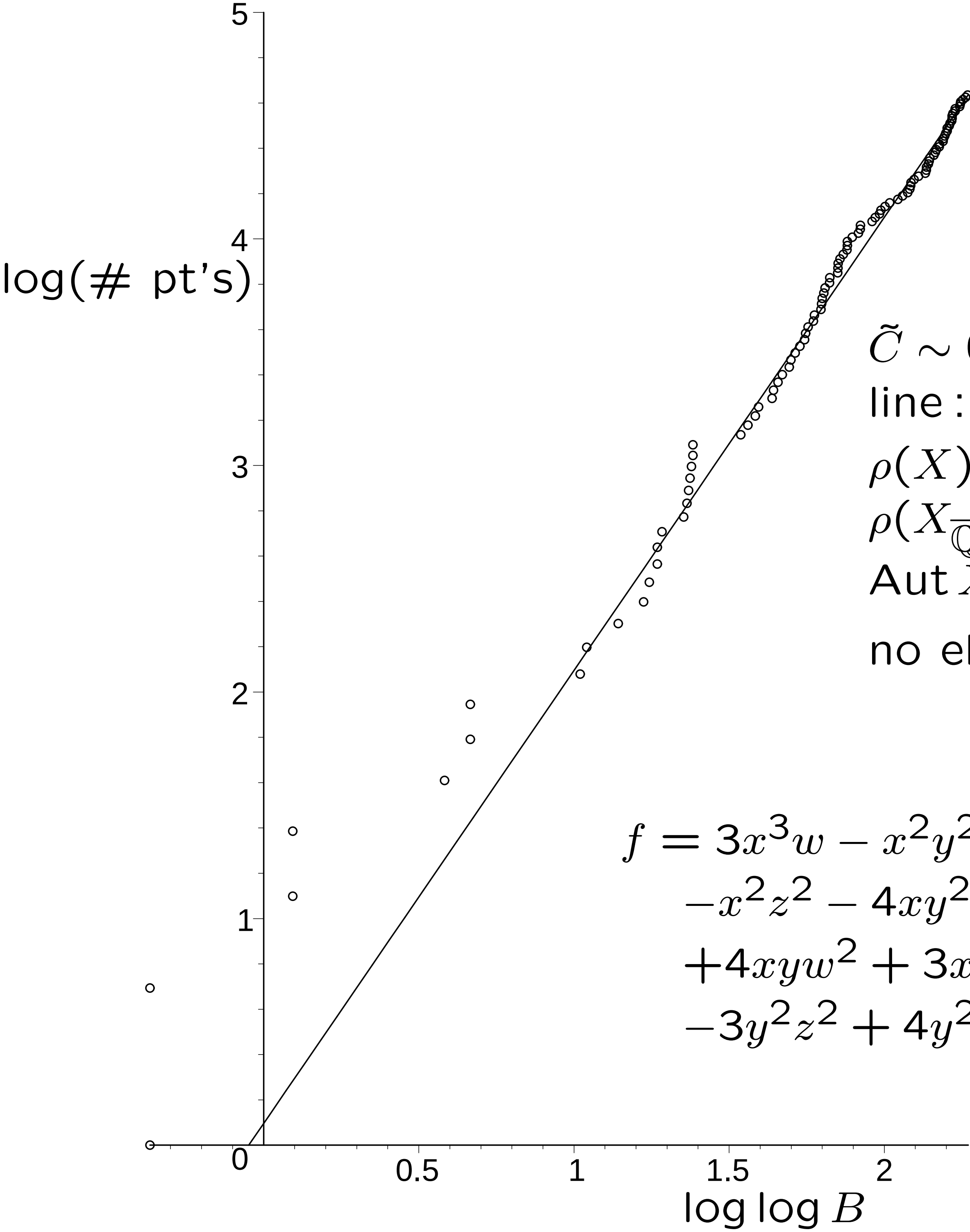
$\mathrm{Aut}\, X_{\overline{\mathbb{Q}}} = \{1\}$

no elliptic fibration

$$f = x^3w - x^2y^2 - 2x^2yz$$
$$-x^2z^2 + x^2zw + xy^2z + xyz^2$$
$$xw^3 + y^3w - y^2z^2 + z^3w$$



pt's
100
80
60
40
20
0
2
4
6
8
10
x
log B
$\tilde{C} \sim 0.96$
parabola : $1.1(\log B)^2$
$\rho(X) = 2$
$\rho(X_{\overline{\mathbb{Q}}}) = 2$
$\mathrm{Aut}\, X_{\overline{\mathbb{Q}}} = \{1\}$
no elliptic fibration
$f = 3x^3w - x^2y^2 - 2x^2yz$
$-x^2z^2 - 4xy^2z + 3xy^2w - 4xyz^2$
$+4xyw^2 + 3xzw^2 + 4xw^3 + y^3w$
$-3y^2z^2 + 4y^2w^2 + 2z^3w + w^4$



$\tilde{C} \sim 0.96$

line: $2(\log\log B) + \log 1.1$

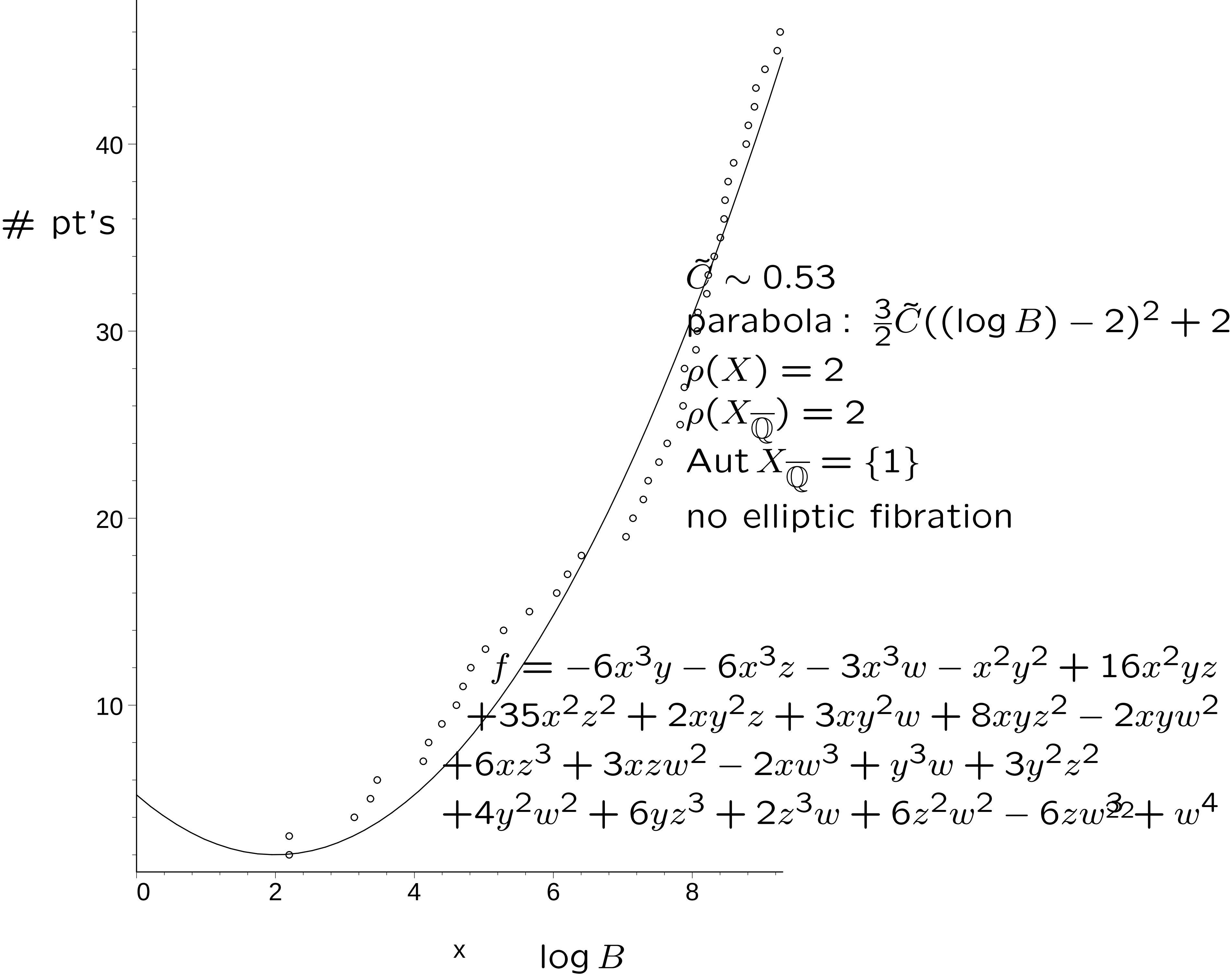
$\rho(X) = 2$

$\rho(X_{\overline{\mathbb{Q}}}) = 2$

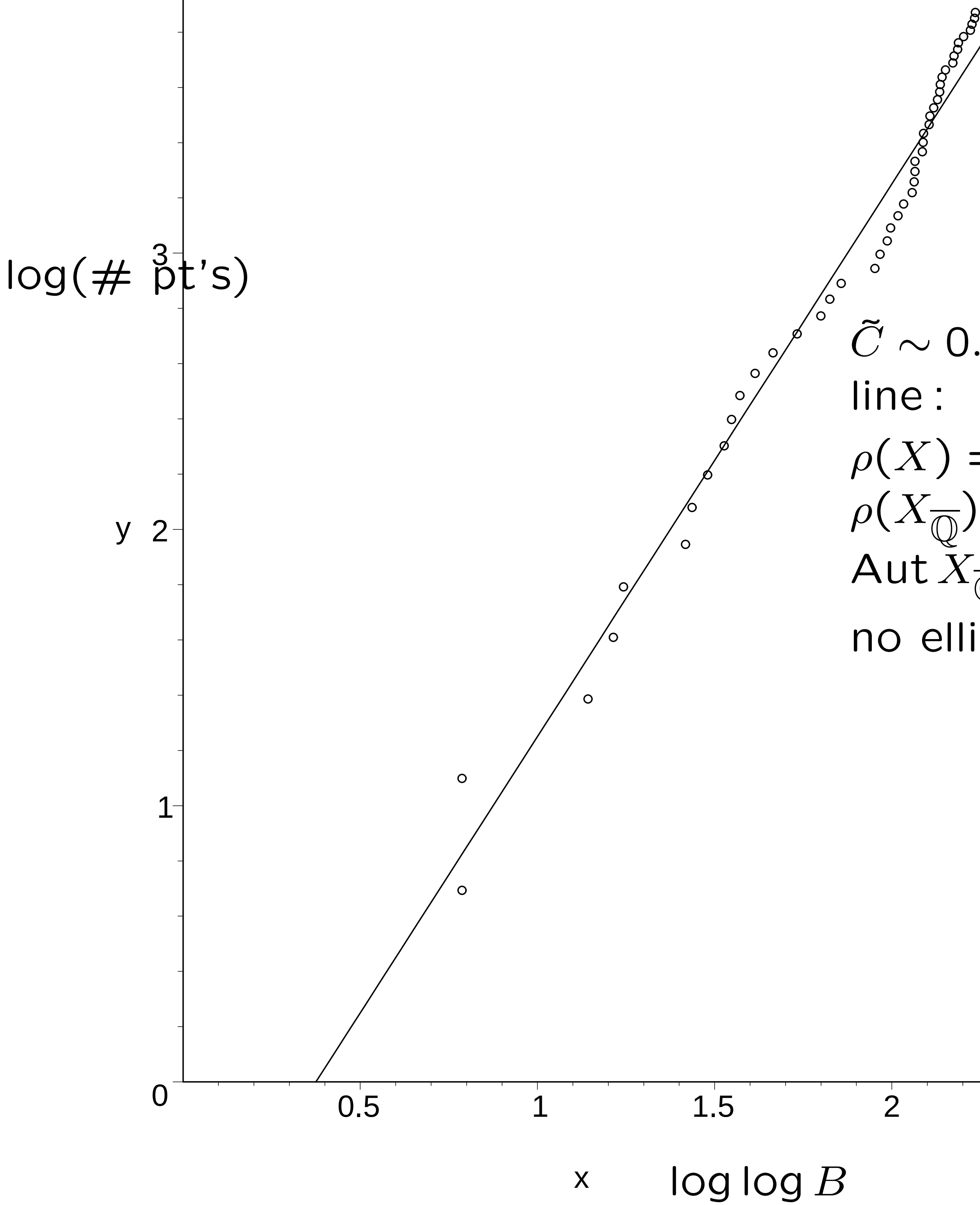
$\operatorname{Aut} X_{\overline{\mathbb{Q}}} = \{1\}$

no elliptic fibration

$$\begin{aligned} f = {} & 3x^3w - x^2y^2 - 2x^2yz \\ & -x^2z^2 - 4xy^2z + 3xy^2w - 4xyz^2 \\ & +4xyw^2 + 3xzw^2 + 4xw^3 + y^3w \\ & -3y^2z^2 + 4y^2w^2 + 2z^3w + w^4 \end{aligned}$$



pt's
40
30
20
10
0
2
4
6
8
x
$\log B$
$\tilde{C} \sim 0.53$
parabola: $\frac{3}{2}\tilde{C}((\log B) - 2)^2 + 2$
$\rho(X) = 2$
$\rho(X_{\overline{\mathbb{Q}}}) = 2$
$\mathrm{Aut}\, X_{\overline{\mathbb{Q}}} = \{1\}$
no elliptic fibration
$f = -6x^3y - 6x^3z - 3x^3w - x^2y^2 + 16x^2yz$
$+35x^2z^2 + 2xy^2z + 3xy^2w + 8xyz^2 - 2xyw^2$
$+6xz^3 + 3xzw^2 - 2xw^3 + y^3w + 3y^2z^2$
$+4y^2w^2 + 6yz^3 + 2z^3w + 6z^2w^2 - 6zw^3 + w^4$



$\tilde{C} \sim 0.53$

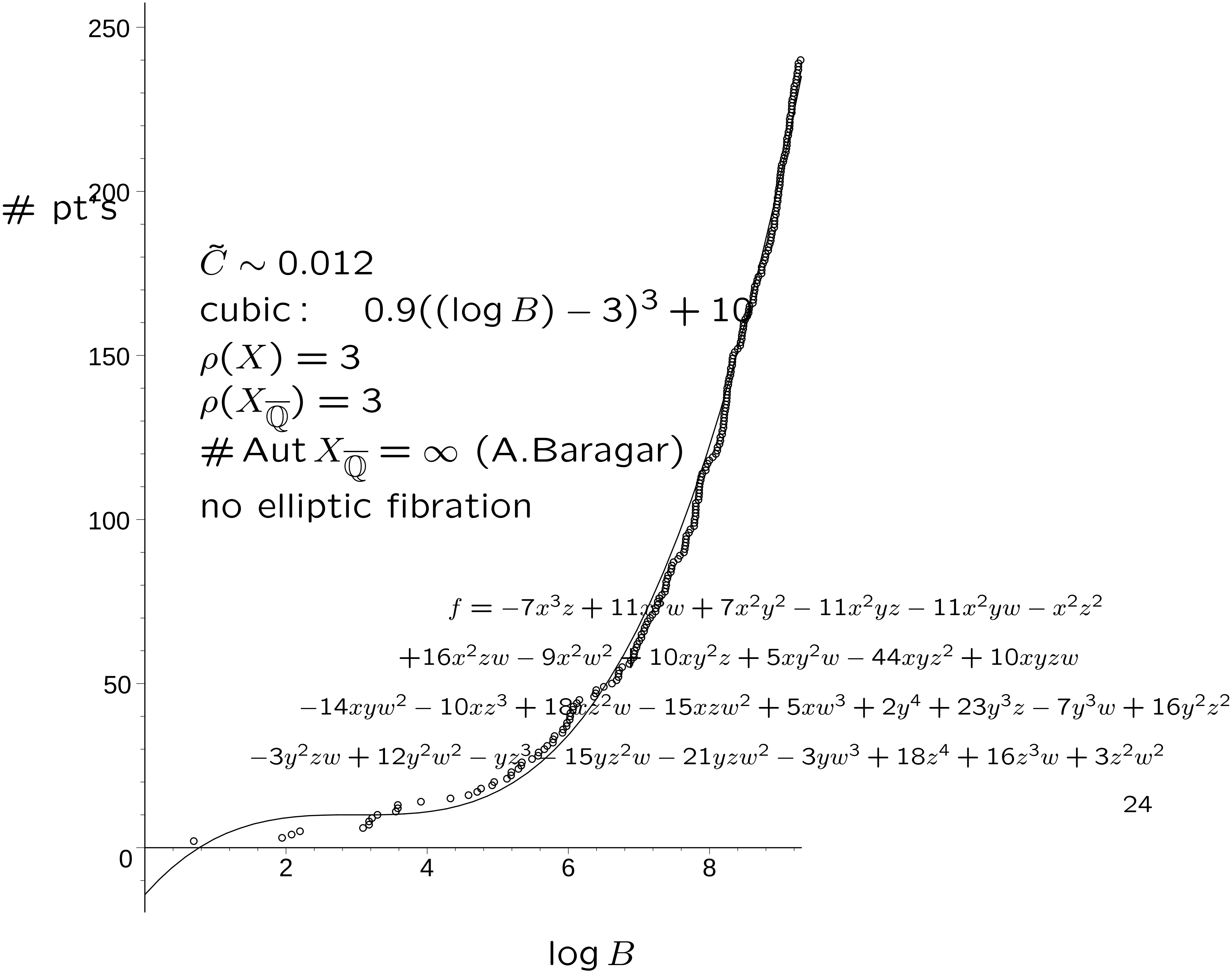
line: $2(\log\log B) - 0.75$

$\rho(X) = 2$

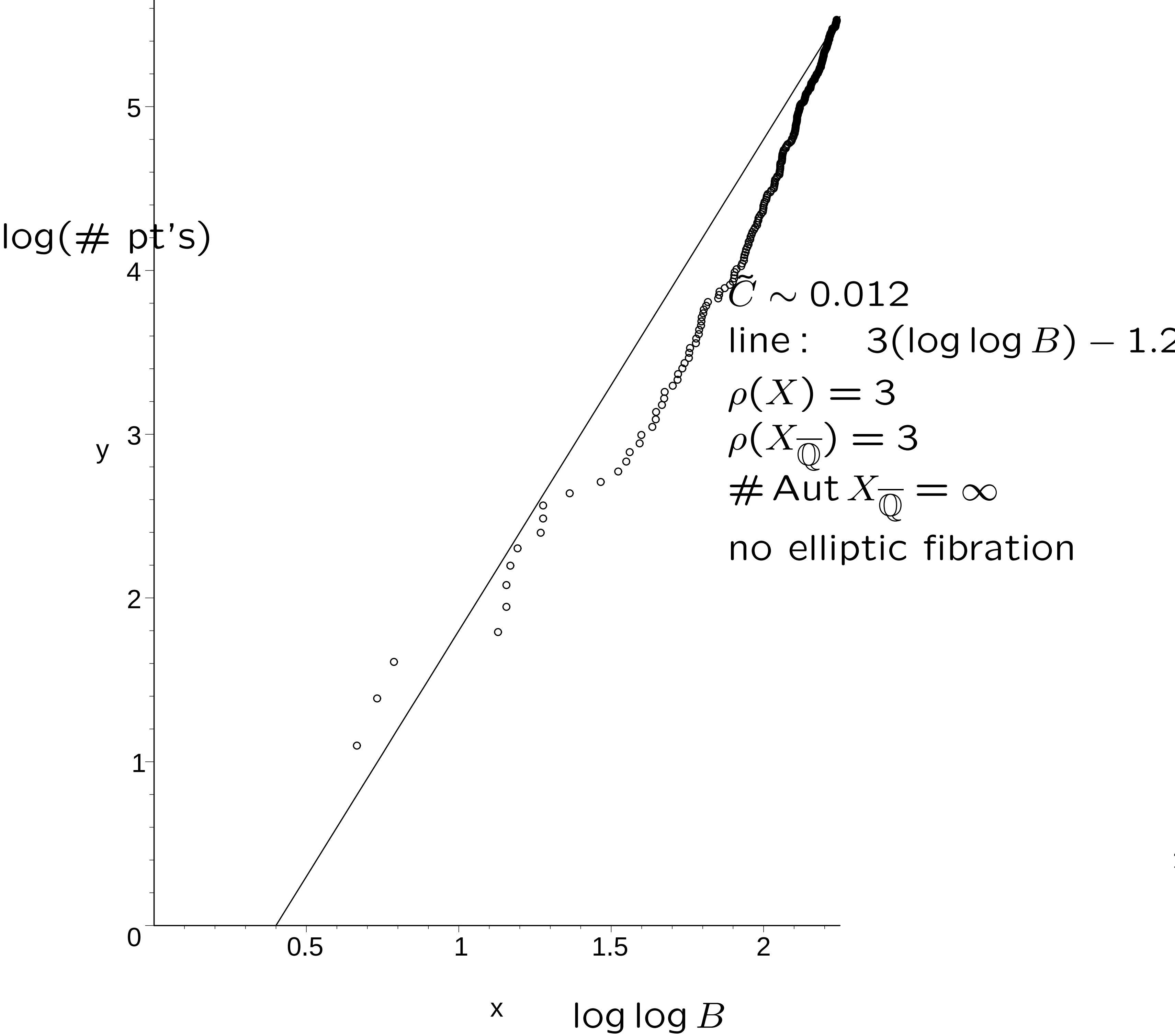
$\rho(X_{\overline{\mathbb{Q}}}) = 2$

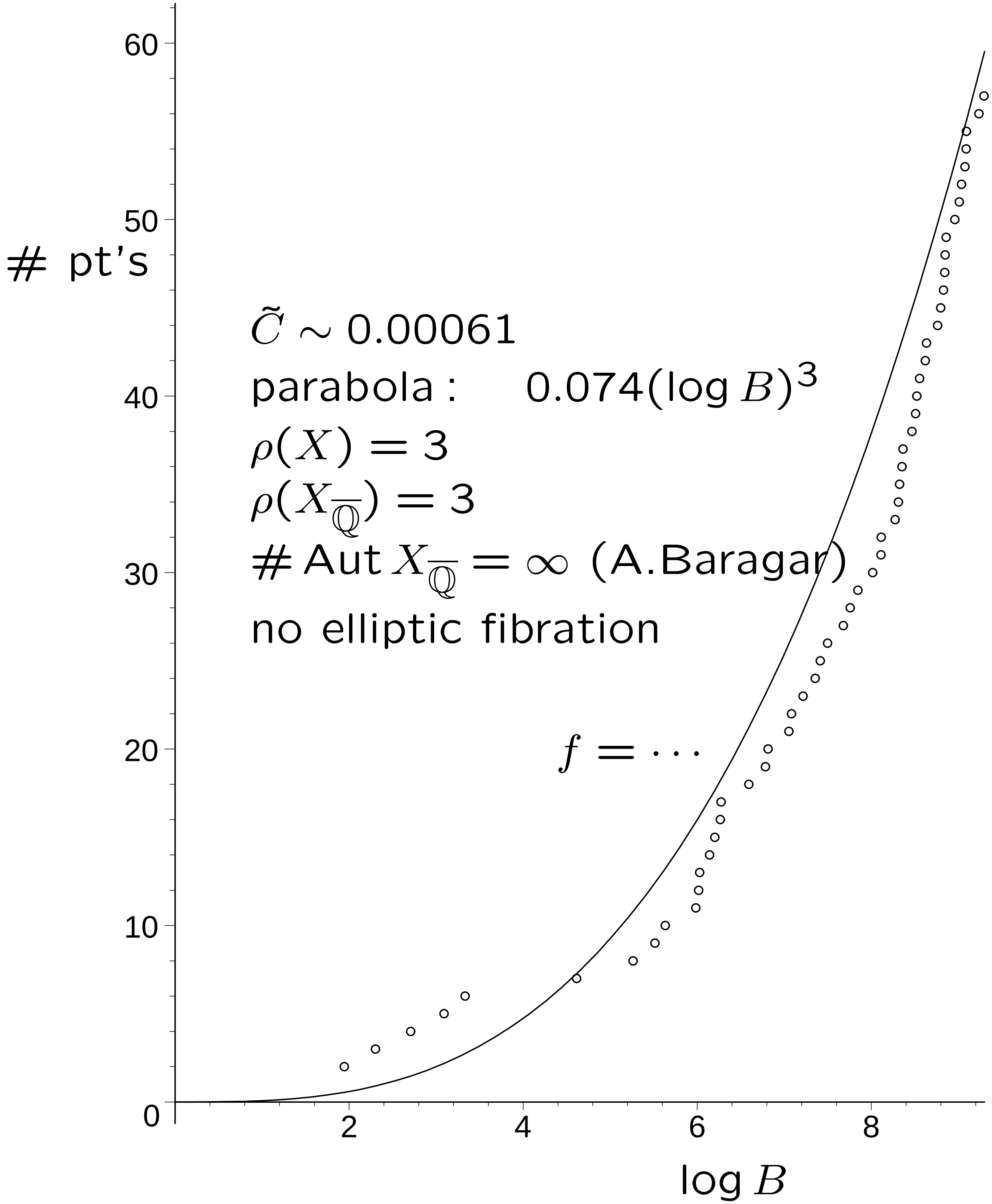
$\mathrm{Aut}\, X_{\overline{\mathbb{Q}}} = \{1\}$

no elliptic fibration

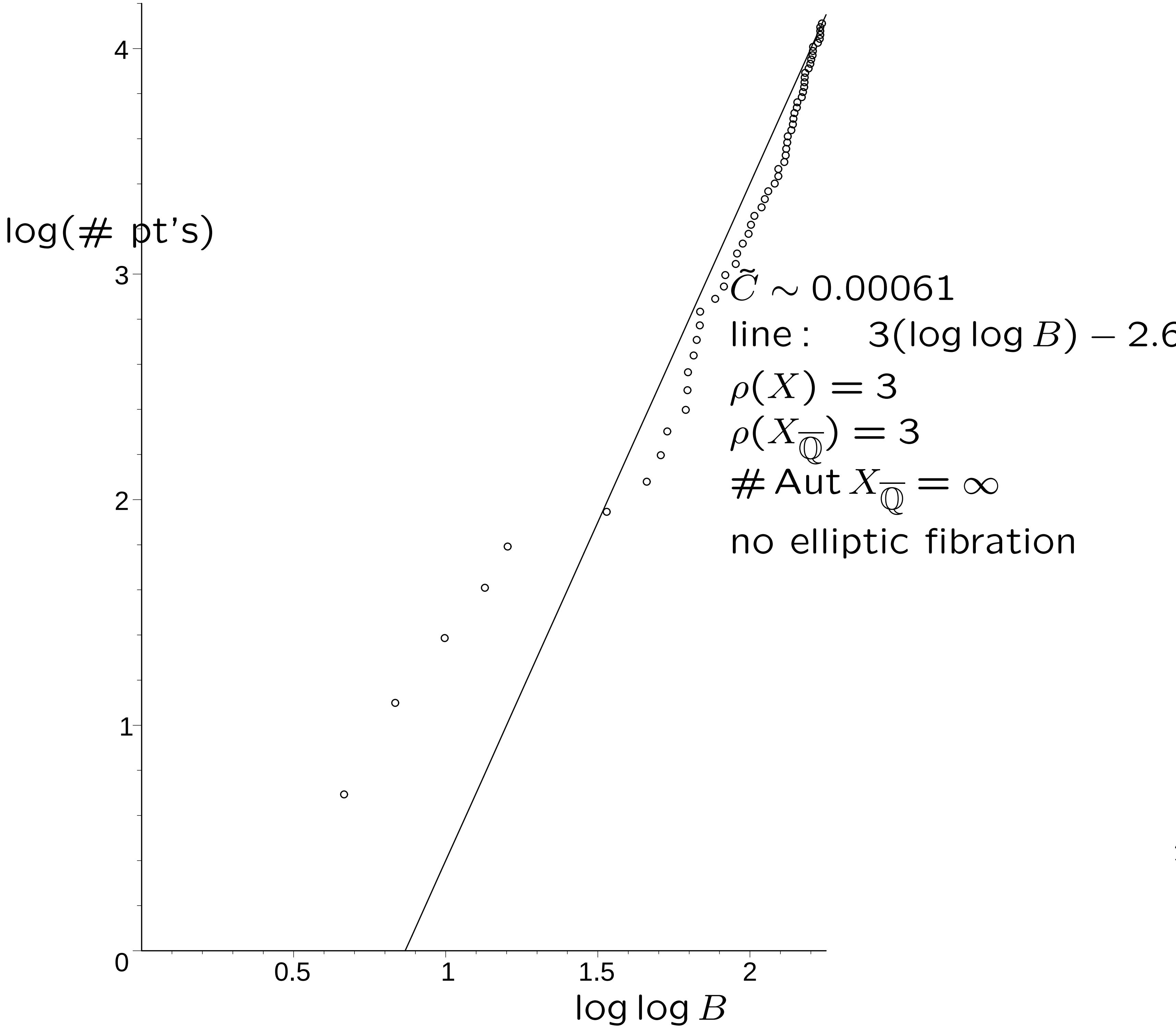


pt's
250
200
150
100
50
0
2
4
6
8
$\tilde{C} \sim 0.012$
cubic: $0.9((\log B) - 3)^3 + 10$
$\rho(X) = 3$
$\rho(X_{\overline{\mathbb{Q}}}) = 3$
$\# \operatorname{Aut} X_{\overline{\mathbb{Q}}} = \infty$ (A.Baragar)
no elliptic fibration
$f = -7x^3z + 11x^3w + 7x^2y^2 - 11x^2yz - 11x^2yw - x^2z^2$
$+16x^2zw - 9x^2w^2 + 10xy^2z + 5xy^2w - 44xyz^2 + 10xyzw$
$-14xyw^2 - 10xz^3 + 18xz^2w - 15xzw^2 + 5xw^3 + 2y^4 + 23y^3z - 7y^3w + 16y^2z^2$
$-3y^2zw + 12y^2w^2 - yz^3 - 15yz^2w - 21yzw^2 - 3yw^3 + 18z^4 + 16z^3w + 3z^2w^2$
$\log B$


log(# pt's)
y
x
log log B
$\widetilde{C} \sim 0.012$
line: $3(\log\log B) - 1.2$
$\rho(X) = 3$
$\rho(X_{\overline{\mathbb{Q}}}) = 3$
$\#\,\mathrm{Aut}\,X_{\overline{\mathbb{Q}}} = \infty$
no elliptic fibration
0
1
2
3
4
5
0.5
1
1.5
2



pt's
$\tilde{C} \sim 0.00061$
parabola : $0.074(\log B)^3$
$\rho(X) = 3$
$\rho(X_{\overline{\mathbb{Q}}}) = 3$
$\# \operatorname{Aut} X_{\overline{\mathbb{Q}}} = \infty$ (A.Baragar)
no elliptic fibration
$f = \cdots$
60
50
40
30
20
10
0
2
4
6
8
log B


log(# pt's)
log log B
$\tilde{C} \sim 0.00061$
line: $3(\log\log B) - 2.6$
$\rho(X) = 3$
$\rho(X_{\overline{\mathbb{Q}}}) = 3$
$\#\,\mathrm{Aut}\,X_{\overline{\mathbb{Q}}} = \infty$
no elliptic fibration