Computing Néron-Severi groups

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Setting

- k a finitely generated field.
- X a nice k-variety (smooth, projective, geometrically integral).
- k^{s} a separable closure of k with Galois group $\Gamma = \mathrm{Gal}(k^{\mathrm{s}}/k)$.
- $X^{\mathrm{s}} = X \times_k k^{\mathrm{s}}$.
- Picard group $\operatorname{Pic} X \subset (\operatorname{Pic} X^{\mathrm{s}})^{\Gamma}$.
- $\operatorname{Pic}^0(X)$ subgroup of classes algebraically equivalent to 0.
- Néron-Severi group $\mathrm{NS}(X) = \operatorname{Pic} X / \operatorname{Pic}^0(X) \subset \mathrm{NS}(X^{\mathrm{s}})^{\Gamma}$.

Goal: Compute $\mathrm{NS}(X^{\mathrm{s}})$ (or its rank).

Special cases

- Elliptic fibrations: map NS to the Mordell-Weil group.
- Fibrations into abelian varieties.
- If a finite group G acts on Y and $X = Y/G$, then

$$\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Q} \to (\mathrm{NS}(Y^{\mathrm{s}}) \otimes \mathbb{Q})^G$$

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  Application (Shioda):
  Delsarte surfaces (given by tetranomials in $\mathbb{P}^3$) are quotients of Fermat surfaces.

K3 surfaces of degree 2

Theorem (Hassett, Kresch, Tschinkel)

There is an algorithm that takes as input a K3 surface X of degree 2 over a number field, and returns $\operatorname{Pic} X^{\mathrm{s}} = \operatorname{NS} X^{\mathrm{s}}$.

Method: Kuga-Satake correspondence.
Ingredients include: abelian variety of dimension 2^{19}.

Tate conjecture(s)

- Fix $0 \leq p \leq \dim X$ and prime $\ell \neq \mathrm{char}\, k$.
- $\mathcal{Z}^p(X)$ is group of codimension-p cycles on X.
- $V^{2p} = \mathrm{H}^{2p}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Q}_\ell(p))$.
- $V^{\mathrm{tate}} \subset V^{2p}$ is set of *Tate classes* (each fixed by some finite-index open subgroup $G \subset \Gamma$).

Conjecture ($T^p(X, \ell)$)

The cycle class map $\mathcal{Z}^p(X^{\mathrm{s}}) \otimes \mathbb{Q}_\ell \to V^{\mathrm{tate}}$ *is surjective.*

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Theorem (Nijgaard, Ogus, Maulik, Charles, Madapusi Pera)

Suppose $\mathrm{char}\, k \neq 2$. If X is a K3 surface, then $T^1(X, \ell)$ holds.

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Conjecture ($E^p(X, \ell)$)

An element in $\mathcal{Z}^p(X^{\mathrm{s}}, \ell)$ is numerically equivalent to 0 if and only if it maps to 0 in V^{2p}.

Remark. $E^p(X, \ell)$ holds for $p = 1$.

Algorithms for general p

- $\mathrm{Num}^p(X)$ is group of codimension-p cycle classes up to numerical equivalence.
- Assuming $E^p(X,\ell)$, the map $\mathrm{Num}^p(X,\ell)\otimes\mathbb{Q}_\ell \hookrightarrow V^{\mathrm{tate}}$ is an injection that is an isomorphism if and only if $T^p(X,\ell)$ holds.
- For $p=1$ we have $\mathrm{Num}^1(X)\cong \mathrm{NS}(X)/\,\mathrm{NS}(X)_{\mathrm{tors}}$ and an injection $\mathrm{NS}(X)\otimes\mathbb{Q}_\ell \to V^{\mathrm{tate}}$.

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Strategy for bounding $\mathrm{rk}\,\mathrm{Num}^p(X^{\mathrm{s}})$.

1. List cycles to find lower bound (also cycles to intersect with).
2. Bound $\dim_{\mathbb{Q}_\ell} V^{\mathrm{tate}}$ from above for upper bound.
   Trivial upper bound: Betti number b_{2p}.

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Problem with computing $V^{\mathrm{tate}}\subset V^{2p}=\mathrm{H}^{2p}_{\mathrm{et}}(X^{\mathrm{s}},\mathbb{Q}_\ell(p))$ is that $\mathbb{Q}_\ell$ requires infinite precision and k^{s} may not be finitely generated.

Hypothesis. Can compute $T^i_{\ell^n} = \mathrm{H}^i_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}/\ell^n\mathbb{Z})$ as Γ-module.
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Theorem (Poonen, Testa, vL)

Assume the hypothesis. Then there is an algorithm that takes as input (k, p, X, ℓ) as before, such that, assuming $E^p(X, \ell)$, the algorithm terminates if and only if $T^p(X, \ell)$ holds, and if the algorithm terminates, it returns $\mathrm{rk}\,\mathrm{Num}^p(X^{\mathrm{s}})$.

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Sketch of proof of upper bound for $r = \dim V^{\mathrm{tate}}$.

1. Extend k so that Γ acts trivially on $T^{2p}_{\ell'}(p)$ with $\ell' = \ell$ for $\ell > 2$ and $\ell' = 4$ for $\ell = 2$.
2. Γ acts trivially on M/M_{tors} with $M = \mathrm{H}^{2p}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(p))^{\mathrm{tate}}$ (Minkovski's Lemma on finite-order elements in $\mathrm{GL}_n(\mathbb{Z}_\ell)$).
3. Compute t such that ℓ^t kills $\mathrm{H}^{2p}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(p))_{\mathrm{tors}}$ (Wittenberg).
4. $\ell^{r(n-t)} \le \#\ell^t M/\ell^n M \le \#(M/\ell^n M)^\Gamma \le \# T^{2p}_{\ell^n}(p)^\Gamma = \mathcal{O}(\ell^{rn})$.
5. Sequence $\left\lfloor \frac{\log \# T^{2p}_{\ell^n}(p)^\Gamma}{\log \ell^{n-t}} \right\rfloor$ (with $n \ge 1$) has minimum r.

Finite fields

Suppose k is finite. Let $V_\mu \subset V^{2p} = \mathrm{H}^{2p}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Q}_\ell(p))$ be the largest subspace on which all eigenvalues of Frobenius are roots of unity.

$$\mathrm{Num}^p(X^{\mathrm{s}}) \otimes \mathbb{Q}_\ell \to V^{\mathrm{tate}} \subset V_\mu$$

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Theorem

Assuming $E^p(X, \ell)$, the following are equivalent.

1. $\mathrm{rk}\,\mathrm{Num}^p(X^{\mathrm{s}}) = \dim V^{\mathrm{tate}}$.
2. *Conjecture $T^p(X, \ell)$ holds.*
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2. *Conjecture $T^p(X, \ell)$ holds.*
3. $\mathrm{rk}\,\mathrm{Num}^p(X^{\mathrm{s}}) = \dim V^{\mathrm{tate}} = \dim V_\mu$.

Proof. $1 \Leftrightarrow 2$. Under $E^p(X, \ell)$, the first map is injective, so it is surjective if and only if 1 holds.
$2 \Rightarrow 3$. $V^{\mathrm{tate}} = V_\mu$ follows as $E^p(X, \ell)$ and $T^p(X, \ell)$ together imply that Frobenius acts semi-simple on V_μ. $3 \Rightarrow 1$. Obvious.

Finite fields

Theorem

There is an algorithm that takes as input (k, p, X, ℓ), *with* k *a finite field, and that, assuming* $E^p(X, \ell)$, *terminates if and only if* $T^p(X, \ell)$ *holds, and if it terminates, it returns* $\mathrm{rk}\,\mathrm{Num}^p\, X^{\mathrm{s}}$.

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Proof. By searching for cycles, we get a lower bound for $\mathrm{rk}\,\mathrm{Num}^p X^{\mathrm{s}}$ that eventually is sharp. To verify that it is, it suffices to compute $\dim V_\mu$. Say $k = \mathbb{F}_q$. The degree of the zeta-function

$$Z_X(T) = \prod_{i=0}^{2\dim X} \left(\det\left(1 - T \cdot \mathrm{Frob}^* | \, \mathrm{H}^i_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Q}_\ell)\right)\right)^{(-1)^{i+1}}$$

is bounded by the sum of Betti numbers: computable bound B.

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is bounded by the sum of Betti numbers: computable bound B. Computing $\#X(\mathbb{F}_{q^n})$ for $n = 1, \ldots, 2B$ gives enough information to determine $Z_X(T)$. Then $\dim V_\mu$ equals the number of poles of $Z_X(T)$ that are roots of unity times q^{-p}.

Finite fields

Example. Let $X \subset \mathbb{P}^3$ over $\mathbb{F}_2$ be given by $\det M = 0$ with $M =$

$$\begin{pmatrix} x_0 & x_2 & x_1 + x_2 & x_2 + x_3 \\ x_1 & x_2 + x_3 & x_0 + x_1 + x_2 + x_3 & x_0 + x_3 \\ x_0 + x_2 & x_0 + x_1 + x_2 + x_3 & x_0 + x_1 & x_2 \\ x_0 + x_1 + x_3 & x_0 + x_2 & x_3 & x_2 \end{pmatrix}.$$

$$\Phi = \mathrm{Frob}^* |\, \mathrm{H}^2(X^{\mathrm{s}}, \mathbb{Q}_\ell(1)) \qquad t_n = \mathrm{Tr}\, \Phi^n$$

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$$\#X(\mathbb{F}_{2^n}) = 1 + 2^n t_n + 2^{2n}$$

n	1	2	3	4	5	6	7	8	9	10
$\#X(\mathbb{F}_{2^n})$	6	26	90	258	1146	4178	17002	64962	260442	1044786
t_n	$\frac{1}{2}$	$\frac{9}{4}$	$\frac{25}{8}$	$\frac{1}{16}$	$\frac{121}{32}$	$\frac{81}{64}$	$\frac{617}{128}$	$-\frac{575}{256}$	$-\frac{1703}{512}$	$-\frac{3791}{1024}$

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$$\begin{aligned} f_\Phi &= \text{palindromic or antipalindromic} \\ &= t^{22} - \tfrac{1}{2}t^{21} - t^{20} - \tfrac{1}{2}t^{19} + t^{18} - \tfrac{1}{2}t^{15} + t^{14} + \tfrac{1}{2}t^{13} - 2t^{11} + \dots \\ &= (t-1)^2(t^{20} + \tfrac{3}{2}t^{19} + t^{18} - \tfrac{1}{2}t^{13} + t^{11} + 2t^{10} + \dots). \end{aligned}$$

Conclusion. We have $\mathrm{rk}\,\mathrm{NS}(X^{\mathrm{s}}) = 2$.

Surfaces over global fields

- Global field K, discrete valuation ring $R \subset K$, residue field k.
- X a nice surface over K, integral model $\mathcal{X}$ over R with good reduction.

$$\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{NS}(\mathcal{X}_{k^{\mathrm{s}}}) \otimes \mathbb{Q}_\ell$$

$$\mathrm{rk}\, \mathrm{NS}(X^{\mathrm{s}}) \leq \mathrm{rk}\, \mathrm{NS}(\mathcal{X}_{k^{\mathrm{s}}})$$

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Problem.
If $T^1(\mathcal{X}_{k^{\mathrm{s}}}, \ell)$ holds, then $\operatorname{rk} \mathrm{NS}(\mathcal{X}_{k^{\mathrm{s}}}) \equiv b_2(X) \pmod 2$.

Proof.
The roots of f_Φ that are **not** roots of unity come in conjugate pairs.

Question.
How to ever prove $\operatorname{rk} \mathrm{NS}(X^{\mathrm{s}}) = 1$ for a K3 surface over $K = \mathbb{Q}$?

The injection

$$\mathrm{Num}^1(X^{\mathrm{s}}) \hookrightarrow \mathrm{Num}^1(\mathcal{X}_{k^{\mathrm{s}}})$$

respects the intersection pairing.

Lemma. If Λ' is a sublattice of finite index of Λ, then we have

$$\operatorname{disc}\Lambda' = [\Lambda : \Lambda']^2 \operatorname{disc}\Lambda.$$

Hence, $\operatorname{disc}\Lambda = \operatorname{disc}\Lambda'$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.

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Corollary (vL). If v, w are two places of good reduction with
1) $\operatorname{rk}\mathrm{Num}^1(X_{k(v)^{\mathrm{s}}}) = r = \operatorname{rk}\mathrm{Num}^1(X_{k(w)^{\mathrm{s}}})$, and
2) $\operatorname{disc}\mathrm{Num}^1(X_{k(v)^{\mathrm{s}}}) \neq \operatorname{disc}\mathrm{Num}^1(X_{k(w)^{\mathrm{s}}})$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$,
then $\operatorname{rk}\mathrm{Num}^1(X^{\mathrm{s}}) < r$.

If $\operatorname{rk}\mathrm{Num}^1(X^{\mathrm{s}}) = r - 1$, then this equality is verifiable.

Example. Let $X \subset \mathbb{P}^3_{\mathbb{Q}}$ be given by

$$wf = 3pq - 2zg$$

with $f \in \mathbb{Z}[x, y, z, w]$ and $g, p, q \in \mathbb{Z}[x, y, z]$ equal to

$$\begin{aligned} f &= x^3 - x^2y - x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw + y^3 \\ &\quad + y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3, \\ g &= xy^2 + xyz - xz^2 - yz^2 + z^3, \\ p &= z^2 + xy + yz, \\ q &= z^2 + xy. \end{aligned}$$

Then $\operatorname{rk} \operatorname{NS}(X^{\mathrm{s}}) = 1$.

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Proof. Two primes of good reduction: 2 and 3.
For both we obtain $\dim V_\mu = 2$ as before.
Reduction $X_{\mathbb{F}_2}$ contains conic C given by $w = p = 0$.
Reduction $X_{\mathbb{F}_3}$ contains line L given by $w = z = 0$.

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Then $\mathrm{Num}^1(X_{\mathbb{F}_2})$ en $\mathrm{Num}^1(X_{\mathbb{F}_3})$ contain finite-index sublattices

$$\begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}$$

with discriminants -12 and -9 in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.
Not the same, so $\mathrm{rk}\,\mathrm{NS}(X^{\mathrm{s}}) = 1$.

Extension by Kloosterman

- $k = \mathbb{F}_q$.
- X/k a nice surface.
- $\Phi = \mathrm{Frob}^* |\, \mathrm{H}^2(X^{\mathrm{s}}, \mathbb{Q}_\ell)$.
- $f_\Phi(T) = \det(1 - T \cdot \Phi)$.
- $\rho = \mathrm{rk}\, \mathrm{Num}^1(X)$ and $\Delta = \mathrm{disc}\, \mathrm{Num}^1(X)$.
- $b_2 = b_2(X)$ and $\alpha = \chi(X, \mathcal{O}_X) - 1 - \dim \mathrm{Pic}^0(X)$.

Conjecture (Artin–Tate).

$$\lim_{T \to q^{-1}} \frac{f_\Phi(T)}{(1 - qT)^\rho} = \frac{(-1)^{\rho-1} \cdot \#\,\mathrm{Br}\, X \cdot \Delta}{q^\alpha (\#\,\mathrm{NS}(X)_{\mathrm{tors}})^2}.$$

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Facts.
$T^1(X, \ell) \Rightarrow$ Artin–Tate.
$T^1(X, \ell) \Rightarrow \#\,\mathrm{Br}\,X \in (\mathbb{Q}^*)^2$ (Liu–Lorenzini–Raynaud).

Conclusion. We may compute $\Delta \in \mathbb{Q}^*/\mathbb{Q}^{*2}$, assuming $T^1(X, \ell)$.

Application

Theorem (Kloosterman)

The elliptic K3 surface $\pi : X \to \mathbb{P}^1$ *over* $\overline{\mathbb{Q}}$ *given by*

$$y^2 = x^3 + 2(t^8 + 14t^4 + 1)x + 4t^2(t^8 + 6t^4 + 1)$$

has $\operatorname{rk} \operatorname{NS}(X) = 17$ *and Mordell-Weil rank* 15*.*

Extension by Elsenhans-Jahnel, I

Main idea. If you consider Galois action, you may not need $r = \mathrm{rk}\,\mathrm{Num}^1(X^{\mathrm{s}}) + 1$.

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Example. (Elsenhans, Jahnel)
Let $X\colon w^2 = f(x, y, z)$ be a K3 surface of degree 2 over $\mathbb{Q}$ with

$$f \equiv y^6 + x^4y^2 - 2x^2y^4 + 2x^5z + 3xz^5 + z^6 \pmod 5$$

and

$$f \equiv 2x^6 + x^4y^2 + 2x^3y^2z + x^2y^2z^2 + x^2yz^3 + 2x^2z^4 + xy^4z + xy^3z^2 + xy^2z^3 + 2xz^5 + 2y^6 + y^4z^2 + y^3z^3 \pmod 3.$$

Then $\mathrm{rk}\,\mathrm{NS}(X^{\mathrm{s}}) = 1$.

Extension by Elsenhans-Jahnel, I

Let L denote the pull-back of a line in $\mathbb{P}^2(x, y, z)$.

The characteristic polynomial of Frobenius acting on the space

$$(\mathrm{NS}\, X_{\overline{\mathbb{F}}_3} \otimes \mathbb{Q})/\langle L \rangle$$

equals $(t-1)(t^2+t+1)$, so only finitely many Galois-invariant subspaces of $\mathrm{NS}\, X_{\overline{\mathbb{F}}_3} \otimes \mathbb{Q}$ containing L; dimensions are $1, 2, 3, 4$.

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equals $(t-1)(t^2+t+1)$, so only finitely many Galois-invariant subspaces of $\mathrm{NS}\, X_{\overline{\mathbb{F}}_3} \otimes \mathbb{Q}$ containing L; dimensions are $1, 2, 3, 4$.

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Extension by Elsenhans-Jahnel, I

Let L denote the pull-back of a line in $\mathbb{P}^2(x, y, z)$.

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Only common dimensions are 1 and 2. Comparing discriminants up to squares of the subspaces of dimension 2 yields $\mathrm{rk}\,\mathrm{NS}(X^{\mathrm{s}}) = 1$.

Extension by Elsenhans-Jahnel, II

- $p \neq 2$ prime.
- X a scheme that is proper and flat over $\mathbb{Z}$.

Theorem (Elsenhans-Jahnel). If the special fiber X_p is nonsingular, then the cokernel of the specialization homomorphism

$$\mathrm{sp}_{\overline{\mathbb{Q}}} \colon \mathrm{Pic}(X_{\overline{\mathbb{Q}}}) \to \mathrm{Pic}(X_{\overline{p}})$$

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Corollary (E-J, Hassett–Várilly-Alvarado).
Let X be a double cover of $\mathbb{P}^2$, ramified over a smooth plane sextic C. Let p, p' denote two odd primes of good reduction. Assume that there is a tritangent line ℓ to the curve C_p. Suppose $\mathrm{Pic}(X_p^{\mathrm{s}})$ has rank 2 and is generated by the components in the pull-back of ℓ. If there are no tritangent lines to $C_{p'}$, then $\mathrm{rk}\,\mathrm{Pic}(X^{\mathrm{s}}) = 1$.

“It works” by Charles

Question.
1) Given a nice surface X over a number field k, is there always a prime p of good reduction with $\mathrm{rk}\,\mathrm{Num}^1(X_p^{\mathrm{s}}) \leq \mathrm{rk}\,\mathrm{Num}^1(X^{\mathrm{s}}) + 1$?
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Not always, but if not, then the minimal jumps are still controllable!

Consequence (Charles).
There is an algorithm with input a K3 surface X over a number field that either returns $\operatorname{rk}\operatorname{NS}(X^{\mathrm{s}})$ or does not terminate.
If $X \times X$ satisfies the Hodge conjecture for codimension 2 cycles, then the algorithm applied to X terminates.

Saturation

Theorem (Poonen, Testa, vL)

There is an algorithm that takes k, X, and a finite set $\mathcal{D}$ of divisors as input, and computes the saturation inside $\mathrm{NS}(X^{\mathrm{s}})$ of the Γ-submodule generated by the classes of divisors in $\mathcal{D}$.

Method. Hilbert scheme computations.

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If not primitive, then G has nontrivial index in its saturation $\tilde{G}$, so there is a prime $r \mid [\tilde{G} : G]$ with $r^2 \mid [\tilde{G} : G]^2 \cdot \operatorname{disc} \tilde{G} = \operatorname{disc} G$. Then $G \otimes \mathbb{F}_r \to \mathrm{Num}^1(X^{\mathrm{s}}) \otimes \mathbb{F}_r$ is not injective.

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For all primes p of good reduction and $H \subset \mathrm{Num}^1(X_p^{\mathrm{s}})$ the map

$$\mathrm{Num}^1(X^{\mathrm{s}}) \to \mathrm{Hom}(H, \mathbb{Z})$$

induces a non-injective composition

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Sufficient for primitivity. Find for each r with $r^2 \mid \mathrm{disc}\, G$ a prime p and a subgroup $H \subset \mathrm{Num}^1(X_p^{\mathrm{s}})$ for which the composition is injective (linear algebra over $\mathbb{F}_r$).

Application

Theorem (Mizukami ($m = 4$), Schütt–Shioda–vL ($m \leq 100$))

For any integer $1 \leq m \leq 100$ the Néron-Severi group of the Fermat surface $S_m \subset \mathbb{P}^3$ over $\mathbb{C}$ given by

$$x^m + y^m + z^m + w^m = 0$$

is generated by the lines on S_m if and only if $m \leq 4$ or $(m, 6) = 1$.