Density of rational points on Del Pezzo surfaces of degree one

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Surface: smooth, projective, geometrically integral scheme of finite type over a field, of dimension 2.

Surface X is Del Pezzo if anticanonical divisor $-K_X$ is ample.

Degree of Del Pezzo surface X is intersection number $d = K_X \cdot K_X$.

Examples:

- Intersection of two quadrics in $\mathbb{P}^4$ ($-K_S$ very ample; $d = 4$).
- A smooth cubic surface in $\mathbb{P}^3$ ($-K_S$ very ample; $d = 3$).
- Smooth double cover of $\mathbb{P}^2$, ramified over a quartic ($d = 2$). (Anticanonical map is the projection to $\mathbb{P}^2$).

Theorem (Segre, Manin, Kollár, Pieropan, Salgado–Testa–Várilly-Alvarado, Festi-vL).
Let S be a Del Pezzo surface of degree $d \geq 2$ over a field k. Suppose $P \in S(k)$ is a rational point. If $d = 2$ and k is infinite, then suppose, furthermore, that P does not lie on four exceptional curves, nor on the ramification locus of the anticanonical map. Then S is unirational over k.

Theorem (Kollár, Mella)
Let S be a Del Pezzo surface of degree $d = 1$ over a field k of characteristic not equal to 2. If S admits a (certain?) conic bundle structure, then S is unirational.

Remark. When these are minimal, Picard number $\rho(S) = 2$.

Question 1. Is there a DP1 with $\rho = 1$ that is unirational?

Question 2. Is there a DP1 with $\rho = 1$ that is **not** unirational?

Conjecture (Batyrev–Manin–Peyre). Suppose that S is a Del Pezzo surface of degree 1 over a number field k with a rational point and Picard number ρ. Then there is a nonzero constant c such that for every small enough nonempty open subset $U \subset S$ we have

$$\#\{ P \in U(k) \ : \ H_{-K}(P) \leq B \} \sim cB(\log B)^{\rho-1}$$

as $B \to \infty$.

Nonzeroness comes from Peyre's description of c or Colliot-Thélène's conjecture that Brauer–Manin is the only obstruction to weak approximation on rational varieties.

Conclusion. There should be lots of rational points!

Every Del Pezzo surface S/k of degree $d = 1$ is isomorphic to a smooth sextic in $\mathbb{P}(2,3,1,1)$, with coordinates x, y, z, w, given by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_i \in k[z,w]$ homogeneous of degree i. (And vice versa.)

Linear system $|-K_S|$ induces rational map $S \dashrightarrow \mathbb{P}^1(z,w)$.

Unique base point $O = [1:1:0:0]$.

Curves in $|-K_S|$ are fibers of $\mathrm{Bl}_{\mathcal{O}}(S) \to \mathbb{P}^1$ (anticanonical fiber). Almost all are elliptic fibers, all are geometrically integral.

Theorem (Várilly-Alvarado). Let A, B be nonzero integers, and let S be the Del Pezzo surface of degree 1 over $\mathbb{Q}$ given by

$$y^2 = x^3 + Az^6 + Bw^6.$$

Assume that Tate–Shafarevich groups of elliptic curves over $\mathbb{Q}$ with j-invariant 0 are finite. If $3AB$ is not a square, or if A and B are relatively prime and $9 \nmid AB$, then $S(\mathbb{Q})$ is Zariski dense in S.

Question. What about $y^2 = x^3 + 243z^6 + 16w^6$?

Theorem (Salgado–vL). Let $S \subset \mathbb{P}(2,3,1,1)$ be a Del Pezzo surface of degree 1 over a number field k. Suppose F is a smooth anticanonical fiber with a point $P \in F(k)$ of order > 2 that does not lie on six exceptional curves. Set $U = \mathbb{P}(2,3,1,1) \setminus Z(z,w)$ and let $\mathcal{C}$ denote the curve of those sections of the projection $U \to \mathbb{P}^1(z,w)$ that meet S at the point P with multiplicity at least 5. If $\#\mathcal{C}(k) = \infty$, then $S(k)$ is Zariski dense.

Remark. If the order of P is 3, then $\#\mathcal{C}(k) = \infty$ for free.
For Várilly-Alvarado's example

$$y^2 = x^3 + 243z^6 + 16w^6$$

there is a 3-torsion point $[0:4:0:1]$, but it lies on nine exceptional curves...

The coefficients of the curve $\mathcal{C}$ associated to the point $[-63:14:1:5]$ are too large to find points...

Elkies' proof for Várilly-Alvarado's example. Take affine part

$$y^2 = x^3 + 243 + 16t^6,$$

and set $y = v + 4t^3$ to obtain

$$v^2 + 8t^3 v = x^3 + 243.$$

Now projection onto v-line gives fibration into cubics. The point $(t, x, y) = (5, -63, 14)$ gives a cubic with infinitely many rational points, most of which have infinite order on their anticanonical fiber. Done!

Elkies: When I tried to generalise this construction, it turned out I needed a 3-torsion point. Maybe it can be done for all rational torsion points (on their anticanonical fiber).

Theorem (Bulthuis-vL). Let S be a Del Pezzo surface of degree 1 over a number field k. Suppose F is a smooth anticanonical fiber with a point $P \in F(k)$ of finite order $n > 1$. Then the linear system

$$|-nK_S - nP| = \{ \ D \in |-nK_S| \ \ : \ \ \mu_P(D) \geq n \ \}$$

induces a fibration $\varphi\colon \ \mathrm{Bl}_P(S) \to \mathbb{P}^1$ of curves of genus 1. If

1) some irreducible fiber of φ has ∞ many k-rational points, or
2) there is a $Q \in S(k) \setminus \{P\}$ such that the fiber $G_Q = \varphi^{-1}(\varphi(Q))$ is smooth, and on the elliptic curve (G_Q, Q), the sum of the points above P has infinite order,

then $S(k)$ is Zariski dense.

Moreover, there is a nonempty Zariski open subset $U \subset S$ such that every $Q \in U(k)$ satisfies the conditions of 2).

Consequence. Suppose S has a point that has finite order on its anticanonical fiber. If $S(k)$ is Zariski dense in S, then there is an easy proof that this is the case.

Set $\pi\colon S' = \mathrm{Bl}_P(S) \to S$, except. curve E, strict transf. F' of F.

1. $|-nK_S - nP| \leftrightarrow |-n\pi^*K_S - nE| = |n\pi^*F - nE| = |nF'|$.
2. $\dim|mF'| = 0$ for $0 \leq m < n$, and $\dim|nF'| = 1$.
3. Base locus of $|nF'|$ is empty: 1-dim'l components are F', contradicting 2. No base points as $F'^2 = (\pi^*F - E)^2 = 0$.
4. Bertini Theorem: almost all curves in $|nF'|$ smooth.
5. All curves connected: Stein factorisation gives $S' \to \mathbb{P}^1$ with fibers H with $aH \sim nF'$ for some a, contradicting 2.
6. Almost all curves geometrically integral.
7. Genus of smooth $D \in |nF'|$ is 1: we have $-K_{S'} \sim F'$, so $2g(D) - 2 = D(D + K_{S'}) = D(D - F') = 0$.
8. Fibers of φ are not in torsion locus of anticanonical fibration: torsion-locus does not self intersect in smooth fibers.
9. Equivalent condition on Q: the divisor $n(Q) - \sum_{P'|P}(P') \in \mathrm{Jac}(G_Q)$ is not torsion.

To find $\dim|mF'| = \dim \mathrm{H}^0(S', \mathcal{O}_{S'}(mF')) - 1$, we consider the embedding $\iota\colon F' \hookrightarrow S'$ and exact sequence (idea Adam Logan)

$$0 \to \mathcal{O}_{S'}(-F') \to \mathcal{O}_{S'} \to \iota_*\mathcal{O}_{F'} \to 0$$

of sheaves on S'. Twisting by $\mathcal{O}_{S'}(mF')$ gives

$$0 \to \mathcal{O}_{S'}((m-1)F') \to \mathcal{O}_{S'}(mF') \to \iota_*\mathcal{O}_{F'} \otimes \mathcal{O}_{S'}(mF') \to 0\,.$$

Let F_∞ be an other anticanonical fiber and $F'_\infty = \pi^* F_\infty$ its strict transform. Then $F' \sim F'_\infty - E$, so

$$\iota_*\mathcal{O}_{F'} \otimes \mathcal{O}_{S'}(mF') \cong \iota_*\mathcal{O}_{F'} \otimes \mathcal{O}_{S'}(mF'_\infty - mE) \cong \iota_*\mathcal{O}_{F'}(mO - mP).$$

We obtain

$$0 \to \mathcal{O}_{S'}((m-1)F') \to \mathcal{O}_{S'}(mF') \to \iota_*\mathcal{O}_{F'}(mO - mP) \to 0\,.$$

Now we take the associated long exact sequence.

$$0 \to \mathrm{H}^0(S', \mathcal{O}((m-1)F')) \to \mathrm{H}^0(S', \mathcal{O}(mF')) \to \mathrm{H}^0(F', mO - mP)$$
$$\to \mathrm{H}^1(S', \mathcal{O}((m-1)F')) \to \mathrm{H}^1(S', \mathcal{O}(mF')) \to \mathrm{H}^1(F', mO - mP)$$

Using (for $i \in \{0, 1\}$)

$$\dim \mathrm{H}^i(F', mO - mP) = \begin{cases} 1 & \text{if } n|m \\ 0 & \text{otherwise} \end{cases}$$

and

$$\dim \mathrm{H}^i(S', \mathcal{O}_{S'}) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases},$$

we find, by induction, that (for $i \in \{0, 1\}$ and $0 < m < n$)

$$\dim \mathrm{H}^i(S', \mathcal{O}_{S'}(mF')) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases}.$$

Finally, for $m = n$, we obtain

$$\dim \mathrm{H}^i(S', \mathcal{O}_{S'}(nF')) = 2.$$

Thanks!