Toward an explicit 2-descent on the Jacobian of a generic curve of genus 2

Ronald van Luijk
CRM, Montreal
MSRI, Berkeley

Joint work in progress with

Adam Logan
CRM, Montreal
Waterloo, Canada

January 15, 2006
San Antonio

Goals:

(1) Computing Mordell-Weil groups of Jacobians

(2) Constructing nontrivial elements of Shafarevich-Tate groups

Tools:

(a) 2-descent on Jacobians

(b) Brauer-Manin obstruction to the existence of rational points

Let C be a smooth, geometrically irreducible curve of genus 2 over a number field K, and J the Jacobian of C.

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Primary goal:
Compute $J(K) \cong J(K)_{\text{tors}} \oplus \mathbb{Z}^r$.

- $J(K)_{\text{tors}}$: finite, easy to compute.
- $J(K)_{\text{tors}}$ and r known $\Rightarrow$ $J(K)$ computable.
- The rank r can be read off from

$$J(K)_{\text{tors}} \quad \& \quad J(K)/2J(K).$$

There are cohomologically defined finite groups

$\mathrm{Sel}^{(2)}(K, J)$,	the 2-Selmer group,
$Ш(K, J)$,	the Shafarevich-Tate group,

with

$$0 \to J(K)/2J(K) \to \mathrm{Sel}^{(2)}(K, J) \to Ш(K, J)[2] \to 0.$$

2-descent: compute $\mathrm{Sel}^{(2)}(K, J)$ and decide which of its elements come from $J(K)/2J(K)$ (i.e., map to 0).

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Assumption: We can compute $\text{Sel}^{(2)}(K, J)$.

Remaining goal: Which elements of $\text{Sel}^{(2)}(K, J)$ map to 0?

Element of Sel$^{(2)}(K, J)$: a twist $\pi\colon Y \to J$ of the map $[2]\colon J \to J$ (over $\overline{K}$ there is an isomorphism σ such that

$$\begin{array}{ccc} Y_{\overline{K}} & \overset{\cong}{\underset{\sigma}{\dashrightarrow}} & J_{\overline{K}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle [2]} \\ J_{\overline{K}} & = & J_{\overline{K}} \end{array}$$

commutes), where Y is locally soluble everywhere.

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The element $Y \to J$ maps to 0 in $\text{Ш}(K, J)[2]$ iff $Y(K) \neq \emptyset$.

Problem: The surfaces Y are described by 72 quadrics in $\mathbb{P}^{15}\ldots$

Solution: A quotient of Y.

$[-1]$ on J commutes with translation by 2-torsion points $\Rightarrow$ it induces a unique involution ι of $Y_{\overline{K}}$, defined over K. Set $X = Y/\iota$.

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Advantages:

- X is a complete intersection of 3 quadrics in $\mathbb{P}^5$.
- $X(K) = \emptyset \Rightarrow Y(K) = \emptyset$

Disadvantage:

- This only gives sufficient conditions for $Y(K) = \emptyset$.

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Situation: Such K3 surfaces are everywhere locally soluble, but may still satisfy $X(K) = \emptyset$. Do they?

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For any scheme Z we set $\operatorname{Br} Z = H^2_{\text{ét}}(Z, \mathbb{G}_m)$.

For any K-algebra S and any S-point $x\colon \operatorname{Spec} S \to X$, we get a homomorphism $x^*\colon \operatorname{Br} X \to \operatorname{Br} S$, yielding a map

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Apply this to K and to the ring of adèles

$$\mathbb{A}_K = \prod_{v \in M_K}{}' K_v \qquad \text{(almost all coordinates are integral).}$$

From class field theory (and comparison theorems) we have

$$0 \to \quad \mathrm{Br}\, K \quad \to \quad \mathrm{Br}\, \mathbb{A}_K \quad \to \quad \mathbb{Q}/\mathbb{Z}$$

Applying $\mathrm{Hom}(\mathrm{Br}\, X, \_)$ we find ...

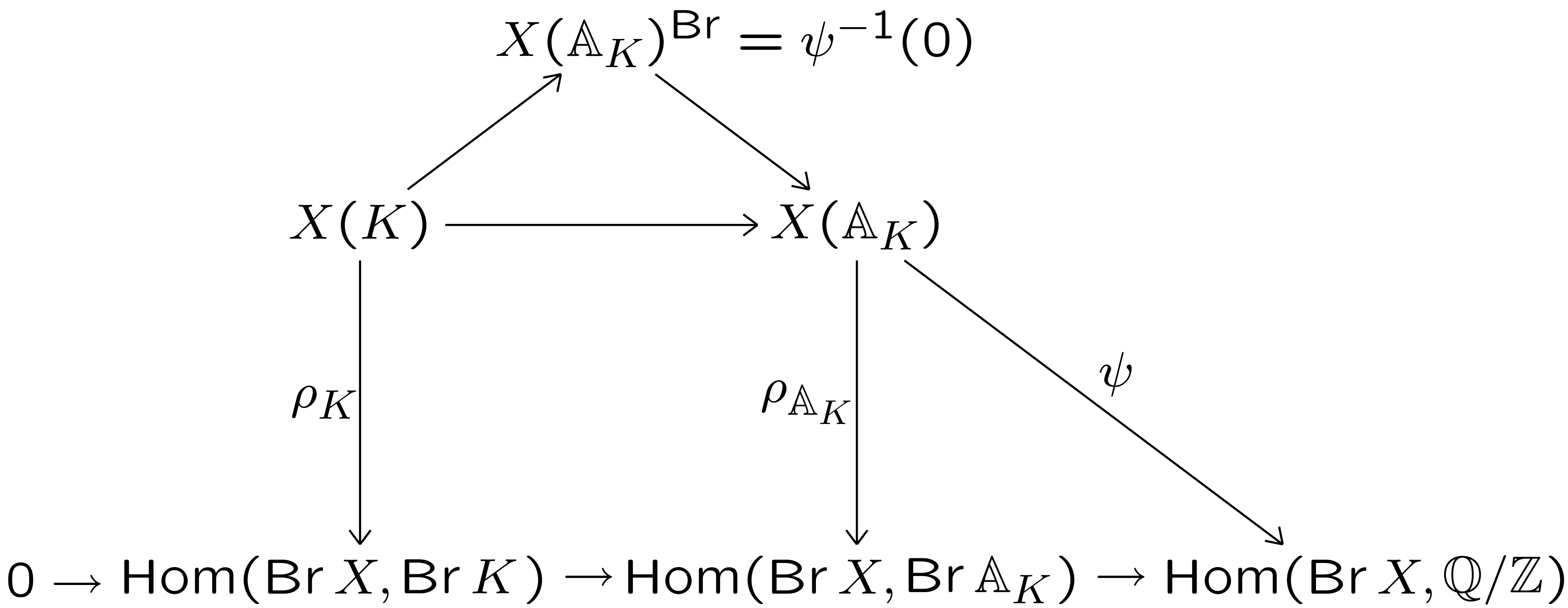
$$0 \to \mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}\, K) \to \mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}\, \mathbb{A}_K) \to \mathrm{Hom}(\mathrm{Br}\, X, \mathbb{Q}/\mathbb{Z})$$

$$\begin{array}{ccccc} & & X(K) & & X(\mathbb{A}_K) & & \\ & & \Big\downarrow{\scriptstyle \rho_K} & & \Big\downarrow{\scriptstyle \rho_{\mathbb{A}_K}} & & \\ 0 \to & & \mathrm{Hom}(\mathrm{Br}\,X, \mathrm{Br}\,K) & \to & \mathrm{Hom}(\mathrm{Br}\,X, \mathrm{Br}\,\mathbb{A}_K) & \to & \mathrm{Hom}(\mathrm{Br}\,X, \mathbb{Q}/\mathbb{Z}) \end{array}$$

$$
\begin{array}{ccccccc}
 & & X(K) & \longrightarrow & X(\mathbb{A}_K) & & \\
 & & \downarrow{\scriptstyle \rho_K} & & \downarrow{\scriptstyle \rho_{\mathbb{A}_K}} & \overset{\psi}{\searrow} & \\
0 & \rightarrow & \mathrm{Hom}(\mathrm{Br}\,X, \mathrm{Br}\,K) & \rightarrow & \mathrm{Hom}(\mathrm{Br}\,X, \mathrm{Br}\,\mathbb{A}_K) & \rightarrow & \mathrm{Hom}(\mathrm{Br}\,X, \mathbb{Q}/\mathbb{Z})
\end{array}
$$

$$\begin{array}{ccccc} & & X(\mathbb{A}_K)^{\mathrm{Br}} = \psi^{-1}(0) & & \\ & \nearrow & & \searrow & \\ X(K) & & \longrightarrow & & X(\mathbb{A}_K) & & \\ \downarrow \rho_K & & & & \downarrow \rho_{\mathbb{A}_K} & \searrow \psi & \\ 0 \to \mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}\, K) & & \longrightarrow & & \mathrm{Hom}(\mathrm{Br}\, X, \mathrm{Br}\, \mathbb{A}_K) & \longrightarrow & \mathrm{Hom}(\mathrm{Br}\, X, \mathbb{Q}/\mathbb{Z}) \end{array}$$

$$X(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset \quad \Rightarrow \quad X(K) = \emptyset$$



$$X(\mathbb{A}_K)^{\mathrm{Br}_1} = \emptyset \quad \Rightarrow \quad X(K) = \emptyset$$

$$\begin{array}{ccccc}
 & & X(\mathbb{A}_K)^{\mathrm{Br}_1} = \psi_1^{-1}(0) & & \\
 & \nearrow & & \searrow & \\
X(K) & & \longrightarrow & & X(\mathbb{A}_K) \\
\downarrow \rho_K & & & & \downarrow \rho_{\mathbb{A}_K} \quad \searrow \psi_1 \\
0 \to \mathrm{Hom}(\mathrm{Br}_1 X, \mathrm{Br}\, K) & \to & \mathrm{Hom}(\mathrm{Br}_1 X, \mathrm{Br}\, \mathbb{A}_K) & \to & \mathrm{Hom}(\mathrm{Br}_1 X, \mathbb{Q}/\mathbb{Z})
\end{array}$$

$$\mathrm{Br}_1\, X = \ker(\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X})$$

$X(\mathbb{A}_K)^{\mathrm{Br}_1} = \emptyset \quad \Rightarrow \quad X(K) = \emptyset$.

Two steps:

- Compute $\mathrm{Br}_1 Z / \mathrm{Br}\, K$ for the desingularization(!) Z of $X = Y/\iota$.

The Hochschild-Serre spectral sequence gives

$$\mathrm{Br}_1 Z / \mathrm{Br}\, K \;\cong\; H^1(G_K, \mathrm{Pic}\, \overline{Z}).$$

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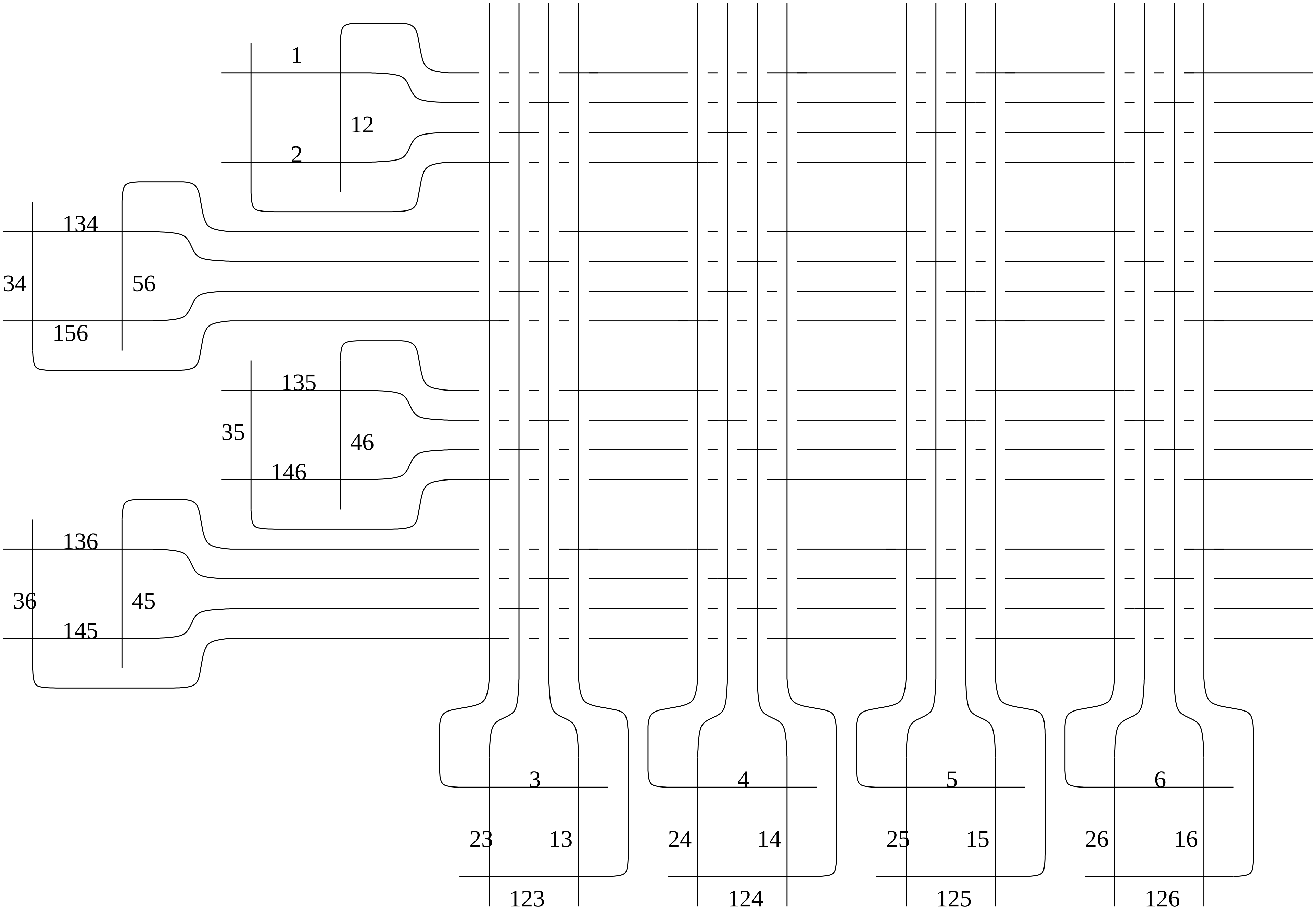
- Compute $Z(\mathbb{A}_K)^{\mathrm{Br}_1}$ (easier for elliptic fibrations).

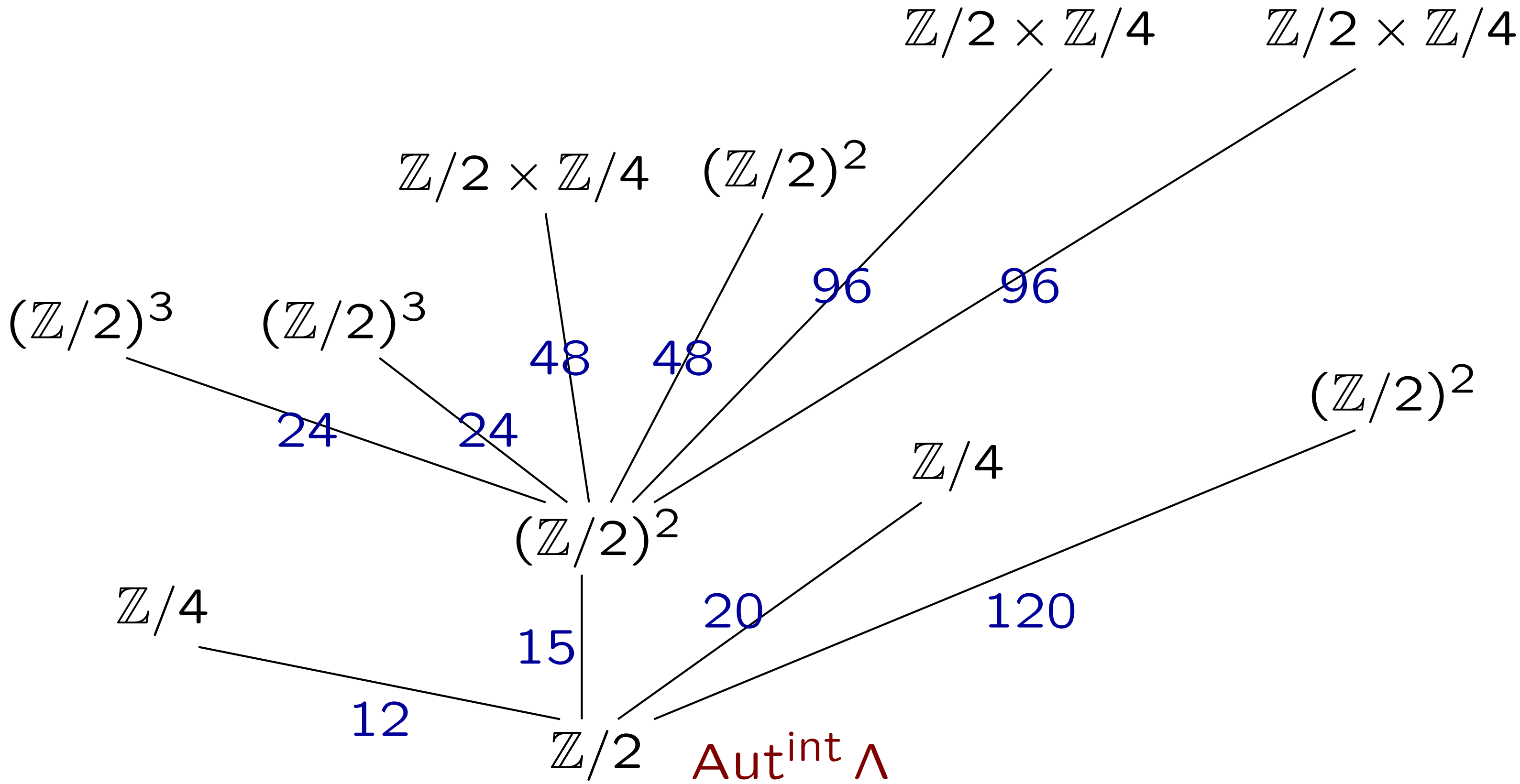
Intersection among 32 lines on $\overline{Z}$

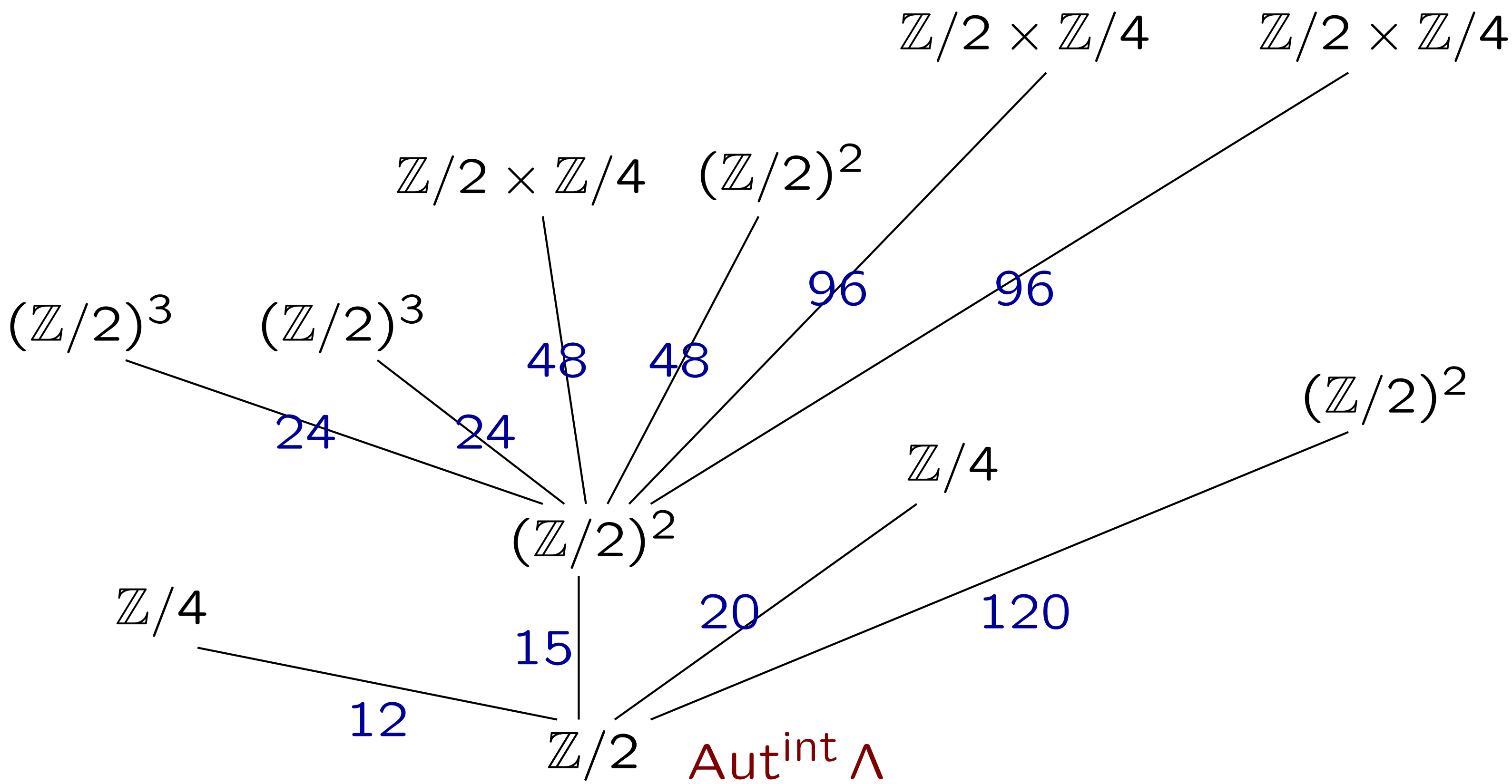


Proposition: Generically the group $\mathrm{Pic}\,\overline{Z}$ has rank 17, generated by the set Λ of 32 lines.

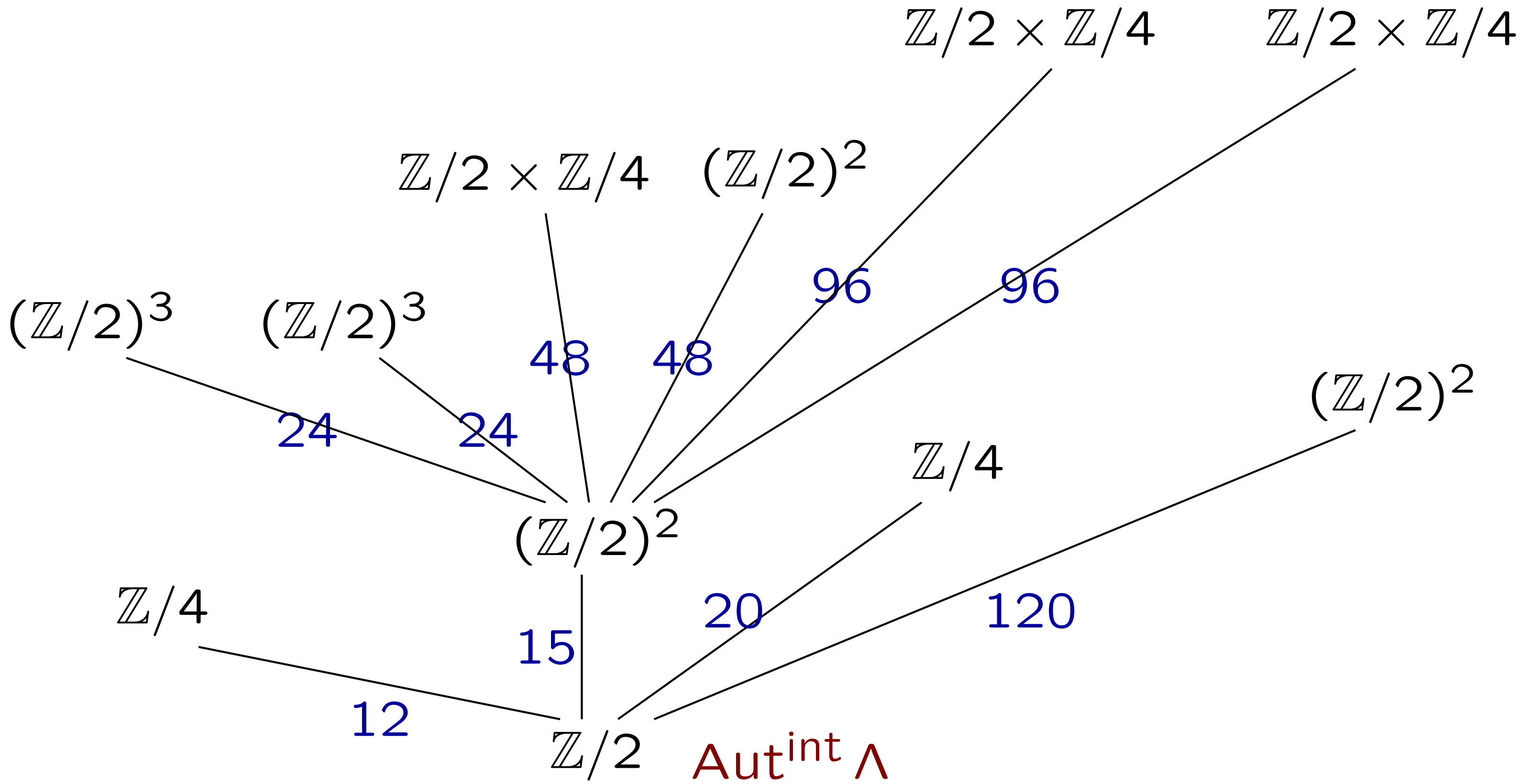
Corollary: G_K acts on $\mathrm{Pic}\,\overline{Z}$ through a subgroup of $\mathrm{Aut}^{\mathrm{int}}\,\Lambda$ (which has size 23040).

We can compute $H^1(G, \mathrm{Pic}\,\overline{Z})$ for all 2455 possible subgroups G of $\mathrm{Aut}^{\mathrm{int}}\,\Lambda$ (up to conjugacy).


ℤ/2 × ℤ/4
ℤ/2 × ℤ/4
ℤ/2 × ℤ/4
(ℤ/2)²
96
96
(ℤ/2)³
(ℤ/2)³
48
48
24
24
(ℤ/2)²
ℤ/4
(ℤ/2)²
ℤ/4
15
20
120
12
ℤ/2
Aut^int Λ



These 11 subgroups, including $\mathrm{Aut}^{\mathrm{int}}\,\Lambda$, induce all nontrivial Brauer elements.



Step 2, Computing $Z(\mathbb{A}_K)^{\mathrm{Br}_1}$, is difficult

There is a group E of order 384 such that if the Galois action factors through E, then Z has an elliptic fibration over K.

Results:

- We can write down this fibration generically,
- Computing $Z(\mathbb{A}_K)^{\mathrm{Br}_1}$ is easier,
- There are 6 subgroups like the 11 before,
- For one, an algorithm for computing $Z(\mathbb{A}_K)^{\mathrm{Br}_1}$ is implemented.

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Non-result:

- We are expecting our first example soon.