THE ACTION OF THE WEYL GROUP ON THE E_8 ROOT SYSTEM

ROSA WINTER AND RONALD VAN LUIJK

Abstract. Let Γ be the graph on the roots of the E_8 root system, where any two distinct vertices e and f are connected by an edge with color equal to the inner product of e and f. For any set c of colors, let Γ_c be the subgraph of Γ consisting of all the 240 vertices, and all the edges whose color lies in c. We consider cliques, i.e., complete subgraphs, of Γ that are either monochromatic, or of size at most 3, or a maximal clique in Γ_c for some color set c, or whose vertices are the vertices of a face of the E_8 root polytope. We prove that, apart from two exceptions, two such cliques are conjugate under the automorphism group of Γ if and only if they are isomorphic as colored graphs. Moreover, for an isomorphism f from one such clique K to another, we give necessary and sufficient conditions for f to extend to an automorphism of Γ, in terms of the restrictions of f to certain special subgraphs of K of size at most 7.

1. Introduction

Let Λ be the E_8 lattice, that is, the unique positive-definite, even, unimodular lattice of dimension 8. More concretely, let Λ be given by

$$\Lambda = \left\{ a \in \mathbb{Z}^8 + \left\langle \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) \right\rangle \;\middle|\; \sum_{i=1}^{8} a_i \in 2\mathbb{Z} \right\}.$$

Consider the E_8 root system E in Λ given by

$$E = \{a \in \Lambda \mid \|a\| = \sqrt{2}\}.$$

In this artice we study a graph on the elements in E, which we call *roots*. By a *graph* we mean a pair (V, D), where V is a set of elements called *vertices*, and D a subset of the powerset of V of which every element has cardinality 2; elements in D are called *edges*, and the *size* of the graph is the cardinality of V. By a *colored graph* we mean a graph (V, D) together with a map $\varphi\colon D \longrightarrow C$, where C is any set, whose elements we call *colors*; for an element $d \in D$ we call $\varphi(d)$ its color. If (V, D) is a colored graph with color function φ, we define a *colored subgraph* of (V, D) to be a pair (V', D') with a map φ', such that V' is a subset of V, while D' is a subset of the intersection of D with the powerset on V', and φ' is the restriction of φ to D'. Finally, we define a *clique* of a colored graph to be a complete colored subgraph.

Let Γ be the complete colored graph whose vertex set is E, of which the color function on the edge set is induced by the dot product. The different colors of the edges in Γ are $-2, -1, 0, 1$. For a subset $c \subseteq \{-2, -1, 0, 1\}$, we denote by Γ_c the colored subgraph of Γ with vertex set E and all edges whose color is an element in c.

Let W be the automorphism group of Γ. It is clear that if two cliques in Γ are conjugate under the action of W, they must be isomorphic. The converse is not always true, and in general it can be hard to determine whether two cliques in Γ

are conjugate under the action of W. Dynkin and Minchenko studied in [DM10] the bases of subsystems of E_8, and classified for which isomorphism classes of these bases being isomorphic implies being conjugate. They call these bases *normal*. In this article, we extend this classification to a large set of cliques in Γ (more specifically, cliques of type I, II, III, or IV, as defined below). In Theorem 1.1 we show that with two exceptions, two such cliques are isomorphic if and only if they are conjugate. One of the exceptions, which is the clique described in Theorem 1.1 (i), is one of the bases (of the system $4A_1$) that was also found as not being normal in [DM10], Theorem 4.7. Additionally, in [DM10] the authors determine when a homomorphism of two bases of subsytems extends to a homomorphism of the whole root system. We answer the same question for cliques of type I, II, III, or IV in Theorem 1.2.

Although the classification of different types of cliques and their orbits is a finite problem, because of the size of Γ it is practically impossible to naively let a computer find and classify the cliques according to their W-orbit. In fact, we avoid using a computer for our computations as much as possible.

The E_8 root polytope is the convex polytope in $\mathbb{R}^8$ whose vertices are the roots in E. By a *face* of the root polytope we mean a non-empty intersection of a hyperplane in $\mathbb{R}^8$ and the root polytope, such that the root polytope lies entirely on one side of the hyperplane. If the dimension of this intersection is k then we call this a k-face, and a 7-face is called a *facet*. We study the following cliques in Γ, and their orbits under the action of W.

(I) Monochromatic cliques

(II) Cliques whose vertices are the vertices of a face of the E_8 root polytope

(III) Cliques of size at most three

(IV) For all $c \neq \{-1, 0, 1\}$, the maximal cliques in Γ_c

More specifically, we prove the following theorem.

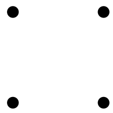
Theorem 1.1. *Let K_1, K_2 be two cliques in Γ of types I, II, III, or IV. Then the following hold.*

(i) If both K_1 and K_2 are of type I with color 0 and of size 4, then K_1 and K_2 are conjugate under the action of W if and only if the vertices sum to an element in 2Λ for both K_1 and K_2, or for neither.

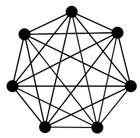
(ii) If both K_1 and K_2 are of type I with color 1 and of size 7, then K_1 and K_2 are conjugate under the action of W if and only if the vertices sum to an element in 2Λ for both K_1 and K_2, or for neither; this is equivalent to K_1 and K_2 both being maximal or both being non-maximal, respectively, under inclusion in Γ_1.

(iii) In all other cases, K_1 and K_2 are conjugate under the action of W if and only if they are isomorphic as colored graphs.

Furthermore, we give conditions for an isomorphism of two cliques of types I, II, III or IV to extend to automorphisms of the lattice Λ. To this end we introduce the following colored graphs.



A



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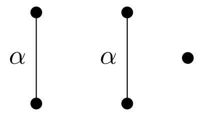
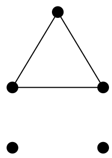


C_α



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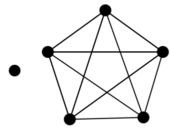


F

Here α is either -1 or 1, two disjoint vertices have an edge of color 0 between them, and all other edges have color 1.

Theorem 1.2. *Let K_1, K_2 be two cliques in Γ of types I, II, III, or IV, and let $f\colon K_1 \longrightarrow K_2$ be an isomorphism between them. The following hold.*

(i) The map f extends to an automorphism of Λ if and only if for every ordered sequence $S = (e_1, \dots, e_r)$ of distinct roots in K_1 such that the colored graph on them is isomorphic to A, B, C_α, D, or F, its image $f(S) = (f(e_1), \dots, f(e_r))$ is conjugate to S under the action of W.

(ii) If $S = (e_1, \dots, e_r)$ is a sequence of distinct roots in K_1 such that the colored graph on them is isomorphic to either A or B, then S and $f(S)$ are conjugate under the action of W if and only if the sets $\{e_1, \dots, e_r\}$ and $\{f(e_1), \dots, f(e_r)\}$ are.

(iii) If K_1 and K_2 are maximal cliques, both in $\Gamma_{-1,0}$ or both in $\Gamma_{-2,-1,0}$, and $S = (e_1, \dots, e_5)$ is a sequence of roots in K_1 such that the colored graph on them is isomorphic to C_{-1} with $e_1 \cdot e_4 = e_2 \cdot e_5 = -1$, then S and $f(S)$ are conjugate under the action of W if and only if both $e = e_1 + e_2 + e_3 - e_4 - e_5$ and $f(e)$ are in the set $\{2f_1 + f_2 \mid f_1, f_2 \in E\}$, or neither are.

(iv) If K_1 and K_2 are maximal cliques in $\Gamma_{-2,0,1}$, and $S = (e_1, \dots, e_r)$ is a sequence of distinct roots in K_1 such that the colored graph G on them is isomorphic to C_1, D, or F, then S and $f(S)$ are conjugate under the action of W if and only if the sets $\{e_1, \dots, e_r\}$ and $\{f(e_1), \dots, f(e_r)\}$ are, or equivalently, if and only if the following hold.

- *If $G \cong C_1$, both $\sum_{i=1}^{5} e_i$ and $\sum_{i=1}^{5} f(e_i)$ are in the set $\{2f_1 + f_2 \mid f_1, f_2 \in E\}$, or neither are.*
- *If $G \cong D$, both $\sum_{i=1}^{5} e_i$ and $\sum_{i=1}^{5} f(e_i)$ are in $\{2f_1 + 2f_2 \mid f_1, f_2 \in E\}$, or neither are.*
- *If $G \cong F$, then both $\sum_{i=1}^{6} e_i$ and $\sum_{i=1}^{6} f(e_i)$ are in 2Λ, or neither are.*

Remark 1.3. Note that to apply Theorem 1.2 (i) to an isomorphism f, we have to know whether certain ordered sequences or roots are conjugate. Theorem 1.2 (ii), in combination with Theorem 1.1 (i) and (ii), tells us how to verify this when the colored graph on the roots in an ordered sequence is isomorphic to A or B. Theorem 1.2 (iii) and (iv) tells us how to verify this when the colored graph on the roots in an ordered sequence is isomorphic to C_α, D, or F.

Remark 1.4. In the proof of Theorem 1.2, we will specify for each type of K_1 and K_2 which of the graphs A, B, C_α, D, and F, are needed to check whether an isomorphism f extends. Of course one can see this partially from the size and the colors, but it turns out that we can make stronger statements. For example, surprisingly, an isomorphism between two maximal graphs in $\Gamma_{0,1}$ always extends, and even uniquely (Lemma 5.33). In the table in Remark 6.1 we show the requirements for each type of K_1 and K_2.

As we mentioned before, because of the size of Γ it is practically impossible to naively let a computer find and classify all cliques of the above types according to their W-orbit. This holds mainly for the results in Section 5, where we study cliques of type IV. This is the only section where we use a computer program, but without using results from the previous sections to minimize the computations it would have been practically undoable. Checking that two cliques are isomorphic is easily done by hand for types I, II, and III, since with one exception of size fourteen, they are all of size at most eight (see Sections 3 and 4). For type IV we give necessary and sufficient invariants to check if two large cliques are isomorphic in Section 5.

The orbits of the faces of the E_8 root polytope under the action of W are described in [Cox30], Section 7.5. These include all monochromatic cliques of color 1 (see Proposition 2.4). We give a different, more group-theoretical proof that W acts transitively on one type of the facets, see Corollary 3.16. The orbits of *ordered sequences* of the vertices in the faces (except for one type of facets) have been described in [Man74], Corollary 26.8. We summarize his results in Proposition 2.12. Monochromatic cliques of color 0 are orthogonal sets, and their orbits under the action of W are described in [DM10], Corollary 3.3. We describe the action of W on the *ordered sequences* of orthogonal roots in Proposition 4.4.

Our inspiration to study the E_8 root system and the cliques in Γ is the connection to del Pezzo surfaces of degree one. Such surfaces have exactly 240 lines, and there is a bijection between these lines and a root system that is isomorphic to E_8. We have studied the maximal number of lines on these surfaces that go through one point, which will be published in future work. This led us to studying cliques in the colored intersection graph on these lines (which is isomorphic to Γ). A good reference for these surfaces and their lines is [Man74], Chapter IV. In Remarks 2.8, 3.5, 3.22, 4.11, and 5.1, we explain how some of our results translate to this geometric view.

We split the article in chapters that deal with one or more of the types I, II, III, or IV. Note that these four types do not exclude each other, and some results in one section may be part of a result in another section. We ordered the sections such that each section builds as much on the previous ones as possible.
Section 2 states all the needed definitions as well as many known results about E_8 and the action of the Weyl group, and the relation with del Pezzo surfaces. We also set up the notation for the rest of this article. The reader that is familiar with root systems, and with E_8 in particular, can skip this section. Section 3 contains all results on the facets of the E_8 root polytope, and cliques of type III. Section 4 deals with cliques of type I. Section 5 classifies all cliques of type IV. This is the biggest

section, and the only section where we use a computer for some of the results (from Section 5.3 onwards). The results from this section are summarized in the tables in the appendices. Finally, we prove Theorems 1.1 and 1.2 in Section 6.

All computations are done in `magma` ([BCP97]). The code that we used can be found in [Win]. We want to thank David Madore, who gave us useful references for results on E_8 and the action of W. Moreover, there is a great interactive view of E_8 on his website `http://www.madore.org/~david/math/e8w.html`, which has been very insightful.

2. Background: the Weyl group and the E_8 root polytope

Let Λ, E, Γ, and W be as defined in the introduction. In this section we recall some well-known results about these objects, the Weyl group, and the E_8 root polytope. We also make a first step in proving Theorems 1.1 and 1.2, by showing that for two cliques of type I, II, III, or IV in Γ that are isomorphic as colored graphs, there is a type that they both belong to (Lemma 2.13).

Useful references for root systems and the Weyl group are [Bou81], Chapter 6, and [Hum72], Chapter III.

The subgroup of the isometry group of $\mathbb{R}^8$ that is generated by the reflections in the hyperplanes orthogonal to the roots in E is called the Weyl group, and denoted by W_8. This group permutes the elements in E, and since these roots span $\mathbb{R}^8$, the action of W_8 on E is faithful. The Weyl group is therefore finite: it has order $696729600 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$. It is equal to the automorphism group of the E_8 root system ([Hum72], section 12.2), hence also to the automorphism group of the root lattice Λ, and to the group W.

Lemma 2.1. *The Weyl group acts transitively on the E_8 root system.*

Proof. [Hum72], Section 10.4, Lemma C. □

Note that the roots in E are of two types. Either they are of the form $\left(\pm\frac{1}{2}, \dots, \pm\frac{1}{2}\right)$, where an even number of entries is negative (giving $2^7 = 128$ roots), or exactly two entries are non-zero, and they can independently be chosen to be -1 or 1 (giving $4 \cdot \binom{8}{2} = 112$ roots).

Proposition 2.2. *The absolute value of the dot product of any two elements in E is at most 2. Let $e \in E$ be a root. Then e has dot product 2 only with itself, and dot product -2 only with its inverse $-e$. There are exactly 56 roots $f \in E$ with $e \cdot f = 1$, there are exactly 56 roots $g \in E$ with $e \cdot g = -1$, and there are exactly 126 roots in E that are orthogonal to e.*

Proof. From Cauchy-Schwarz it follows that for $e, e' \in E$ we have

$$|e \cdot e'| \leq \|e\| \cdot \|e'\| = 2,$$

and equality holds if and only if e, e' are a scalar multiple of each other. Since all roots are primitive, it follows that $e \cdot e' = 2$ if and only if $e = e'$, and $e \cdot e' = -2$ if and only if $e = -e'$. Since W acts transitively on E (Lemma 2.1), to count the other cases it suffices to prove this for one element in E. Take $e = (1, 1, 0, 0, 0, 0, 0, 0) \in E$.

The roots $f \in E$ with $e \cdot f = 1$ are of the form $f = (a_1, \ldots, a_8)$ with $a_1 + a_2 = 1$. So for these roots we either have $a_1 = a_2 = \frac{1}{2}$, which gives 32 different roots, or $\{a_1, a_2\} = \{0, 1\}$, which gives 24 different roots. This gives a total of 56 roots.
For $f \in E$, we have $e \cdot f = 1$ if and only if $e \cdot -f = -1$, so this gives also 56 roots $g \in E$ with $e \cdot g = -1$.
The roots in E that are orthogonal to e are of the form $f = (a_1, \ldots, a_8)$ with $a_1 + a_2 = 0$. So for these roots we have $a_1 = a_2 = 0$, which gives 60 roots, or $\{a_1, a_2\} = \{-1, 1\}$, which gives 2 roots, or $\{a_1, a_2\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$, which gives 64 roots. This gives a total of 126 roots. □

We continue with results on the E_8 root polytope. Coxeter described all faces of the E_8 root polytope, which he called the 4_{21} polytope, in [Cox30]. The faces come in two types: k-simplices (for $k \leq 7$), given by $k+1$ vertices with angle $\frac{\pi}{3}$ and distance $\sqrt{2}$ between any two of them, and k-crosspolytopes (for $k = 7$), given by $2k$ vertices where every vertex is orthogonal to exactly one other vertex, and has angle $\frac{\pi}{3}$ and distance $\sqrt{2}$ with all the other vertices. We summarize his results in Propositions 2.4 and 2.5.

Lemma 2.3. *Two vertices in the E_8 root polytope have distance $\sqrt{2}$ between them if and only if their dot product is one.*

Proof. For $e, f \in E$ we have $\|e - f\|^2 = e^2 - 2 \cdot e \cdot f + f^2 = 4 - 2 \cdot e \cdot f$. □

Proposition 2.4. *For $k \leq 7$, the set of k-simplices in the E_8 root polytope is given by*

$$\{\{e_1, \ldots, e_{k+1}\} \mid \forall i : e_i \in E;\ \forall j \neq i : e_i \cdot e_j = 1\},$$

where a k-simplex is identified with the set of its vertices. For $k \leq 6$, the k-simplices in the E_8 root polytope are exactly its k-faces.

Proof. The vertices in a k-simplex have dot product 1 by the previous lemma. The fact that the k-faces are exactly the k-simplices for $k \leq 6$ is in [Cox30], section 7.5 or the table on page 414. □

Proposition 2.5. *The facets of the E_8 root polytopes are exactly the 7-simplices and the 7-crosspolytopes contained in it. The set of 7-crosspolytopes is given by*

$$\left\{ \{\{e_1, f_1\}, \ldots, \{e_7, f_7\}\} \;\middle|\; \begin{array}{l} \forall i \in \{1, \ldots, 7\} : e_i, f_i \in E;\ e_i \cdot f_i = 0; \\ \forall j \neq i : e_i \cdot e_j = e_i \cdot f_j = f_i \cdot f_j = 1. \end{array} \right\},$$

where a 7-crosspolytope is identified by the set of its 7 pairs of orthogonal roots.

Proof. The facets are the 7-simplices and the 7-crosspolytopes by [Cox30], Section 7.5 or see the table on page 414. The dot products follow from Lemma 2.3. □

Remark 2.6. We also show that the 7-simplices and the 7-crosspolytopes in the E_8 root polytope are facets in Remarks 3.6 and 3.18.

Corollary 2.7. *The E_8 root polytope has 6720 1-faces, 60480 2-faces, 241920 3-faces, 483840 4-faces, 483840 5-faces, 207360 6-faces, 17280 7-simplices, and 2160 7-crosspolytopes.*

Proof. See [Cox30], p.414. □

Remark - analogy with geometry 2.8. Let us give a quick analogy with geometry, which was our motivation to study the E_8 root lattice. More on this can be found in [Man74], Chapter IV, and in a lot more detail than sketched here.
Let X be a del Pezzo surface of degree one over an algebraically closed field k. Then X is isomorphic to the blow up of $\mathbb{P}^2_k$ in eight points in general position (meaning no three on a line, no six on a conic, and no eight on a cubic that is singular at one of them). Let K_X be the class in Pic X of the anticanonical divisor of X, and let $K_X^\perp$ be the orthogonal complement of K_X in the lattice Pic X. Let $\left(\mathbb{R}\otimes K_X^\perp, \langle\cdot,\cdot\rangle\right)$ be the Euclidean vector space with inner product $\langle\cdot,\cdot\rangle$ defined by the *negative* of the intersection pairing in Pic X. Classes in $K_X^\perp$ with self intersection -2 (so inner product 2 in $\mathbb{R}\otimes K_X^\perp$) form a root system within this vector space, and this root system is isomorphic to E_8.
It is well known that Pic X contains 240 classes c with $c^2 = c \cdot K_X = -1$, called exceptional classes. Let C be the set of exceptional classes in Pic X. For $c \in C$ we have $c+K_X \in K_X^\perp$ and $\langle c+K_X, c+K_X\rangle = 2$, and this gives a bijection between C and the root system in $\mathbb{R}\otimes K_X^\perp$, such that $\langle c_1+K_X, c_2+K_X\rangle = 1 - c_1\cdot c_2$. Therefore the group of permutations of C that preserves the intersection multiplicity is isomorphic to the Weyl group W_8. Moreover, studying the colored intersection graph of C, where colors are given by the intersection multiplicities, is equivalent to studying the colored graph of the E_8 root system, where colors are given by the dot products. Throughout this article, we will remark on some of the analogies of the results for the set C.
For example, the vertices of a k-simplex in the E_8 root polytope correspond to a sequence of $k+1$ exceptional classes in C that are pairwise disjoint. Moreover, for r pairwise disjoint exceptional curves $e_1,\dots,e_r$ (for $1 \le r \le 7$), the exceptional curves that are disjoint to $e_1,\dots,e_r$ are isomorphic to the exceptional curves of the del Pezzo surface of degree $r+1$ that is obtained by blowing down $e_1,\dots,e_r$. We know the number of exceptional curves on del Pezzo surfaces (see [Man74], Table IV.9), and we can use this to compute the number of k-faces of the E_8 root polytope for $k \le 5$.

Remark 2.9. For $k \le 5$, the statement in Corollary 2.7 also follows from the last part of Remark 2.8 and Table (IV.9) in [Man74]: we have

$$\frac{240\cdot 56}{2} = 6720, \quad \frac{240\cdot 56\cdot 27}{3!} = 60480, \quad \frac{240\cdot 56\cdot 27\cdot 16}{4!} = 241920,$$

and so on. For k equal to 6 and for the 7-simplices, the statement is in Proposition 4.7. For the 7-crosspolytopes it follows from Lemma 3.14, see Remark 3.15.

The following propositions state results about the action of the Weyl group on the faces of the E_8 root polytope.

Proposition 2.10. *The group W acts transitively on the set of k-faces for $k \le 5$. There are two orbits of facets.*

Proof. In [Cox30], Section 7.5, it is shown that all k-simplices are conjugate for $k \le 5$, and that any two facets of the same type are conjugate as well. We know that there are two types of facets from Proposition 2.5. □

Remark 2.11. There are two orbits of 6-faces, which we describe in Proposition 4.7. See also Remark 4.10.

We know something even stronger, namely, the action of W on the *ordered sequences* of roots in faces of the E_8 root polytope.

Proposition 2.12. *For all $r \leq 8$ such that $r \neq 7$, the group W acts transitively on the set*

$$R_r = \{(e_1, \ldots, e_r) \in E^s \mid \forall i \neq j : e_i \cdot e_j = 1\}.$$

For $r = 7$, there are two orbits under the action of W.

Proof. In Remark 2.8 we describe a bijection between E and the set C of 240 exceptional curves on a del Pezzo surface of degree one, where two elements in E have dot product a if and only if the two corresponding elements in C have intersection product $1 - a$. This bijection respects the action of W, and under this bijection the set R_r corresponds to the set of sequences of length r of disjoint exceptional curves. The statement now follows from [Man74], Corollary 26.8. □

The following lemma is the first step in proving Theorems 1.1 and 1.2.

Lemma 2.13. *Let K_1, K_2 be two cliques in Γ of type I, II, III, or IV that are isomorphic. Then there is a type I, II, III, or IV that they both belong to.*

Proof. If a clique is of type I or III, then any clique that is isomorphic to it is of the same type. If K_1 is of type II, then its vertices form a k-simplex (for $k \leq 7$) or a k-crosspolytope (for $k = 7$) by Proposition 2.4 and Proposition 2.5. In both cases, K_2 is of the same type, again by Proposition 2.4 and Proposition 2.5. Analogousyly, if K_2 is of type II then so is K_1. Finally, if K_1 and K_2 are both not of types I, II, or III, then they are automatically both of type IV. □

We conclude this section by stating a lemma that will be used throughout this article.

Lemma 2.14. *Let H be a group, let A, B be H-sets, and $f \colon A \longrightarrow B$ a morphism of H-sets. Then the following hold.*

(i) If H acts transitively on A, then H acts transitively on $f(A)$.

(ii) If H acts transitively on B, then all fibers of f have the same cardinality.

(iii) If H acts transitively on A and A is finite, then all non-empty fibers of f have the same cardinality, say n, and $|f(A)| = \frac{|A|}{n}$.

(iv) If H acts transitively on $f(A)$, and there is an element $b \in f(A)$ such that H_b acts transitively on $f^{-1}(b)$, then f acts transitively on A.

Proof.

(i) Take $f(a), f(a') \in f(A)$ with $a, a' \in A$. Assume that H acts transitively on A, then there is an $h \in H$ such that $ha = a'$. Since f is a morphism of H-sets, we have $hf(a) = f(ha) = f(a')$, so H acts transitively on $f(A)$.

(ii) Take $b, b' \in B$. Since H acts transitively on B, there is an $h \in H$ such that $hb = b'$, so $|f^{-1}(b')| = |f^{-1}(hb)| = |hf^{-1}(b)| = |f^{-1}(b)|$.

(iii) Take $b, b' \in B$ such that $f^{-1}(b)$ and $f^{-1}(b')$ are non-empty. Then we have $b, b' \in f(A)$. Since H acts transitively on $f(A)$ by (i), it follows from (ii) that

$f^{-1}(b)$ and $f^{-1}(b')$ have the same cardinality, say n. It is now immediate that $|A| = |f^{-1}(B)| = \sum_{b\in f(A)} n = n|f(A)|$, so $|f(A)| = \frac{|A|}{n}$.

(iv) Take $b \in f(A)$ such that H_b acts transitively on $f^{-1}(b)$. Take $a, a' \in A$. Since H acts transitively on $f(A)$, there are $h, h' \in H$ such that $hf(a) = b$ and $h'f(a') = b$. Then ha and $h'a'$ are contained in $f^{-1}(b)$. Since H_b acts transitively on $f^{-1}(b)$, there is an element $g \in H_b$ with $gha = h'a'$. So we have $h'^{-1}gha = a'$ and H acts transitively on A. □

3. Facets of the E_8 root polytope and cliques of size at most three

In this section we study the cliques in Γ of type III, and the facets of the E_8 root polytope. We give an alternative proof for the fact that W acts transitively on the set of facets that are 7-crosspolytopes (Corollary 3.16), and we prove the following propositions.

Proposition 3.1. *For $a \in \{\pm 1, -2, 0\}$, The group W acts transitively on the set*

$$\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = a\}.$$

Proposition 3.2. *For $a, b, c \in \{-2, -1, 0, 1\}$, the group W acts transitively on the set*

$$\{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = a,\ e_2 \cdot e_3 = b,\ e_1 \cdot e_3 = c\},$$

in all cases where it is not empty.

Note that these two propositions describe the orbits under the action of W of *sequences* of the vertices of cliqes in Γ, hence they also prove Theorem 1.2 for cliques of Type III; see Corollary 3.33. The proof of Proposition 3.1 can be found below Proposition 3.13, and the proof of Proposition 3.2 below Lemma 3.32. Throughout this section we do not use any computer programs. More background on the E_8 root polytope can be found in [Cox30] and [Cox48].

We start by some results on the facets of the E_8 root polytope that are 7-simplices. The results on the facets that are 7-crosspolytopes are in Lemmas 3.16 and 3.17. Consider the set

$$U = \{(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \in E^8 \mid \forall i \neq j : e_i \cdot e_j = 1\}.$$

Note that an element in U is a sequence of eight roots that form a 7-simplex. Define the following roots, and note that $(u_1, \ldots, u_8)$ is an element in U.

$$u_1 = (1,1,0,0,0,0,0,0); \qquad u_5 = (1,0,0,0,0,1,0,0);$$
$$u_2 = (1,0,1,0,0,0,0,0); \qquad u_6 = (1,0,0,0,0,0,1,0);$$
$$u_3 = (1,0,0,1,0,0,0,0); \qquad u_7 = (1,0,0,0,0,0,0,1);$$
$$u_4 = (1,0,0,0,1,0,0,0); \qquad u_8 = \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right).$$

Lemma 3.3. *Every element in U generates a sublattice of index 3 of the root lattice Λ, and the group W acts freely on U.*

Proof. By Proposition 2.12, it is enough to check the first statement for one element in U. The matrix whose i-th row is u_i for $i \in \{1, \ldots, 8\}$ has determinant 3, so $u_1, \ldots, u_8$ are linearly independent and generate a sublattice of rank 8 and index 3 in Λ. Take $w \in W$ such that there is an element $u \in U$ with $w(u) = u$. Then w fixes the sublattice generated by u, so for all $x \in \Lambda$ we have $3w(x) = w(3x) = 3x$. Since Λ is torsion free, this implies that w fixes all of Λ. It follows that w is the identity. We conclude that the action of W on U is free. □

Corollary 3.4. *Let $u = (e_1, \dots, e_8)$ be an element in U. Then $\frac{1}{3}\sum_{i=1}^{8} e_i$ is contained in Λ.*

Proof. By Lemma 3.3, we know that the roots $e_1, \dots, e_8$ generate a lattice M of index 3 in Λ. Set $v = \frac{1}{3}\sum_{i=1}^{8} e_i$. Since $v \cdot e_i = 3$ for $i \in \{1, \dots, 8\}$, we have $\frac{1}{3}v \in M^\vee$, where $M^\vee$ is the dual lattice of M. But the dual lattice $\Lambda^\vee$ has index 3 in $M^\vee$, so it follows that $3 \cdot \frac{1}{3}v = v$ is contained in $\Lambda^\vee$. Since Λ is unimodular, it is self dual, so v is contained in Λ. □

Remark - analogy with geometry 3.5. Let X be a del Pezzo surface of degree 1 and K_X its canonical divisor, see Remark 2.8. Lemma 3.3 can be stated in terms of X as follows. For every set of eight pairwise disjoint exceptional classes $c_1, \dots, c_8$ there exists a unique class l such that we have $K_X = -3l + \sum_{i=1}^{8} c_i$ and $(l, c_1, \dots, c_8)$ is a basis for Pic X; one can blow down the exceptional curves corresponding to $c_1, \dots, c_8$ to eight points in $\mathbb{P}^2$, such that l is the class of the pullback of a line in $\mathbb{P}^2$ that does not contain any of these eight points.

Remark 3.6. Let $u = (e_1, \dots, e_8)$ be an element in U. We know that $e_1, \dots, e_8$ define a facet of the E_8 root polytope. This also follows from from Corollary 3.4. Indeed, for $v = \frac{1}{3}\sum_{i=1}^{8} e_i$ we have $v \cdot e_i = 3$ for $i \in \{1, \dots, 8\}$, and we have

$$v \cdot e = \frac{1}{3}\sum_{i=1}^{8} e_i \cdot e \leq \frac{1}{3}\sum_{i=1}^{8} 1 = \frac{8}{3} < 3$$

for $e \in E \setminus \{e_1, \dots, e_8\}$. This implies that the whole E_8 root polytope lies on one side of the hyperplane given by $v \cdot x = 3$, and the intersection of the polytope with this hyperplane, which is exactly given by the convex combinations of $e_1, \dots, e_8$, lies in the boundary of the polytope. Hence $e_1, \dots, e_8$ generate a facet of the E_8 root polytope, and v is the normal vector to this facet.

We will now prove part of Proposition 3.1.

Lemma 3.7. *For any $a \in \{-2, \pm 1\}$, the group W acts transitively on the set*

$$A_a = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = a\}.$$

Proof. The group W acts transitively on A_1 by Proposition 2.12. There is a bijection between the W-sets A_1 and A_{-1} given by

$$f\colon A_1 \longrightarrow A_{-1},\ (e_1, e_2) \longmapsto (e_1, -e_2).$$

It follows from Lemma 2.14 that W acts transitively on A_{-1}, too. Finally, we have a bijection

$$E \longrightarrow A_{-2},\ e \longmapsto (e, -e),$$

so W acts transitively on A_{-2} by Proposition 2.12 and by Lemma 2.14. □

Before we prove the rest of Proposition 3.1, we prove Proposition 3.2 for the cases $(a,b,c) = (-1,-1,-1)$ (Corollary 3.9) and $(a,b,c) = (0,0,1)$ (Lemma 3.11), which we will use later.

Lemma 3.8. *For $e_1, e_2 \in E$ with $e_1 \cdot e_2 = -1$ there is a unique element $e \in E$ with $e \cdot e_1 = e \cdot e_2 = -1$, which is given by $e = -e_1 - e_2$.*

Proof. Take $e_1, e_2, e \in E$ with $e_1 \cdot e_2 = -1$ and $e \cdot e_1 = e \cdot e_2 = -1$. Set $f = e_1 + e_2 + e$. Then we have $\|f\| = 0$, hence $f = 0$, so $e = -e_1 - e_2$. Therefore e is unique if it exists. Moreover, we have $\| -e_1 - e_2\| = \sqrt{2}$, so $-e_1 - e_2$ is an element in E that satisfies the conditions. □

Corollary 3.9. *The group W acts transitively on the W-set*

$$\{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_2 \cdot e_3 = e_1 \cdot e_3 = -1\}.$$

Proof. By Lemma 3.8 there is a bijection between the sets

$$\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = -1\}$$

and

$$\{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_2 \cdot e_3 = e_1 \cdot e_3 = -1\},$$

given by $(e_1, e_2) \longmapsto (e_1, e_2, -e_1 - e_2)$. The statement now follows from Lemma 3.7 and Lemma 2.14. □

Lemma 3.10. *Take $e_1,\ e_2 \in E$ such that $e_1 \cdot e_2 = 1$. Then there are exactly 72 elements of E orthogonal to e_1 and e_2.*

Proof. By Lemma 3.7 it is enough to check this for fixed $e_1, e_2 \in E$ with $e_1 \cdot e_2 = 1$. Set $e_1 = (1,1,0,0,0,0,0,0)$, $e_2 = (1,0,1,0,0,0,0,0)$. Then $e_1 \cdot e_2 = 1$. An element $f \in E$ with $f \cdot e_1 = f \cdot e_2 = 0$ is of the form $f = (a_1, \ldots, a_8)$ with $a_1 + a_2 = 0$ and $a_1 + a_3 = 0$, hence $a_1 = -a_2$ and $a_2 = a_3$. If f is of the form $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$, then there are 32 such possibilities. If f has two non-zero entries, given by ± 1, then a_1, a_2, a_3 should all be zero, which gives 40 possibilities. We find a total of 72 possibilities for f. □

Lemma 3.11. *Consider the set*

$$B = \{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_2 \cdot e_3 = 0;\ e_1 \cdot e_3 = 1\}.$$

We have $|B| = 967680$, and the following hold.

(i) The group W acts transitively on B.

(ii) For every element $b = (e_1, e_2, e_3) \in B$, there are exactly 6 roots that have dot product 1 with e_1, e_2 and e_3. These 6 roots, together with e_1 and e_3, form a facet in the set U.

Proof. From Proposition 2.2 and Lemma 3.10 we have

$$|B| = 240 \cdot 56 \cdot 72 = 967680.$$

Set $e_1 = (1,1,0,0,0,0,0,0)$, $e_2 = (0,0,1,1,0,0,0,0)$, and $e_3 = (1,0,0,0,1,0,0,0)$. Then $b = (e_1, e_2, e_3)$ is an element in B. Let W_b be its stabilizer in W and Wb its orbit in B. Let U_b be the set

$$U_b = \{e \in E \mid e \cdot e_1 = e \cdot e_2 = e \cdot e_3 = 1\}.$$

For an element $e = (a_1, \ldots, a_8) \in U_b$, we have $a_1 + a_2 = a_3 + a_4 = a_1 + a_5 = 1$. From this we find

$$U_b = \left\{ \begin{array}{c} (1,0,0,1,0,0,0,0) \\ (1,0,1,0,0,0,0,0) \\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \frac{1}{2}\right) \\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \frac{1}{2}, -\frac{1}{2}\right) \\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \end{array} \right\}.$$

We conclude that there are 6 roots that have dot product 1 with e_1, e_2, and e_3. It is obvious that these 6 elements, together with e_1 and e_3, form an element of the set U.

We have $\frac{|W|}{|W_b|} = |Wb| \leq |B|$. We want to show that the latter is an equality. The group W_b acts on U_b. Let w be an element of W_b that fixes all the roots in U_b. Since the roots in $\{e_1, e_3\} \cup U_b$ form an element in U, by Lemma 3.3 this implies that w is the identity. Therefore the action of W_b on U_b is faithful. This implies that W_b injects into S_6, so $|W_b| \leq 720$. We now have

$$967680 = \frac{|W|}{720} \leq \frac{|W|}{|W_b|} = |Wb| \leq |B| = 967680,$$

so we have equality everywhere and therefore we have $Wb = B$. We conclude that W acts transitively on B, proving (i). Part (ii) clearly holds for the element b, and from part (i) it follows that it holds for all elements in B. □

We proceed to prove the rest of Proposition 3.1.

Lemma 3.12. *For $e_1 = (1,1,0,0,0,0,0,0), e_2 = (0,0,1,1,0,0,0,0) \in E$, there are 32 elements e in E such that $e \cdot e_1 = 0$ and $e \cdot e_2 = 1$.*

Proof. Take $f \in E$ with $f \cdot e_1 = 0$ and $f \cdot e_2 = 1$. Then f is of the form $f = (a_1, a_2, a_3, a_4, \ldots, a_8)$ with $a_1 + a_2 = 0$ and $a_3 + a_4 = 1$. If f is of the form $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$, then $a_1 = -a_2$ and $a_3 = a_4 = \frac{1}{2}$. There are 16 such possibilities. If f has two non-zero entries given by ± 1, then either $a_3 = 1,\ a_1 = a_2 = a_4 = 0$, or $a_4 = 1,\ a_1 = a_2 = a_3 = 0$. This gives 16 possibilities. We find a total of 32 possibilities for f. □

Proposition 3.13. *The group W acts transitively on the set*

$$A_0 = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\}.$$

Proof. Consider the set $B' = \{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_1 \cdot e_3 = 0;\ e_2 \cdot e_3 = 1\}$. Note that there is a bijection between the W-set B' and the W-set B in Lemma 3.11, given by $(e, f, g) \longmapsto (f, e, g)$. Therefore, the group W acts transitively on B' and we have $|B'| = 967680$ by Lemma 3.11. We have a projection $\lambda \colon B' \longrightarrow A_0$ on the first two coordinates. We show that λ is surjective. Fix the roots $e_1 = (1,1,0,0,0,0,0,0)$ and $e_2 = (0,0,1,1,0,0,0,0)$ in E. Then (e_1, e_2) is an element of A_0. Take $e \in E$, then (e_1, e_2, e) is in B' if and only if $e \cdot e_1 = 0$ and $e \cdot e_2 = 1$. By Lemma 3.12 this gives 32 possibilities for e, so $|\lambda^{-1}((e_1, e_2))| = 32$. Since W acts transitively on B', it follows from Lemma 2.14 that all non-empty fibers of λ have cardinality 32, and $|\lambda(B')| = \frac{|B'|}{32} = 30240$. By Proposition 2.2 we have $|A_0| = 240 \cdot 126 = 30240$. We conclude that $\lambda(B') = A_0$. Hence λ is surjective. Therefore, the group W acts transitively on A_0 by Lemma 2.14. □

Proof of Proposition 3.1. This follows from the previous proposition together with Lemma 3.7.

Before we continue proving Proposition 3.2, we complete our study on the facets of the E_8 root polytope. Define the set

$$C = \left\{ \{\{e_1, f_1\}, \dots, \{e_7, f_7\}\} \;\middle|\; \begin{array}{l} \forall i \in \{1, \dots, 7\} : e_i, f_i \in E;\ e_i \cdot f_i = 0; \\ \forall j \neq i : e_i \cdot e_j = e_i \cdot f_j = f_i \cdot f_j = 1. \end{array} \right\}.$$

Elements in C are facets that are 7-crosspolytopes by Proposition 2.4. We define the following elements $c_1, \dots, c_7, d_1, \dots, d_7$. Note that $\{\{c_1, d_1\}, \dots, \{c_7, d_7\}\}$ is an element in C.

$$\begin{aligned}
c_1 &= (1,1,0,0,0,0,0,0)\,, & d_1 &= (0,0,1,1,0,0,0,0),\\
c_2 &= (1,0,1,0,0,0,0,0)\,, & d_2 &= (0,1,0,1,0,0,0,0),\\
c_3 &= (1,0,0,1,0,0,0,0), & d_3 &= (0,1,1,0,0,0,0,0),\\
c_4 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), & d_4 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right),\\
c_5 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), & d_5 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right),\\
c_6 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\right), & d_6 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right),\\
c_7 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right), & d_7 &= \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\right).
\end{aligned}$$

Lemma 3.14. *For $e_1, e_2 \in E$ with $e_1 \cdot e_2 = 0$, there are exactly 12 elements $e \in E$ with $e \cdot e_1 = e \cdot e_2 = 1$. These 12 elements, together with e_1 and e_2, form an element in C, and this is the unique element in C containing e_1, e_2.*

Proof. By Proposition 3.13, it is enough to check this for fixed $e_1, e_2 \in E$ with $e_1 \cdot e_2 = 0$. Take $e_1 = c_1$, $e_2 = d_1$ in E. For a root $e = (a_1, \dots, a_8)$ in E with $e \cdot c_1 = e \cdot d_1 = 1$, we have either $a_1 = a_2 = a_3 = a_4 = \frac{1}{2}$, which implies $e \in \{c_4, \dots, c_7, d_4, \dots, d_7\}$, or $\{a_1, a_2\} = \{a_3, a_4\} = \{0, 1\}$, which implies $e \in \{c_2, c_3, d_2, d_3\}$. Therefore there are exactly 12 possibilities $\{c_2, \dots, c_7, d_2, \dots, d_7\}$ for e, and we conclude that $\{\{c_1, d_1\}, \dots, \{c_7, d_7\}\}$ is the unique element in C containing c_1, d_1. □

Remark 3.15. Since elements in C correspond to 7-crosspolytopes, we know that $|C| = 2160$ from Corollary 2.7. This also follows from the previous lemma. Recall the set $A_0 = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\}$. By Lemma 3.14, for every element (e_1, e_2) in A_0 there is a unique element in C containing e_1, e_2. But every element in C contains seven pairs f_1, f_2 such that (f_1, f_2) and (f_2, f_1) are in A_0, so the map $A_0 \longrightarrow C$ is fourteen to one. Hence we have $|C| = \frac{|A_0|}{14} = \frac{240 \cdot 126}{14} = 2160$.

Corollary 3.16. *The group W acts transitively on C.*

Proof. Consider the set $A_0 = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\}$. By Proposition 3.13, the group W acts transitively on A_0. By Lemma 3.14 there is a map $A_0 \longrightarrow C$, sending (e_1, e_2) to the unique element in C that contains e_1 and e_2. This map is clearly surjective. It follows from Lemma 2.14 that W acts transitively on C. □

Lemma 3.17. *Every element in C generates a sublattice of finite index in Λ.*

Proof. By Corollary 3.16, it is enough to check this for one element in C. Take the element $\{\{c_1, d_1\}, \dots, \{c_7, d_7\}\}$ in C, where the c_i, d_i are defined above Lemma 3.14. The matrix whose rows are the vectors $c_1, \dots, c_7, d_1, \dots, d_7$ has rank 8, so these 14 elements generate a sublattice L of finite index in Λ. □

Remark 3.18. Let $\{\{e_1, f_1\}, \dots, \{e_7, f_7\}\}$ be an element in C, and let c be the set $c = \{e_1, \dots, e_7, f_1, \dots, f_7\}$. We know that the elements in c are the vertices of a facet of the E_8 root polytope. We show how this also follows from the previous lemma. Take $i \in \{1, \dots, 7\}$, then we have $(e_i + f_i) \cdot e = 2$ for all $e \in c$. Since the elements in c generate a full rank sublattice, this implies that $e_i + f_i = e_j + f_j$ for all $i, j \in \{1, \dots, 7\}$. So the vector $n = \frac{1}{7}\sum_{i=1}^{7}(e_i + f_i) = e_1 + f_1$ is an element in Λ with $n \cdot e = 2$ for $e \in s$. Take $e \in E \setminus s$, and note that e can not have dot product 1 with both e_1 and f_1 by Lemma 3.14. It follows that we have $n \cdot e < 2$, so the entire E_8 root polytope lies on one side of the affine hyperplane given by $n \cdot x = 2$. Moreover, this hyperplane intersects the E_8 root polytope in its boundary, and exactly in the convex combinations of the roots $e_1, \dots, e_7, f_1, \dots, f_7$. Therefore these roots are the vertices of a facet of the E_8 root polytope with normal vector n.

We continue with Proposition 3.2, and prove it for $(a, b, c) = (0, 0, 0)$. Consider the sets

$$V_3 = \{(e_1, e_2, e_3) \in E^3 \mid \forall i \neq j : e_i \cdot e_j = 0\}$$

and

$$V_4 = \{(e_1, e_2, e_3, e_4) \in E^4 \mid \forall i \neq j : e_i \cdot e_j = 0\}.$$

We begin by studying V_4. To this end, recall the set U defined above Lemma 3.3, and define the set

$$Z = \{(\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\}) \mid \forall i : e_i \in E;\ \forall j \neq i : e_i \cdot e_j = 1\}.$$

Remark 3.19. We have a surjective map $U \longrightarrow Z$ by simply forgetting the order of e_i and e_{i+1} for $i \in \{1, 3, 5, 7\}$. Since W acts transitively on U (Proposition 2.12), it follows from Lemma 2.14 that W acts transitively on Z. By Lemma 3.3, the action of W on U is free, so we have $|U| = |W|$, and $|Z| = \frac{|U|}{2^4} = \frac{|W|}{2^4} = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7$.

We want to define a map $\alpha \colon Z \longrightarrow V_4$. To do this we use the following lemma.

Lemma 3.20. *For an element $z = (\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})$ in Z, there are unique roots $f_1, f_2, f_3, f_4 \in E$ with*

$$f_1 \cdot e_i = 0,\ f_1 \cdot e_j = 1 \ \text{ for } i \in \{1, 2\},\ j \notin \{1, 2\};$$
$$f_2 \cdot e_i = 0,\ f_2 \cdot e_j = 1 \ \text{ for } i \in \{3, 4\},\ j \notin \{3, 4\};$$
$$f_3 \cdot e_i = 0,\ f_3 \cdot e_j = 1 \ \text{ for } i \in \{5, 6\},\ j \notin \{5, 6\};$$
$$f_4 \cdot e_i = 0,\ f_4 \cdot e_j = 1 \ \text{ for } i \in \{7, 8\},\ j \notin \{7, 8\}.$$

For these f_1, f_2, f_3, f_4 *we have* $f_i \cdot f_j = 0$ *for* $i \neq j$*, and* $3\sum_{i=1}^{4} f_i = \sum_{i=1}^{8} e_i$.

Proof. By Lemma 3.3, the elements $e_1, \ldots, e_8$ generate a full rank sublattice of Λ, so an element $f \in E$ is uniquely determined by the intersection numbers $f \cdot e_i$ for $i \in \{1, \ldots, 8\}$. We will show existence. Set $v = \frac{1}{3}\sum_{i=1}^{8} e_i$. By Corollary 3.4, the vector v is an element in Λ. We have $\|v\| = \sqrt{8}$, and $v \cdot e_i = 3$ for $i \in \{1, \ldots, 8\}$. For $i \in \{1, 2, 3, 4\}$, set $f_i = v - e_{2i-1} - e_{2i}$. Then $\|f_i\| = \sqrt{2}$, so $f_i \in E$. Moreover, f_1, f_2, f_3, f_4 satisfy the conditions in the lemma. □

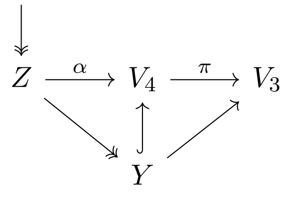
We now define a map $\alpha \colon Z \longrightarrow V_4$, $(\{e_1, e_2\}, \ldots, \{e_7, e_8\}) \longmapsto (f_1, f_2, f_3, f_4)$, where f_1, f_2, f_3, f_4 are the unique elements found in Lemma 3.20.

Corollary 3.21. *If* (f_1, f_2, f_3, f_4) *is an element in the image of* α*, then* $x = \sum_{i=1}^{4} f_i$ *is a primitive element of* Λ *with norm* $\sqrt{8}$.

Proof. Take (f_1, f_2, f_3, f_4) in the image of α, and let $(\{e_1, e_2\}, \ldots, \{e_7, e_8\}) \in Z$ be such that $(f_1, f_2, f_3, f_4) = \alpha((\{e_1, e_2\}, \ldots, \{e_7, e_8\}))$. Set $x = \sum_{i=1}^{4} f_i$. Then we have $3x = \sum_{i=1}^{8} e_i$ by Lemma 3.20. It follows that $\|3x\|^2 = 72$, hence $\|x\|^2 = 8$. Moreover, for any $i \in \{1, \ldots, 8\}$ we have $3x \cdot e_i = 9$, hence $x \cdot e_i = 3$. This implies that if we have $x = m \cdot x'$ for some $m \in \mathbb{Z}$, $x' \in \Lambda$, then $m|2$ and $m|3$, so $m = 1$ and x is primitive. □

Remark - analogy with geometry 3.22. Let X be a del Pezzo surface of degree one over an algebraically closed field, and C the set of exceptional classes in Pic X. The map α has a nice description in the geometric setting, through the bijection $C \longrightarrow E$, $c \longmapsto c + K_X$. Take $z = (\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})$ an element in Z. The roots $e_1, \ldots, e_8$ correspond to classes $c_1, \ldots, c_8$ in C with $c_i \cdot c_j = 0$ for all $i \neq j \in \{1, \ldots, 8\}$. These classes correspond to pairwise disjoint curves on X that can be blown down to points $P_1, \ldots, P_8$ in $\mathbb{P}^2$ such that c_i is the class of the exceptional curve above P_i for $i \in \{1, \ldots, 8\}$ (See [Man74]). The conditions for f_i in Lemma 3.20 are equivalent with f_i being the strict transform on X of the line in $\mathbb{P}^2$ through P_{2i-1} and P_{2i} for $i \in \{1, 2, 3, 4\}$. This geometrical argument immediately proves the uniqueness of f_i.

Let $\pi \colon V_4 \longrightarrow V_3$ be the projection on the first three coordinates. From the maps π and α, transitivity on V_3 will follow (Proposition 3.27). Let Y be the image of α. We will show that V_4 has two orbits under the action of W, given by Y and $V_4 \setminus Y$ (Proposition 3.28). The following commutative diagram shows the maps and sets that are defined.



Lemma 3.23. *The map* α *is injective.*

Proof. Consider the roots in E given by

$$f_1 = (1,1,0,0,0,0,0,0)\,, \qquad f_3 = (0,0,0,0,1,1,0,0)\,,$$
$$f_2 = (0,0,1,1,0,0,0,0)\,, \qquad f_4 = (1,-1,0,0,0,0,0,0)\,.$$

Then $v = (f_1, f_2, f_3, f_4)$ is an element in V_4. Let $(\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})$ be an element in the fiber of α above v. Then we have

$$\begin{aligned} &e_1 \cdot f_1 = e_2 \cdot f_1 = 0 \text{ and } e_1 \cdot f_i = e_2 \cdot f_i = 1 \text{ for all } i \neq 1;\\ &e_3 \cdot f_2 = e_4 \cdot f_2 = 0 \text{ and } e_3 \cdot f_i = e_4 \cdot f_i = 1 \text{ for all } i \neq 2;\\ &e_5 \cdot f_3 = e_6 \cdot f_3 = 0 \text{ and } e_5 \cdot f_i = e_6 \cdot f_i = 1 \text{ for all } i \neq 3;\\ &e_7 \cdot f_4 = e_8 \cdot f_4 = 0 \text{ and } e_7 \cdot f_i = e_8 \cdot f_i = 1 \text{ for all } i \neq 4. \end{aligned} \tag{1}$$

Write $e_1 = (a_1, \ldots, a_8)$. Then (1) implies $a_1 + a_2 = 0$ and $a_1 - a_2 = 1$, and $a_3 + a_4 = a_5 + a_6 = 1$. So e_1 is $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$ or $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, and e_2 is the other. Analogously we find:

$$\begin{aligned} \{e_3, e_4\} &= \{(1,0,0,0,0,1,0,0)\,, (1,0,0,0,1,0,0,0)\}\,,\\ \{e_5, e_6\} &= \{(1,0,0,1,0,0,0,0)\,, (1,0,1,0,0,0,0,0)\}\,,\\ \{e_7, e_8\} &= \left\{\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)\right\}. \end{aligned}$$

Hence the fiber above v has cardinality one. Since W acts transitively on Z, we conclude from Lemma 2.14 that all non-empty fibers of α have cardinality one, so α is injective. ☐

Remark 3.24. By the previous lemma, there is a bijection between the sets Z and $\alpha(Z) = Y$. Since α is a W-map, it follows that Y is a W-set, and that W acts transitively on Y by Lemma 2.14.

We state two more lemmas before we prove that W acts transitively on V_3.

Lemma 3.25. *Consider the elements in E given by*

$$e_1 = (1,1,0,0,0,0,0,0); \qquad f_1 = (0,0,0,0,0,0,1,1)$$
$$e_2 = (0,0,1,1,0,0,0,0); \qquad f_2 = (0,0,0,0,0,0,-1,-1).$$
$$e_3 = (0,0,0,0,1,1,0,0);$$

Then $v = (e_1, e_2, e_3, f_1)$ and $v' = (e_1, e_2, e_3, f_2)$ are elements in V_4 that are not in Y.

Proof. It is easy to check that v and v' are in V_4. We have

$$e_1 + e_2 + e_3 + f_1 = 2 \cdot \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

and

$$e_1 + e_2 + e_3 + f_2 = 2 \cdot \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right),$$

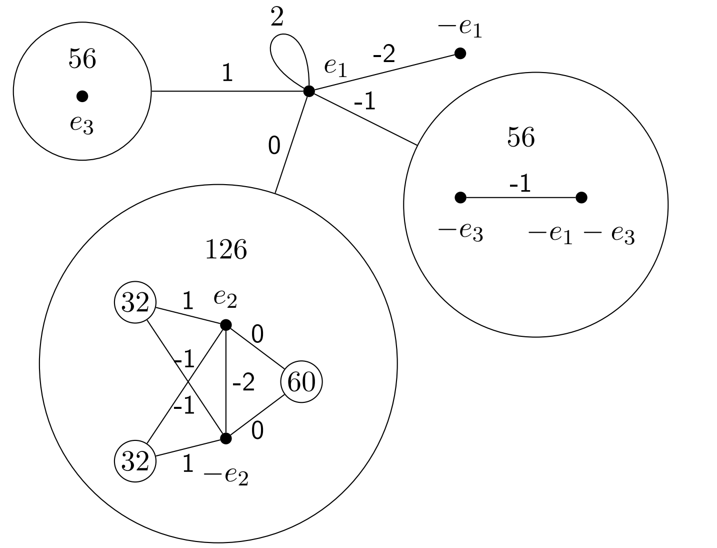
hence both $e_1 + e_2 + e_3 + f_1$ and $e_1 + e_2 + e_3 + f_2$ are not primitive elements in Λ and therefore not contained in Y by Corollary 3.21. ☐

Lemma 3.26. *For two elements $e_1, e_2 \in E^2$ with $e_1 \cdot e_2 = 0$, there are exactly 60 roots $e \in E$ such that $e_1 \cdot e = e_2 \cdot e = 0$.*

Proof. By Proposition 3.13, it is enough to check this for two orthogonal roots e_1, e_2 in E. Set $e_1 = (1,1,0,0,0,0,0,0)$, $e_2 = (0,0,1,1,0,0,0,0)$. An element $f \in E$ with

$f \cdot e_1 = f \cdot e_2 = 0$ is of the form $f = (a_1, a_2, a_3, \ldots, a_8)$ with $a_1 = -a_2$ and $a_3 = -a_4$. If f is of the form $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$, then there are 32 such possibilities. If f has two non-zero entries, given by ± 1, then there are 28 possibilities. We find a total of 60 possibilities for f. □

The following graph summarizes the results in Proposition 2.2 and Lemmas 3.8, 3.12 and 3.26. Vertices are roots, and the number in a subset is its cardinality. The number on an edge between two subsets is the dot product of two roots, one from each subset.



Proposition 3.27. *Let $v = (f_1, f_2, f_3)$ be an element of V_3. The following hold.*

(i) We have $|V_3| = 1814400$, and the group W acts transitively on V_3.

(ii) We have $|\pi^{-1}(v)| = 26$, and $|\pi^{-1}(v) \cap Y| = 24$.

(iii) For $\{(f_1, f_2, f_3, u), (f_1, f_2, f_3, u')\} = \pi^{-1}(v) \setminus Y$, we have $u = -u'$, and for $(f_1, f_2, f_3, e) \in \pi^{-1}(v) \cap Y$, we have $e \cdot u = e \cdot u' = 0$.

Proof. From Proposition 2.2 and Lemma 3.26 it follows that

$$|V_3| = 240 \cdot 126 \cdot 60 = 1814400.$$

Consider the map $\lambda = \pi \circ \alpha \colon Z \to V_3$. Note that λ is a W-map, since both π and α are. We want to show that λ is surjective. Set

$$f_1 = (1,1,0,0,0,0,0,0),\ f_2 = (0,0,1,1,0,0,0,0),\ f_3 = (0,0,0,0,1,1,0,0).$$

Then we have $v = (f_1, f_2, f_3) \in V_3$. Define the roots

$$e_1 = \left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right), \qquad e_5 = (1,0,0,1,0,0,0,0),$$

$$e_2 = \left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right), \qquad e_6 = (1,0,1,0,0,0,0,0),$$

$$e_3 = (1,0,0,0,0,1,0,0), \qquad e_7 = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

$$e_4 = (1,0,0,0,1,0,0,0), \qquad e_8 = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right).$$

Note that for $i \neq j$ we have $e_i \cdot e_j = 1$, so $(\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})$ is an element in Z. We have

$$f_1 \cdot e_1 = f_1 \cdot e_2 = 0 \text{ and } f_1 \cdot e_i = 1 \text{ for all } i \notin \{1, 2\};$$
$$f_2 \cdot e_3 = f_2 \cdot e_4 = 0 \text{ and } f_2 \cdot e_i = 1 \text{ for all } i \notin \{3, 4\};$$
$$f_3 \cdot e_5 = f_3 \cdot e_6 = 0 \text{ and } f_3 \cdot e_i = 1 \text{ for all } i \notin \{5, 6\},$$

so $\lambda\left((\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\})\right) = v$. Hence the fiber of λ above v is not empty, and we want to compute its cardinality. We first compute the cardinality of the fiber of π above v. For an element $f = (a_1, \ldots, a_8) \in E$, we have $(f_1, f_2, f_3, f) \in V_4$ if and only if $a_1 + a_2 = a_3 + a_4 = a_5 + a_6 = 0$. This gives 16 possibilities for f with $a_i \in \left\{\pm\frac{1}{2}\right\}$ for $i \in \{1, \ldots, 8\}$, and 10 possibilities for f where the two non-zero entries are ± 1. We conclude that $|\pi^{-1}(v)| = 26$. Set $g_1 = (0, 0, 0, 0, 0, 0, 1, 1)$ and $g_2 = (0, 0, 0, 0, 0, 0, -1, -1)$, then $u = (f_1, f_2, f_3, g_1)$ and $u' = (f_1, f_2, f_3, g_2)$ are both elements in $\pi^{-1}(v)$. By Lemma 3.25, we know that the fibers of α above u and u' are empty. Since α is injective, this implies $|\lambda^{-1}(v)| \leq 24$. Since $\lambda^{-1}(v)$ is not empty, by Lemma 2.14, we have $|\lambda(Z)| = \frac{|Z|}{|\lambda^{-1}(v)|}$. Combining this, we find

$$\frac{|Z|}{24} \leq \frac{|Z|}{|\lambda^{-1}(v)|} = |\lambda(Z)| \leq |V_3| = 1814400 = \frac{|Z|}{24}.$$

So we have equality everywhere, hence $|\lambda^{-1}(v)| = 24$, and $|\lambda(Z)| = |V_3|$, so λ is surjective. Since W acts transitively on Z, we conclude from Lemma 2.14 that W acts transitively on V_3, too. This proves (i). To prove (ii), note that we showed that $|\pi^{-1}(v)| = 26$ and $|\lambda^{-1}(v)| = 24$, and since α is injective, we have the equality $|\pi^{-1}(v) \cap Y| = |\lambda^{-1}(v)| = 24$. Since π is a W-map, and W acts transitively on V_3, the result holds for all elements in V_3. Finally, (iii) is an easy check for the element v, after writing down the 26 elements in $\pi^{-1}(v)$. Since W acts transitively on V_3, this holds for all elements in V_3. $\square$

Proposition 3.28. *The set V_4 has two orbits under the action of W, which are Y and $V_4 \setminus Y$. We have $|Y| = 43545600$ and $|V_4 \setminus Y| = 3628800$. An element $(e_1, \ldots, e_4)$ is in $V_4 \setminus Y$ if and only if $\sum_{i=1}^{4} e_i \in 2\Lambda$.*

Proof. From Remark 3.24 it follows that Y is an orbit under the action of W on V_4. Therefore $O = V_4 \setminus Y$ is also a W-set. Consider the restriction $\pi|_O$ of π to O. Let e_1, e_2, e_3, f_1, f_2 be as in Lemma 3.25, and set $v = (e_1, e_2, e_3)$, $u = (e_1, e_2, e_3, f_1)$, and $u' = (e_1, e_2, e_3, f_2)$. Then we have $v \in V_3$, and $u, u' \in \pi|_O^{-1}(v)$ by Lemma 3.25. From Proposition 3.27 we know that $|\pi^{-1}(v) \cap Y| = 24$, so $\left|\pi|_O^{-1}(v)\right| = 2$. This implies $\pi|_O^{-1}(v) = \{u, u'\}$. Consider the element r in W given by the reflection in the hyperplane that is orthogonal to f_1. Since e_1, e_1, e_3 are contained in this hyperplane, the reflection r is contained in the stabilizer W_v in W of v. Moreover, since $f_2 = -f_1$, the reflection r interchanges f_1 and f_2, hence W_v acts transitively on $\pi|_O^{-1}(v)$. Since W acts transitively on V_3 by Proposition 3.27, we conclude that W acts transitively on O from Lemma 2.14. From Proposition 3.27 it follows that $|Y| = |V_3| \cdot 24 = 43545600$, and $|O| = |V_3| \cdot 2 = 3628800$. By Corollary 3.21, for every element (g_1, g_2, g_3, g_4) in Y the sum $\sum_{i=1}^{4} g_i$ is primitive. On the other hand, $u = (e_1, e_2, e_3, f_1)$ is an element in O, and $e_1 + e_2 + e_3 + f_1 = 2 \cdot \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

So the sum of the coordinates of u is an element in 2Λ, and since W acts transitively on O, this holds for every element in O. □

Now that we proved that W acts transitively on V_3, there is one last case of Proposition 3.2 that we prove separately (Lemma 3.32). We state two auxiliary lemmas first.

Lemma 3.29. *Let r be a positive integer, and let G be a graph with vertex set $\{v_1, \ldots, v_r, w_1 \ldots, w_r\}$, and edge set $\{\{v_i, w_i\} \mid i \in \{1, \ldots, r\}\}$. Let A be the automorphism group of G. For an element $a \in A$ and for $i \in \{1, \ldots, r\}$, define an integer a_i by $a_i = 1$ if $a(v_i) \in \{v_1, \ldots, v_r\}$, and $a_i = -1$ otherwise. There exists an isomorphism $\varphi\colon A \xrightarrow{\sim} \mu_2^r \rtimes S_r$, where μ_2 is the multiplicative group with two elements and S_r the symmetric group on r elements, acting on μ_2^r by permuting the coordinates, given by*

$$\varphi(a) = ((a_1, \ldots, a_r), (i \mapsto j \text{ for } a(v_i) \in \{v_j, w_j\})).$$

Proof. Let a be an element in A. Note that for all i, the image $a(v_i)$ of v_i is only connected to $a(w_i)$, so there is a j such that $\{a(v_i), a(w_i)\} = \{v_j, w_j\}$. Therefore we have a group homomorphism $\gamma\colon A \longrightarrow S_r$, given by

$$a \longmapsto (i \mapsto j \text{ for } a(v_i) \in \{v_j, w_j\}).$$

Note that γ is surjective, and its kernel consists of all elements $a \in A$ such that, for all $i \in \{1, \ldots, r\}$, either $a(v_i) = v_i$, or $a(v_i) = w_i$. We conclude that the kernel of γ is isomorphic to the group μ_2^r. So we have a short exact sequence

$$1 \longrightarrow \mu_2^r \longrightarrow A \xrightarrow{\gamma} S_r \longrightarrow 1.$$

Moreover, we have a section $S_r \longrightarrow A$ given by $g \longmapsto \{v_i \mapsto v_{g(i)},\ w_i \mapsto w_{g(i)}\}$, so the statement follows. □

Lemma 3.30. *Let $c = \{\{e_1, f_1, \}, \ldots, \{e_7, f_7\}\}$ be an element in the set C that is defined above Lemma 3.14, and let s be the set $\{e_1, \ldots, e_7, f_1, \ldots, f_7\}$. Let A the automophism group of the colored graph associated to s, and let $\varphi\colon A \xrightarrow{\sim} \mu_2^7 \rtimes S_7$ be the isomorphism from Lemma 3.29. Let W_s be the stabilizer in W of s. Then there is an injective map $W_s \longrightarrow A$, whose image has index 2 in A, and its image after composing with φ is given by*

$$\left\{((m_1, \ldots, m_7), g) \in \mu_2^7 \rtimes S_7 \mid \prod_{i=1}^{7} m_i = 1\right\}.$$

Proof. Elements in W_s respect the dot product, so we have a map $\beta\colon W_s \longrightarrow A$. If an element $w \in W_s$ fixes every element in s, then it fixes a sublattice of Λ of finite index by Lemma 3.17, and since Λ is torsion free this implies that w is the identity. So the action of W_s on s is faithful, hence β is injective, and $|\beta(W_s)| = |W_s|$. Since W acts transitively on C by Corollary 3.16, and $|C| = 2160$ by Remark 3.15, we have $|W_s| = |W_c| = \frac{|W|}{|C|} = \frac{|W|}{2160} = 322560$. Moreover, we have $|A| = 2^7 \cdot 7! = 645120$, so $|\beta(W_s)| = |W_s| = 322560 = \frac{1}{2} \cdot |A|$. Hence $\beta(W_s)$ is a subgroup of index two in A. We will now determine which subgroup. Note that $\|e_1 - e_2\| = \sqrt{2}$, so $e_1 - e_2$ is an element $e \in E$, and the reflection in the hyperplane orthogonal to e gives an element in W, say r_{12}. Note that $e_1 + f_1 = e_2 + f_2$ by Remark 3.18, so $e_1 - e_2 = f_2 - f_1$. Therefore r_{12} interchanges e_1 with e_2 and f_1 with f_2. Moreover, since all roots in

$\{e_3, \ldots, e_7, f_3, \ldots, f_7\}$ are orthogonal to e, the element r_{12} acts trivially on them. Analogously, for $i, j \in \{1, \ldots, 7\}$, $i \neq j$, the reflection r_{ij} is an element in W_s that interchanges e_i and e_j, and f_i with f_j. Let $\gamma\colon A \longrightarrow S_7$ be the projection of $\varphi(A)$ to S_7, then it follows that $\gamma(\beta(W_s)) = S_7$. Now consider for $i, j \in \{1, \ldots, 7\}, i \neq j$, the element $e_i - f_j$. Again, this is an element in E, and the reflection t_{ij} in the hyperplane orthogonal to it is an element in W_s interchanging e_i with f_j, and e_j with f_i, and leaving all other roots in s fixed. It follows that the composition $t_{ij} \circ r_{ij}$ is an element in W_s with $\varphi(\beta(t_{ij} \circ r_{ij})) = ((-1, -1, 1, 1, 1, 1, 1), \mathrm{id}) \in \mu_2^7 \rtimes S_7$. By composing these automorphisms $t_{ij} \circ r_{ij}$ for different i, j, we see that $\varphi(\beta(W_c))$ contains all elements $((m_1, \ldots, m_7), g) \in \mu_2^7 \rtimes S_7$ with $\prod_{i=1}^7 m_i = 1$. Therefore, the reflections r_{ij}, t_{ij} generate a subgroup of A of order $7! \cdot 2^6 = \frac{1}{2}A$, and we conclude that this is all of W_s. □

Corollary 3.31. *Let K_1 and K_2 be two cliques in Γ whose vertices correspond to a 7-crosspolytope in the E_8 root polytope. Let $f\colon K_1 \longrightarrow K_2$ be an isomorphism between them. Then f extends to an automorphism of Λ if and only if for every subclique $S = \{e_1, \ldots, e_7\}$ of K_1 of 7 vertices that are pairwise connected with edges of color 1, the vectors $\sum_{i=1}^7 e_i$ and $\sum_{i=1}^7 f(e_i)$ are either both in 2Λ, or neither are.*

Proof. Consider the set $H = \{c_1 \ldots, c_7, d_1, \ldots, d_7\}$, where the elements are defined above Lemma 3.14. Note that the vertices in H correspond to a 7-crosspolytope, and since W acts transitively on the set of cliques corresponding to 7-crosspolytopes (Corollary 3.16), there are elements α, β in W such that $\alpha(K_1) = \beta(K_2) = H$. So $\beta \circ f \circ \alpha^{-1}$ is an element in the automorphism group $\mathrm{Aut}(H)$ of H. Of course, f extends to an element in W if and only if $\beta \circ f \circ \alpha^{-1}$ does. Moreover, since α and β are automorphisms of Λ, the two sums $\sum_{i=1}^7 f(e_i)$ and $\sum_{i=1}^7 (\beta \circ f \circ \alpha^{-1})(e_i)$ are either both in or both not in 2Λ. We conclude that we can reduce to the case where $K_1 = K_2 = H$, and f is an element in $\mathrm{Aut}(H)$.
Let W_H be the stabilizer of H in W. By Lemma 3.30, there is an injective map $\psi\colon W_H \longrightarrow \mathrm{Aut}(H)$, whose image has index 2 in $\mathrm{Aut}(H)$. Of course, for all elements w in the image of ψ, and for all cliques $S = \{s_1, \ldots, s_7\}$ as in the statement, the sums $\sum_{i=1}^7 s_i$ and $\sum_{i=1}^7 w(s_i)$ are either both in, or both not in 2Λ. We will show that this completely determines the image of ψ, that is, we will show that every element in $\mathrm{Aut}(H) \setminus \psi(W_H)$ does not have this property for all cliques S as in the statement. To this end, consider the element h in $\mathrm{Aut}(H)$ that exchanges c_1 and d_1, and fixes all other vertices. Since h exchanges an odd number of c_i with d_i, it is not in the image of ψ. Note that $S = \{c_1, \ldots, c_7\}$ is a clique as in the statement. The sum $\sum_{i=1}^7 c_i = (5, 3, 3, 3, -1, 1, 1, 1)$ is an element in 2Λ, and its image under h, which is $\sum_{i=1}^7 h(c_i) = d_1 + \sum_{i=2}^7 c_i = (4, 2, 4, 4, -1, 1, 1, 1)$, is not. Since all elements in $\mathrm{Aut}(H) \setminus \psi(W_H)$ are compositions of h with elements in W_H, we conclude that for all elements a in $\mathrm{Aut}(H) \setminus \psi(W_H)$, the sum $\sum_{i=1}^7 a(c_i)$ is not an element in 2Λ. Since the image of ψ consists exactly of those elements in $\mathrm{Aut}(H)$ extending to an element in W, this finishes the proof. □

Lemma 3.32. *The group W acts transitively on the set*

$$B = \{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = 0,\ e_2 \cdot e_3 = e_1 \cdot e_3 = 1\}.$$

Proof. By Proposition 2.2 and Lemma 3.14, we have $|B| = 240 \cdot 126 \cdot 12 = 362880$. Let c, s, A be as defined in Lemma 3.30, and note that $b = (e_1, f_1, e_2)$ is an element in B. Let W_b be the stabilizer in W of b. Then we have

$$|W_b| = \frac{|W|}{|Wb|} \geq \frac{|W|}{|B|} = 1920.$$

We want to show that this is an equality.
Since c is the unique element in C containing e_1, f_1 by Lemma 3.14, the stabilizer W_b of b acts on the set s. If an element $w \in W_b$ fixes all the roots in s, then it fixes a full rank sublattice of finite index in Λ, and since Λ is torsion free this implies that w is the identity. Therefore the action of W_b on s is faithful, so there is an injective map $W_b \longrightarrow W_s$. Note that f_2 is uniquely determined in s as the root that is orthogonal to e_2, so every element in W_b fixes e_1, e_2, f_1, f_2, hence W_b acts faithfully on $s' = \{e_3, \ldots, e_7, f_3, \ldots, f_7\}$. Let A' be the automorphism group of the colored graph associated to s'. We know there is an isomorphism $\varphi' : A' \longrightarrow \mu_2^5 \rtimes S_5$ by Lemma 3.29. Since elements in W_b respect the dot product, we have an injective map $\beta' \colon W_b \longrightarrow A'$. Let $\beta \colon W_s \longrightarrow A$ be the injective map from Lemma 3.30. together with the injective maps $W_b \longrightarrow W_s$ and $A' \longrightarrow A$, we have the following commutative diagram.

$$\begin{array}{ccccc} W_b & \overset{\beta'}{\hookrightarrow} & A' & \underset{\sim}{\overset{\varphi'}{\longrightarrow}} & \mu_2^5 \rtimes S_5 \\ \downarrow & & \downarrow & & \downarrow \\ W_s & \overset{\beta}{\hookrightarrow} & A & \underset{\sim}{\overset{\varphi}{\longrightarrow}} & \mu_2^7 \rtimes S_7 \end{array}$$

By Lemma 3.30, the image $\varphi(\beta(W_s))$ is a subset of index 2 in $\mu_2^7 \rtimes S_7$, given by subset $\left\{((m_1, \ldots, m_7), g) \in \mu_2^7 \rtimes S_7 \mid \prod_{i=1}^7 m_i = 1\right\}$. Intersecting this subset with $\mu_2^5 \rtimes S_5$ gives a subset of index 2 in $\mu_2^5 \rtimes S_5$, so by the diagram above, the image $\varphi'(\beta'(W_b))$ has index at least 2 in $\mu_2^5 \rtimes S_5$. We find $|W_b| \leq \frac{1}{2} \cdot 2^5 \cdot 5! = 1920$, so together with the inequality above we conclude that $|W_b| = 1920$. So we find $|Wb| = \frac{|W|}{|W_b|} = 362880 = |B|$, and W acts transitively on B. □

We can now prove Proposition 3.2.

Proof of Proposition 3.2. Note that for a, b, c fixed and σ any permutation of them, there is a bijection between the sets $V_{a,b,c}$ and $V_{\sigma(a),\sigma(b),\sigma(c)}$, so if we prove that W acts transitively on one of them, then W also acts transitively on the other by Lemma 2.14. Therefore, we only consider the sets $V_{a,b,c}$ where $a \leq b \leq c$.
There are 4 different sets with $a = b = c$. There are 12 different sets where two of a, b, c are equal to each other and unequal to the third, and 4 different sets with a, b, c all distinct. So there are 20 different sets $V_{a,b,c}$ with $a \leq b \leq c$.

- If $V_{a,b,c}$ is a non-empty set with $a = -2$, then every element (e_1, e_2, e_3) in $V_{a,b,c}$ has $e_1 = -e_2$, so $b = -c$. Therefore the set $V_{a,b,c}$ is empty for (a, b, c) in
$$\{(-2,-2,-2), (-2,-2,-1), (-2,-2,0), (-2,-2,1), \\ (-2,-1,-1), (-2,-1,0), (-2,0,1), (-2,1,1)\}.$$
- We have proved that W acts transitively on the sets $V_{-1,-1,-1}$ (Corollary 3.9), $V_{0,0,0}$ (Proposition 3.27), $V_{0,0,1}$ (Lemma 3.11), $V_{0,1,1}$ (Lemma 3.32), and $V_{1,1,1}$ (Proposition 2.12).

- We have the following bijections.

$$\{(e_1,e_2)\in E^2 \mid e_1\cdot e_2=-1\} \longrightarrow V_{-2,-1,1}, \qquad (e_1,e_2)\longmapsto(-e_1,e_1,e_2);$$
$$\{(e_1,e_2)\in E^2 \mid e_1\cdot e_2=0\} \longrightarrow V_{-2,0,0}, \qquad (e_1,e_2)\longmapsto(-e_1,e_1,e_2);$$
$$V_{0,1,1}\longrightarrow V_{-1,-1,0}, \qquad (e_1,e_2,e_3)\longmapsto(e_1,-e_3,e_2);$$
$$V_{1,1,1}\longrightarrow V_{-1,-1,1}, \qquad (e_1,e_2,e_3)\longmapsto(e_1,-e_2,e_3);$$
$$V_{0,0,1}\longrightarrow V_{-1,0,0}, \qquad (e_1,e_2,e_3)\longmapsto(-e_1,e_3,e_2);$$
$$V_{0,1,1}\longrightarrow V_{-1,0,1}, \qquad (e_1,e_2,e_3)\longmapsto(-e_3,e_2,-e_1);$$
$$V_{-1,-1,-1}\longrightarrow V_{-1,1,1}, \qquad (e_1,e_2,e_3)\longmapsto(e_1,e_2,-e_3).$$

  We proved that W acts transitively on the six different sets on the left-hand sides. From Lemma 2.14 it follows that W acts transitively on $V_{-2,-1,1}$, $V_{-2,0,0}$, $V_{-1,-1,0}$, $V_{-1,-1,1}$, $V_{-1,0,0}$, $V_{-1,0,1}$, and $V_{-1,1,1}$, too.

Since we proved that $V_{a,b,c}$ is either empty or W acts transitively on it for 20 different sets, we conclude that we proved the proposition.

The following corollary proves Theorem 1.2 for cliques of Type III.

Corollary 3.33. *Let K_1 and K_2 be two cliques of type III, and let $f\colon K_1 \longrightarrow K_2$ be an isomorphism between them. Then f extends to an automorphism of Λ.*

Proof. Since W acts transitively on the set of ordered sequences of n roots for $1\leq n\leq 3$ by Propositions 3.1 and 3.2, there extists an automorphism $w\in W$ of Λ such that w restricted to K_1 equals f. □

4. Monochromatic cliques

In this section we study the cliques of type I, that is, cliques in Γ_{-2}, Γ_{-1}, Γ_0, and Γ_1. We describe the orbits under the action of W of *sequences* of roots that form a clique, thus obtaining the results in Theorem 1.2 for cliques of type I (see Corollaries 4.5 and 4.9). We also describe all maximal cliques per color. For Γ_{-2} and Γ_{-1}, everything follows from the previous sections. For Γ_1 we already have Proposition 2.12; we show moreover that there are no cliques of size bigger than eight, and describe the maximal cliques in Proposition 4.7. Finally, in this section we prove that W acts transitively on ordered sequences of orthogonal roots of length r for $r\geq 5$. The result is in Proposition 4.4. Throughout this section we do not use any computer.

Cliques in Γ_{-2}

The maximal size of a clique in Γ_{-2} is two, since such a maximal clique consists of an element in E and its inverse (see Proposition 2.2). There are therefore 120 such cliques. In Lemma 3.7 we showed that W acts transitively on the set of ordered pairs $\{(e_1,e_2)\in E^2 \mid e_1=-e_2\}$, so W acts transitively on the set of maximal cliques in Γ_{-2}.

Cliques in Γ_{-1}

In Γ_{-1}, the maximal size of a clique is three, and there are no maximal cliques of smaller size, by Lemma 3.8. From Proposition 2.2 and Lemma 3.8 it follows that there are $\frac{240\cdot 56}{3!}=2240$ maximal cliques. By Corollary 3.9, the group W acts transitively on the set of sequences $\{(e_1,e_2,e_3)\in E^3 \mid e_1\cdot e_2=e_2\cdot e_3=e_1\cdot e_3=-1\}$, so W acts transitively on the set of maximal cliques in Γ_1. By Lemma 3.7, the group W acts transitively on the set $\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = -1\}$, so W acts also transitively on the set of cliques of size two in Γ_{-1}, of which there are $\frac{240 \cdot 56}{2} = 6720$ (Proposition 2.2).

Cliques in Γ_0

Cliques in Γ_0 are studied in [DM10], where they are called *orthogonal subsets*. In their article, the authors show that the maximal size of cliques in Γ_0 is eight ([DM10], Table 1), that two cliques of the same size r are conjugate if and only if $r \neq 4$, and that there are two orbits of cliques of size 4 ([DM10], Corollary 2.3). In the previous section we showed that W acts transitively on the set of *ordered sequences* of length at most 3 of orthogonal roots, and that there are two orbits of sequences of length 4. In this section we use this to conclude the same results as in [DM10] for cliques of size $r \leq 4$, and we compute the number of these cliques. Moreover, we study the action of W on ordered sequences of length ≥ 5 of orthogonal roots (Proposition 4.4), and compute the number of cliques of size ≥ 5 (Proposition 4.6).

The following proposition deals with the cliques of size at most 4.

Proposition 4.1.

(i) There are 15120 *cliques of size two in* Γ_0*, and the group* W *acts transitively on the set of all of them.*

(ii) There are 302400 *cliques of size three in* Γ_0*, and the group* W *acts transitively on the set of all of them.*

(iii) There are 1965600 *cliques of size four in* Γ_0*, and they from two orbits under the action of* W*: one of size* 151200*, in which all cliques have vertices whose roots sum up to a vector in* 2Λ*, and one of size* 1814400*, in which all cliques have vertices whose roots sum op to a vector that is not in* 2Λ*.*

Proof.

(i) We have shown that the group W acts transitively on the set

$$A_0 \;=\; \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\}$$

(Proposition 3.13), and $|A_0| = 240 \cdot 126 = 30240$ (Proposition 2.2). It follows that there are $\frac{30240}{2} = 15120$ cliques of size two in Γ_0, and the group W acts transitively on the set of all of them.

(ii) The group W acts transitively on the set

$$V_3 \;=\; \{(e_1, e_2, e_3) \in E^3 \mid \forall i \neq j : e_i \cdot e_j = 0\},$$

and we have $|V_3| = 1814400$ (Proposition 3.27 (i)). It follows that there are $\frac{1814400}{6} = 302400$ cliques of size three in Γ_0, and the group W acts transitively on the set of all of them.

(iii) By Proposition 3.28 there are two orbits under the action of W on the set

$$V_4 = \{(e_1, e_2, e_3, e_4) \in E^4 \mid \forall i \neq j : e_i \cdot e_j = 0\};$$

one of size 3628800 where all elements have coordinates that sum up to a vector that is in 2Λ, and one orbit of size 43545600 where all elements have coordinates that sum up to a vector that is not in 2Λ. Since the orbit in which an element is contained does not depend on the order of its coordinates, we conclude that this also gives two orbits with the same properties under the action of W on the set of all cliques of size four in Γ_0, of sizes $\frac{3628800}{4!} = 151200$ and $\frac{43545600}{4!} = 1814400$, respectively. $\square$

We continue by studying the sequences of orthogonal roots of length greater than four. Recall the set V_4 and its orbits under the action of W, given by Y of size 43545600 and $O = V_4 \setminus Y$ of size 3628800 (Proposition 3.28).

Lemma 4.2. *For an element $y = (e_1, \ldots, e_4) \in Y$, define the set*

$$C_y = \{e \in E \mid e \cdot e_i = 0 \text{ for } i \in \{1,2,3,4\}\}.$$

The following hold.

(i) The set C_y is the union of four sets $\{f_1, -f_1\}, \{f_2, -f_2\}, \{f_3, -f_3\}, \{f_4, -f_4\}$ with $f_i \cdot f_j = 0$ for $i \neq j$. For such a set $\{f_i, -f_i\}$, there is exactly one triple $\{e_{i_1}, e_{i_2}, e_{i_3}\}$ of elements in y such that the permutations of $(e_{i_1}, e_{i_2}, e_{i_3}, f_i)$ (or equivalently of $(e_{i_1}, e_{i_2}, e_{i_3}, -f_i)$) form elements in O. Moreover, for $j \neq i$, and j_1, j_2, j_3 such that the permutations of $(e_{j_1}, e_{j_2}, e_{j_3}, f_j)$ form elements in O, the sets $\{e_{i_1}, e_{i_2}, e_{i_3}\}$ and $\{e_{j_1}, e_{j_2}, e_{j_3}\}$ are different.

(ii) The stabilizer of y is generated by the reflections in the hyperplanes orthogonal to f_i for $i \in \{1,2,3,4\}$.

Proof. Since W acts transtively on Y, it suffices to show this for a fixed element $y \in Y$. Set

$$e_1 = (1,1,0,0,0,0,0,0), \qquad e_3 = (0,0,0,0,1,1,0,0),$$
$$e_2 = (0,0,1,1,0,0,0,0), \qquad e_4 = (1,-1,0,0,0,0,0,0).$$

Then (e_1, e_2, e_3, e_4) is an element in V_4 and since $\sum_{i=1}^4 e_i \notin 2\Lambda$, it is an element in Y as well by Proposition 3.28. Take $e = (a_1, \ldots, a_8) \in E$ such that $e \cdot e_i = 0$ for $i \in \{1,2,3,4\}$. Then we have $a_1 + a_2 = a_1 - a_2 = a_3 + a_4 = a_5 + a_6 = 0$. We find the following possibilities.

$$\pm f_1 = \pm(0,0,0,0,0,0,1,-1), \qquad \pm f_3 = \pm(0,0,1,-1,0,0,0,0),$$
$$\pm f_2 = \pm(0,0,0,0,1,-1,0,0), \qquad \pm f_4 = \pm(0,0,0,0,0,0,1,1).$$

It is an easy check that $f_i \cdot f_j = 0$ for $i \neq j$, and for $i, k \in 1,2,3,4$, the sum $\left(\sum_{j \neq i} e_j\right) \pm f_k$ is contained in 2Λ if and only if $i = k$. This proves (i). We continue with (ii). Take $i \in \{1,2,3,4\}$. Since f_i is orthogonal to the elements in y the reflection r_i in the hyperplane orthogonal to f_i is an element of W_y. For $i \neq j$, the reflections r_i and r_j commute, since f_i and f_j are orthogonal. Therefore the elements r_1, r_2, r_3, r_4 generate a subgroup of W_y of order 16. Since we have

$$|W_y| = \frac{|W|}{|Y|} = \frac{696729600}{43545600} = 16,$$

they generate the whole group W_y. $\square$

Corollary 4.3. *Set $n_5 = 1$, $n_6 = 3$, $n_7 = 7$, and $n_8 = 14$. Let K be a clique of size $r \in \{5,6,7,8\}$ in Γ_0. Then the number of sets of four vertices e_1, e_2, e_3, e_4 in K such that the permutations of (e_1, e_2, e_3, e_4) are elements in O is equal to n_r.*

Proof. First let K be a clique of size 5 in Γ_0. Assume by contradiction that there are two distinct subsets, say y_1, y_2, of four vertices in K that form an element in O. Then there are three vertices of K, say e_1, e_2, e_3, that are contained both in y_1 and y_2. Write $y_1 = \{e_1, e_2, e_3, f_1\}$, $y_2 = \{e_1, e_2, e_3, f_2\}$. By applying Proposition 3.27 (iii) to the triple (e_1, e_2, e_3), it follows that $f_1 = -f_2$, so $f_1 \cdot f_2 = -2$. But this gives a contradiction, since f_1, f_2 are both in K. So the number of sets of four vertices in K that form an element in O is at most 1, which means that there is at least one subset $\{g_1, g_2, g_3, g_4\}$ of K of four roots such that (g_1, g_2, g_3, g_4) is an element in Y. For the fifth element in K, say g_5, it follows from the previous lemma that there is exactly one triple $\{g_\alpha, g_\beta, g_\gamma\}$ of elements in $\{g_1, \ldots, g_4\}$ that it forms an element in O with. We conclude that there is exactly 1 set of four vertices in K that form an element in O; this proves the statement for $r = 5$.
We proceed by induction. Take $s \in \{6, 7, 8\}$. Assume that the statement holds for $5 \leq r < s$, and let $K = \{e_1, \ldots, e_s\}$ be a clique of size s in Γ_0. By induction we know that $\{e_1, \ldots, e_{s-1}\}$ contains n_{s-1} subsets of size four that form an element in O. That means that there are $\binom{s-1}{4} - n_{s-1}$ subsets of size four in $\{e_1, \ldots, e_{s-1}\}$ that form an element in Y. By Lemma 4.2, each of these $\binom{s-1}{4} - n_{s-1}$ subsets contains exacty three elements that, together with e_s, form an element in O. Let d_1, d_2, d_3 be three elements in $\{e_1, \ldots, e_{s-1}\}$ such that (d_1, d_2, d_3, e_s) is an element in O. Then for every element $d \in \{e_1, \ldots, e_{s-1}\} \setminus \{d_1, d_2, d_3\}$, the set $\{d_1, d_2, d_3, e_s, d\}$ forms a clique of size 5 in Γ_0, and since $n_5 = 1$, it follows that (d_1, d_2, d_3, d) is an element in Y. This means that every set of three roots in $\{e_1, \ldots, e_{s-1}\}$ that forms an element in O with e_s forms an element in Y with all other roots in $\{e_1, \ldots, e_{s-1}\}$. Since every set of three roots in $\{e_1, \ldots, e_{s-1}\}$ is contained in $(s-1)-3$ subsets of size four of $\{e_1, \ldots, e_{s-1}\}$, this gives $\frac{\binom{s-1}{4} - n_{s-1}}{s-4}$ distinct sets of three that form an element in O with e_s. In total this gives $n_{s-1} + \frac{\binom{s-1}{4} - n_{s-1}}{s-4}$ sets of four vertices in K that form an element in O. This is exactly equal to n_s for $s = 6, 7, 8$. □

For $1 \leq r \leq 8$, let V_r be the set

$$V_r = \{(e_1, \ldots, e_r) \in E^r \mid \forall i \neq j : e_i \cdot e_j = 0\}.$$

Proposition 4.4. *For $5 \leq r \leq 8$, two elements $(e_1, \ldots, e_r)$, $(f_1, \ldots, f_r)$ in V_r are in the same orbit under the action of W if and only if for all $1 \leq i < j < k < l \leq r$, the elements (e_i, e_j, e_k, e_l) and (f_i, f_j, f_k, f_l) are conjugate in V_4 under the action of W.*

Proof. For $5 \leq r \leq 8$, define the relation $\sim$ on V_r by $(e_1, \ldots, e_r) \sim (f_1, \ldots, f_r)$ if and only if for all $1 \leq i < j < k < l \leq r$, the elements (e_i, e_j, e_k, e_l) and (f_i, f_j, f_k, f_l) are conjugate in V_4. Note that $\sim$ is an equivalence relation on V_r, and the group W acts on the equivalence classes. Our goal is to show that each equivalence class is an orbit in V_r under the action of W. We do this by induction on r.
For $r = 5$, Let $X_5 \subset V_5$ be an equivalence class with respect to $\sim$. We distinguish two cases. If for every element in X_5 the first four coordinates form an element in Y, we let $p \colon X_5 \longrightarrow Y$ be the projection on the first four coordinates. Note that this is surjective by Lemma 4.2. Set $y = (y_1, \ldots, y_4) \in Y$. Since the elements in the fiber $p^{-1}(y)$ are equivalent under $\sim$, there are exactly two elements $(y_1, \ldots, y_4, f), (y_1, \ldots, y_4, -f)$ in $p^{-1}(y)$ by Lemma 4.2 (i). Moreover, the stabilizer W_y acts transitively on these two elements by Lemma 4.2 (ii). From Lemma 2.14 it follows that W acts transitively on X_5. If, on the other hand, for every element

in X_5 the first four coordinates form an element in O, then the last four coordinates of every element in X_5 form an element in Y by Corollary 4.3. We now let $p\colon X_5 \longrightarrow Y$ be the projection on the last four coordinates, and the proof is the same.

Now assume that $r > 5$, and that each equivalence class in V_{r-1} is an orbit under the action of W. Let X_r be an equivalence class in V_r, and $p_r\colon X_r \longrightarrow V_{r-1}$ the projection on the first $r-1$ coordinates. Then W acts on $p_r(X_r)$, and $p_r(X_r)$ is contained in an equivalence class X_{r-1} with respect to $\sim$ in V_{r-1}. Since W acts transitively on X_{r-1} by hypothesis, it follows that $p_r(X_r) = X_{r-1}$, and W acts transitively on $p_r(X_r)$. Since $r > 5$, by Corollary 4.3 there exist $i, j, k, l \in \{1, \ldots, r-1\}$ such that for all elements $(e_1, \ldots, e_r) \in X_r$ we have $(e_i, e_j, e_k, e_l) \in Y$. Fix such i, j, k, l, and let $v = (v_1, \ldots, v_{r-1})$ be an element in $p_r(X_r)$. Then (v_i, v_j, v_k, v_l) is an element in Y. Let $(v_1, \ldots, v_{r-1}, f)$, $(v_1, \ldots, v_{r-1}, g)$ be elements in the fiber $p_r^{-1}(v)$. Since $(v_1, \ldots, v_{r-1}, f)$ is equivalent to $(v_1, \ldots, v_{r-1}, g)$ with respect to $\sim$, by applying Lemma 4.2 to (v_i, v_j, v_k, v_l) we see that $f = -g$, and the fiber $p_r^{-1}(v)$ consists of the two elements $(v_1, \ldots, v_{r-1}, f)$ and $(v_1, \ldots, v_{r-1}, -f)$. Moreover, the reflection in the hyperplane orthogonal to f fixes $v_1, \ldots, v_{r-1}$, hence is an element in the stabilizer of v that switches f and $-f$. So the stabilizer of v acts transitively on $p_r^{-1}(v)$, and again from Lemma 2.14 we conclude that W acts transitively on X_r. □

Corollary 4.5. *Let K_1 and K_2 be two cliques in Γ_0, and $f\colon K_1 \longrightarrow K_2$ an isomorphism between them. Then f extends to an automorphism of Λ if and only if for every subclique S of size 4 in K_1, the image $f(S)$ in K_2 is conjugate to S under the action of W.*

Proof. If K_1 and K_2 have size ≤ 3, then f extends always by Corollary 3.33. From Proposition 4.4 it follows that if K_1 and K_2 have size at least four, the isomorphism f extends to an element in W exactly when f sends every sequence of four roots that form an element in V_4 to a conjugate element in V_4. By Proposition 3.28, there are two orbits of ordered sequences of four pairwise orthogonal roots, that do not depend on the order of the roots. We conclude that if f and $f(S)$ are conjugate under the action of W for every set S of four vertices in K_1, there exists an automorphism $w \in W$ of Λ such that w restricted to K_1 equals f. □

Theorem 4.6. *In Γ_0, the following hold.*

(i) There are no maximal cliques of size smaller than eight.

(ii) There are 3628800 *cliques of size five,* 3628800 *cliques of size six,* 2073600 *cliques of size seven, and* 518400 *cliques of size eight.*

(iii) The group W acts transitively on the cliques of size 5.

Proof.

(i) We know that every root in E is orthogonal to 126 other roots (Proposition 2.2). Moreover, we know that in Γ_0 every clique of size 2 extends to a clique of size 3 (Lemma 3.26), and every clique of size 3 extends to a clique of size 4 (Proposition 3.27 (ii)). Since $n_5 = 1 < \binom{5}{4}$ by Corollary 4.3, every clique of size 5 in Γ_0 contains both a subclique whose vertices form an element in O, and a subclique whose vertices form an element in Y. Since W acts transitively on O and on Y, and $V_4 = O \cup Y$, this means that every clique of size 4 in Γ_0 extends to a clique of size 5. Moreover, by Lemma 4.2 (i), every clique of size 4 whose vertices form an element in Y can be extended to a clique of size 8. Since every clique of size at least 5 contains a clique of size 4 whose vertices form an element in Y, there are no maximal cliques of size smaller than 8.

(ii) By Lemma 4.2, if we fix an element $y = (e_1, e_2, e_3, e_4) \in Y$, there are exactly 8 elements in V_5, and $8 \cdot 6$ elements in V_6, and $8 \cdot 6 \cdot 4$ elements in V_7, and $8 \cdot 6 \cdot 4 \cdot 2$ elements in V_8, that have e_i as the i^{th} coordinate. We call this number m_r for $r = 5, 6, 7, 8$. For all $5 \le r \le 8$, for S a clique of size r, it follows from Corollary 4.3 that S contains $\binom{r}{4} - n_r$ cliques of size 4 that, together, form $4! \cdot (\binom{r}{4} - n_r)$ different elements in Y; for such a subclique of size 4 in S, the other $r - 4$ elements can be permuted in $(r-4)!$ ways. For all $5 \le r \le 8$, let D_r be the set of cliques of size r in Γ_0. It follows that we have

$$|D_r| = \frac{|Y| \cdot m_r}{4! \cdot (\binom{r}{4} - n_r) \cdot (r-4)!}.$$

We find the following results.

$$|D_5| = \frac{|Y| \cdot 8}{4! \cdot 4} = 3628800, \ |D_6| = \frac{|Y| \cdot 8 \cdot 6}{4! \cdot 12 \cdot 2} = 3628800,$$
$$|D_7| = \frac{|Y| \cdot 8 \cdot 6 \cdot 4}{4! \cdot 28 \cdot 3!} = 2073600, \ |D_8| = \frac{|Y| \cdot 8 \cdot 6 \cdot 4 \cdot 2}{4! \cdot 56 \cdot 4!} = 518400.$$

(iii) Let $K_1 = \{e_1, \ldots, e_5\}$, $K_2 = \{f_1, \ldots, f_5\}$ be two cliques in Γ_0. We have $n_5 = 1$ by Corollary 4.3, so without loss of generality we can assume that e_1, e_2, e_3, e_4 and f_1, f_2, f_3, f_4 are the unique four elements in K_1 and K_2, respectively, that form an element in O. Then $(e_1, e_2, e_3, e_4, e_5)$ and $(f_1, f_2, f_3, f_4, f_5)$ are conjugate under the action of W by Proposition 4.4, hence so are K_1 and K_2. ☐

Cliques in Γ_1

We know that cliques in Γ_1 form k-simplices that are k-faces of the E_8 root polytope (Proposition 2.4), hence Corollary 2.7 states how many cliques of size n there are in Γ_1 for $n \le 8$. Moreover, we know that W acts transitively on these cliques for $n \le 8$, $n \ne 7$ (Proposition 2.12). Proposition 4.7 shows that there are no cliques of size bigger than eight in Γ_1, and that there are two orbits of cliques of size seven (which was already known, for example by [Cox30] and [Man74]); it shows that all maximal cliques are of sizes 7 or 8.

Proposition 4.7. *In Γ_1, the following hold.*

(i) There are only maximal cliques of size 7 and 8.

(ii) There are two orbits of cliques of size 7 in Γ_1; one of size 138240, which is given by non-maximal cliques, and one of size 69120, which is given by maximal cliques. A clique of size seven in Γ_1 is maximal if and only if the sum of its vertices is an element in 2Λ.

(iii) There are 17280 cliques of size 8.

Proof. Consider the clique of size six in Γ_1 given by $\{e_1, \ldots, e_6\}$, where we define

$$e_1 = (1,1,0,0,0,0,0,0), \quad e_4 = (1,0,0,0,1,0,0,0)$$
$$e_2 = (1,0,1,0,0,0,0,0), \quad e_5 = (1,0,0,0,0,1,0,0)$$
$$e_3 = (1,0,0,1,0,0,0,0), \quad e_6 = (1,0,0,0,0,0,1,0).$$

Since W acts transitively on the set of cliques of size smaller than 6 in Γ_1 by Proposition 2.12, it follows that every clique of size smaller than 6 in Γ_1 is contained in a clique of size 6 in Γ_1. The elements in E that have dot product one with all $e_1, \ldots, e_6$ are given by:

$$c_1 = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \quad c_2 = (1,0,0,0,0,0,0,1), \quad c_3 = (1,0,0,0,0,0,0,-1).$$

Note that $c_1 \cdot c_2 = 1$ and $c_3 \cdot c_1 = c_3 \cdot c_2 = 0$, so $\{e_1, \ldots, e_6, c_1, c_2\}$ is a maximal clique of size 8 in Γ_1, and $\{e_1, \ldots, e_6, c_3\}$ is a maximal clique of size 7 in Γ_1. Since W acts transitively on the cliques of size 6 in Γ_1 by Proposition 2.12, all maximal cliques in Γ_1 are of size 7 or 8. This proves part (i). Moreover, it follows that every non-maximal clique of size 7 is contained in a unique clique of size 8, so there are $\frac{138240}{8} = 17280$ cliques of size 8. This proves part (iii). We will now prove (ii). From part (i) it follows that there exist maximal and non-maximal cliques of size 7 in Γ_1. It is obvious that they can not be in the same orbit under the action of W. Moreover, there are two orbits of ordered sequences of length 7, hence at most two orbits of cliques of size 7 by Proposition 2.12. We conclude that the orbits are given exactly by the maximal cliques and the non-maximal cliques. Since there are 483840 cliques of size 6 (Corollary 2.7), from the above it follows that there are $\frac{483840 \cdot 2}{7} = 138240$ non-maximal cliques, and $\frac{483840 \cdot 1}{7} = 69120$ maximal cliques. Now consider the set $\{e_1, \ldots, e_7\}$, where the elements are defined above Lemma 3.14. This is a clique of size 7 in Γ_1, and it is not hard to check that it is maximal. Moreover, we have

$$\sum_{i=1}^{7} e_i = (5,3,3,3,1,1,1,1) \in 2\Lambda.$$

Since W acts transitively on all maximal cliques of size 7 in Γ_1, for all such cliques the sum of the vertices is an element in 2Λ. On the other hand, consider the set $d = \{d_1, \ldots, d_7\}$ as defined above Lemma 3.14. This is a non-maximal clique of size 7 in Γ_1, since the union of d with the root $\left(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right)$ is a clique of size 8 in Γ_1. Moreover, we have

$$\sum_{i=1}^{7} d_i = (2,4,4,4,1,-1,-1,-1) \notin 2\Lambda.$$

Since W acts transitively on all non-maximal cliques of size 7 in Γ_1, for all such cliques the sum of the vertices is not an element in 2Λ. ☐

Remark 4.8. Note that $138240 + 69120 = 207360$, which is the total number of cliques of size 7 by Corollary 2.7.

Corollary 4.9. *Let K_1 and K_2 be two cliques in Γ_1, and $f\colon K_1 \longrightarrow K_2$ an isomorphism between them. If K_1 and K_2 have size unequal to 7, then f extends to an automorphism of Λ. If K_1 and K_2 have size 7, then f extends if and only if the sum of the vertices of K_1 and the sum of the vertices of K_2 are either both an element in 2Λ, or both not.*

Proof. Another way of saying that the morphism f extends, is that for $\{e_1, \ldots, e_7\}$ the roots in K_1, the sequences $(e_1, \ldots, e_7)$ and $(f(e_1), \ldots, f(e_7))$ are conjugate. By Proposition 2.12, for $r \leq 8$, $r \neq 7$, there is only one orbit of ordered sequences of length r of roots that have pairwise dot product 1. This implies that f extends to an element in W if K_1, K_2 have size unequal to 7. Furthermore, by the same proposition, there are two orbits of ordered sequences of roots of length 7. By Proposition 4.7, there two orbits of cliques of size 7, that are distinguished by whether the sum of the 7 roots is an element in 2Λ or not. We conclude that the two orbits of ordered sequences are distinguished in the same way. This implies that f extends if and only if the sum of the vertices in K_1 and the sum of the vertices in $f(K_1) = K_2$ are both in 2Λ or both not. $\square$

Remark 4.10. We know that the cliques of size 7 in Γ_1 are 6-faces of the E_8 root polytope. We can describe the two orbits of these cliques in this framework as well. A 6-face of the polytope is an intersection of two facets. There are two types of facets of the E_8 root polyopte: 7-crosspolytopes and 7-simplices (Proposition 2.5). Since the maximal cliques of size 7 in Γ_1 are not contained in a 7-simplex, these are exactly the intersections of two 7-crosspolytopes.
Consider the set $c = \{c_1, \ldots, c_7, d_1, \ldots, d_7\}$ as defined above Lemma 3.14. Note that $d = \{d_1, \ldots, d_7\}$ is a non-maximal clique of size 7 in Γ_1 that is contained in the 7-crosspolytope with vertices in c, but also in the 7-simplex with vertices $d \cup \left\{\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)\right\}$. It follows that all non-maximal cliques of size 7 in Γ_1 are the intersection of a 7-crosspolytope with a 7-simplex.
From this it also follows that two 7-simplices in the E_8 root polytope never intersect.

Remark - analogy with geometry 4.11. Let X be a del Pezzo surface of degree one over an algebraically closed field, and C the set of exceptional classes in Pic X. Through the bijection between C and E, cliques in Γ_1 are related to sets of exceptional classes that are pairwise disjoint (see Remark 2.8). These are called *exceptional sets*, and can be blown down so that we obtain a del Pezzo surface of higher degree (see [Man74], Chapter IV). Since a del Pezzo surface can have degree at most 9 (in which case it is $\mathbb{P}^2$), it is clear that the maximal size of a clique in Γ_1 is eight. We can also describe the two orbits of size seven in this setting; cliques that are maximal correspond to exceptional sets that blow down to a del Pezzo surface of degree eight that is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and cliques that are not maximal correspond to exceptional sets that blow down to a del Pezzo surface of degree eight that is isomorphic to $\mathbb{P}^1$ blown up in one point ([Man74], remark below Corollary 26.8).

5. Maximal cliques

In this section we describe all maximal cliques in Γ_c for $c \neq \{-1, 0, 1\}$ (cliques of type IV), and their orbits under the action of W. Note that $\Gamma_{-1,0,1}$ is the graph Γ after removing all edges between roots and their inverses. This means that the maximal cliques in $\Gamma_{-1,0,1}$ are all of size 120: for each root you can either choose the root or its inverse. There are therefore 2^{120} maximal cliques in $\Gamma_{-1,0,1}$, and at least $\left\lceil \frac{2^{120}}{|W|} \right\rceil = 1907810427151244719477695595$ orbits in the set of maximal cliques under the action of W. Because of the size of these cliques and their orbits, we did not compute the orbits.

In the first two subsections of this section we describe all maximal cliques in Γ_{-2}, Γ_{-1}, Γ_0, Γ_1, $\Gamma_{-2,-1}$, $\Gamma_{-2,1}$, $\Gamma_{-2,0}$, and $\Gamma_{-2,-1,0,1} = \Gamma$. Cliques in $\Gamma_{-2,-1}$ and $\Gamma_{-2,1}$ are monochromatic (Lemma 5.2), and maximal cliques in $\Gamma_{-2,0}$ are in bijection with maximal cliques in Γ_0 (Lemma 5.4). Therefore, everything before Section 5.3 follows from results in Section 4 and is done without a computer. From Section 5.3 onwards, we used `magma` for some computations. The code that we used can be found in [Win].

Our motivation to study the cliques in Γ comes from del Pezzo surfaces of degree one (see Remark 2.8), and because of that, the maximal cliques in $\Gamma_{-2,0}$ and $\Gamma_{-1,0}$ are of special interest to us, which is explained in Remark 5.1. For these two graphs we have some extra results. We compute the structure of the largest cliques in the graphs, see Propositions 5.6 and 5.20. We also show that for these largest cliques, their stabilizer in W acts transitively on the clique itself (Corollaries 5.9 and 5.22). The techniques in Sections 5.2 and 5.3 show how one could prove similar results for graphs with other colors.

The main results of this section are summarized in the tables in Appendix A and Remark 6.1.

Notation. To denote cliques of Γ in a compact way, we order the root system E as follows. Roots of the form $\left(\pm\frac{1}{2},\ldots,\pm\frac{1}{2}\right)$ are ordered lexicographically and denoted by numbers $1-128$; for example, $\left(-\frac{1}{2},\ldots,-\frac{1}{2}\right)$ is number 1, and $\left(\frac{1}{2},\ldots,\frac{1}{2}\right)$ number 128. Roots that are permutations of $(\pm1,\pm1,0,0,0,0,0,0)$ are ordered lexicographically and denoted by the numbers $129-240$; for example, $(-1,-1,0,0,0,0,0,0)$ is number 129, and $(1,1,0,0,0,0,0,0)$ is number 240.

The table in Appendix A contains the following information.

- Graph: a graph Γ_c where c is a set of colors in $\{-2,-1,0,1\}$.
- K: a clique in Γ_c; we denote vertices by their index as in the notation above.
- $|K|$: the size of K.
- $|W_K|$: the size of the stabilizer of clique K in the group W.
- $|\operatorname{Aut}(K)|$: the size of the automorphism group of K as a colored graph.
- $\#O$: the number of orbits of the set of all maximal cliques of size $|K|$ in Γ_c under the action of W.

For each graph Γ_c, the list of cliques in Γ_c in the table in Appendix A gives exactly one representative for each orbit of the set of maximal cliques in Γ_c under the action of W. The proofs of these results are in Proposition 5.3, Corollary 5.13, Proposition 5.25, Lemma 5.27, Proposition 5.29, and Proposition 5.31.

The following remark shows the connection between del Pezzo surfaces and cliques in $\Gamma_{-2,0}$ and $\Gamma_{-1,0}$.

Remark - analogy with geometry 5.1. Let X be a del Pezzo surface of degree one over an algebraically closed field, and let C be the set of exceptional classes in Pix X. The question that led us to study the E_8 root system was how many elements of C can go through the same point on X. The linear system $|-2K_X|$ realizes X as a double cover of a cone in $\mathbb{P}^3$, ramified over a smooth sextic curve B that does not contain the vertex of the cone. There are 120 hyperplanes that are tritangent to B, and such a hyperplane pulls back to the sum of two elements in C that intersect with multiplicity three. It follows that two elements in C intersecting with multiplicity three correspond to curves on X intersecting in three points on the

ramification curve. Conversely, if an element c in C corresponds to a curve on X that goes trough a point P on the ramification curve, then the unique element $c' \in C$ with $c \cdot c' = 3$ corresponds to a curve on X going through P as well.
Through the bijection $C \longrightarrow E,\ c \longmapsto c + K_X$, two elements in C that intersect with multiplicity a correspond to two roots $e_1, e_2 \in E$ with $e_1 \cdot e_2 = 1 - a$. Therefore, cliques in Γ that correspond to sets of pairwise intersecting lines on X have edges of colors $-2, -1, 0$. Since elements in C with intersection multiplicity 3 correspond to two roots in E with dot product -2, it follows that a set of lines on X that all go through one point on the ramification curve forms a clique in $\Gamma_{-2,0}$, and a set of lines on X that all go through one point outside the ramification curve forms a clique in $\Gamma_{-1,0}$. This motivates why we have studied these two graphs extensively, and especially the biggest cliques in them (with respect to number of vertices).

5.1. Maximal cliques in $\Gamma_{-2}, \Gamma_{-1}, \Gamma_1, \Gamma_{-2,-1}, \Gamma_{-2,1}$, and $\Gamma_{-2,-1,0,1}$

Lemma 5.2. *Cliques in $\Gamma_{-2,-1}$ and in $\Gamma_{-2,1}$ are monochromatic.*

Proof. For an element $e \in E$, its inverse $-e$ is the unique element intersecting it with multiplicity -2 (Proposition 2.2). Take $e, f \in E$ with $e \cdot f = -1$, then $-e \cdot f = 1$, hence $e, f, -e$ do not form a clique in $\Gamma_{-2,-1}$. Therefore all cliques in $\Gamma_{-2,-1}$ are monochromatic. Analogously, the cliques in $\Gamma_{-2,1}$ are monochromatic. □

Proposition 5.3. *For*

$$c \in \{\{-2\}, \{-1\}, \{1\}, \{-2,-1\}, \{-2,1\}, \{-2,-1,0,1\}\},$$

the table in Appendix A gives the complete list of orbits of the maximal cliques in Γ_c, as well as a correct representative for each orbit, the size of its stabilizer in W, and the size of its automorphism group.

Proof. We showed in Section 4 that all maximal cliques in Γ_{-2} have size 2, and that they form one orbit of size 120. We also showed that all maximal cliques in Γ_{-1} have size 3, and they form one orbit of size 2240. In Proposition 4.7 we showed that there are two orbits of maximal cliques in Γ_1; one of size 69120, which consists of cliques of size 7, and one of size 17280, which consists of cliques of size 8. For $\Gamma_{-2,-1}$ and $\Gamma_{-2,1}$ we proved that all cliques are monochromatic in Lemma 5.2, so the maximal cliques and their orbits are found by looking at the monochromatic subgraphs Γ_{-2}, Γ_{-1}, and Γ_1.
It is an easy check that for these five graphs, the cliques in the table are correct representatives of the orbits. The sizes of their stabilizers are found by dividing the order of W by the size of their orbit. Since all the cliques in these five graphs are monochromatic, their automorphism group is the permutation group on their vertices.
Finally, note that $\Gamma_{-2,-1,0,1} = \Gamma$. The only maximal clique in $\Gamma_{-2,-1,0,1}$ is therefore the whole graph, which forms an orbit of size 1 under the action of W. □

5.2. Cliques in Γ_0 and $\Gamma_{-2,0}$

The following lemma describes the maximal cliques in $\Gamma_{-2,0}$.

Lemma 5.4. *In $\Gamma_{-2,0}$, the following hold.*

(i) The maximal size of a clique in $\Gamma_{-2,0}$ is 16, and there are no maximal cliques of smaller size.

(ii) The set of maximal cliques in $\Gamma_{-2,0}$ is given by

$$\{\{e_1, \ldots, e_8, -e_1, \ldots, -e_8\} \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}.$$

Proof. By Theorem 4.6, all maximal cliques in Γ_0 are of size 8. Let $\{e_1, \ldots, e_8\}$ be a maximal clique in Γ_0. Then $\{e_1, \ldots, e_8, -e_1, \ldots, -e_8\}$ is a clique in $\Gamma_{-2,0}$ of size 16. Now assume that $\{c_1, \ldots, c_r\}$ is a clique in $\Gamma_{-2,0}$ of size bigger than 16. Since edges of color -2 connect a root and its inverse, the clique $\{c_1, \ldots, c_r\}$ contains a subclique of size at least $\lceil \frac{r}{2} \rceil$ with only edges of color 0. But this would give a clique in Γ_0 of size at least $\left\lceil \frac{17}{2} \right\rceil = 9$, contradicting Theorem 4.6. We conclude that the maximal size of a clique in $\Gamma_{-2,0}$ is 16. Now assume that S is a maximal clique in $\Gamma_{-2,0}$ of size smaller than 16. Let K be the biggest (with respect to number of vertices) subclique of S with only edges of color 0. Let K' be a maximal clique in Γ_0 containing K, so K' has size 8. Then the clique consisting of all vertices of K' and their inverses is a clique in $\Gamma_{-2,0}$ of size 16 that strictly contains S, contradicting the maximality of S. We conclude that there are no maximal cliques of size smaller than 16 in $\Gamma_{-2,0}$, concluding the proof of (i). Part (ii) is now obvious. □

To show that the group W acts transitively on the maximal cliques in $\Gamma_{-2,0}$, we use the following lemma, which builds on results in previous sections. Recall the set Y as defined above Lemma 3.20.

Lemma 5.5. *The following hold.*

(i) For every element $y = (e_1, \ldots, e_4) \in Y$, there is a unique maximal clique in $\Gamma_{-2,0}$ containing $e_1, \ldots, e_4$.

(ii) Every maximal clique in $\Gamma_{-2,0}$ contains 896 distinct subsets of four roots $e_1, \ldots, e_4$ such that $(e_1, \ldots, e_4)$ is an element in Y.

Proof.

(i) From Lemma 4.2 it follows that an element in Y is contained in a unique clique of size 8 in Γ_0. But such a clique extends uniquely to a maximal clique in $\Gamma_{-2,0}$ by adding all inverses of the roots.

(ii) By Lemma 5.4, a maximal clique in $\Gamma_{-2,0}$ consists of eight pairwise orthogonal roots and their inverses. Let K be such a clique. Eight pairwise orthogonal roots in K contain $\binom{8}{4} - 14 = 56$ distinct subsets of four roots that form an element in Y by Corollary 4.3. Let $D = \{e_1, e_2, e_3, e_4\}$ be such a subset. If we replace a root in D by its inverse, then the roots in D still form an element in Y. This gives $56 \cdot 2^4 = 896$ distinct subsets of K of that form an element in Y. Since a set of four roots that contains both a root and its inverse never forms an element in Y, these are all of them. □

Let $\mathcal{S}$ be the set of all cliques of size 16 in $\Gamma_{-2,0}$. By Lemma 5.4, this is exactly the set of maximal cliques in $\Gamma_{-2,0}$. By Lemma 5.5 we have a surjective map

$$s \colon Y \longrightarrow \mathcal{S}.$$

Corollary 5.6. *The group W acts transitively on $\mathcal{S}$, and we have $|\mathcal{S}| = 2025$.*

Proof. Since the map s is surjective and W acts transitively on Y (Proposition 3.28), it follows from Lemma 2.14 that W acts transitively on $\mathcal{S}$. From Lemma 5.5 it follows that $|\mathcal{S}| = \frac{|Y|}{|896\cdot 4!|} = 2025$. □

Let K be an element of $\mathcal{S}$, and W_K its stabilizer in W. Now that we fully described all maximal cliques in $\Gamma_{-2,0}$ and the action of W on the set of these maximal cliques, we finish the study of $\Gamma_{-2,0}$ by studying the action of W_K on K, and concluding that W acts transitively on cliques of sizes $6, 7, 8$ in Γ_0 in Proposition 5.12. Consider the sets

$$I = \{(e_1, e_2, e_3) \in K^3 \mid e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0\},$$

and

$$J = \{(e_1, e_2) \in K^2 \mid e_1 \cdot e_2 = 0\}.$$

Proposition 5.7. *The group W_K acts transitively on I.*

Proof. Since K consists of eight pairwise orthogonal roots and their inverses by Lemma 5.4, we have $|I| = 16 \cdot 14 \cdot 12 = 2688$. Fix an element $\iota = (e_1, e_2, e_3)$ in I. We want to show that its orbit $W_K\iota$ has size 2688, hence is equal to I. Let $W_{K,\iota}$ be the stabilizer in W_K of ι. We have $|W_K\iota| = \frac{|W_K|}{|W_{K,\iota}|}$, and

$$\frac{|W|}{|W_{K,\iota}|} = \frac{|W|}{|W_K|} \cdot \frac{|W_K|}{|W_{K,\iota}|}.$$

By Corollary 5.6 we have $\frac{|W|}{|W_K|} = |WK| = 2025$. Moreover, we have

$$\frac{|W|}{|W_{K,\iota}|} = \frac{|W|}{|W_\iota|} \cdot \frac{|W_\iota|}{|W_{\iota,K}|}.$$

By Proposition 3.27 we have $\frac{|W|}{|W_\iota|} = |W\iota| = 240\cdot 126\cdot 60 = 1814400$. We now compute $\frac{|W_\iota|}{|W_{\iota,K}|} = |W_\iota K|$. From Proposition 3.27 we know that there are 24 roots $e \in E$ such that (e_1, e_2, e_3, e) is an element in Y. Since W_ι acts transitively on those 24 roots by Proposition 3.28, the orbit $W_\iota K$ contains the cliques $s((e_1, e_2, e_3, e))$ for all 24 roots e. Now fix e and set $y = (e_1, e_2, e_3, e)$, and $L = s(y)$. From Lemma (i) we know that L contains exactly eight roots f such that (e_1, e_2, e_3, f) is an element in Y. Therefore, they determine the same unique clique of size sixteen as e. We conclude that there are $\frac{24}{8} = 3$ different cliques containing ι. So we have $|W_\iota K| \geq 3$, and we find $\frac{|W|}{|W_{K,\iota}|} \geq 1814400 \cdot 3 = 5443200$. It follows that $\frac{|W_K|}{|W_{K,\iota}|} \geq \frac{5443200}{2025} = 2688$. Since on the other hand we have $\frac{|W_K|}{|W_{K,\iota}|} = |W_K\iota| \leq |I| = 2688$, we have equality everywhere and we conclude that $W_K\iota = I$. This finishes the proof. □

Corollary 5.8. *The group W_K acts transitively on J.*

Proof. We have a projection map $\lambda\colon I \longrightarrow J$ on the first two coordinates. Since K consists of eight pairwise orthogonal roots and their inverses, if we fix two elements e_1, e_2 such that $(e_1, e_2) \in J$, there are $16 - 4 = 12$ elements $e \in K$ such that $(e_1, e_2, e) \in I$. Therefore, λ is surjective. From Proposition 5.7 and Lemma 2.14, it follows that W_K acts transitively on J. □

Corollary 5.9. *The group W_K acts transitively on K.*

Proof. We have a projection map $\lambda\colon J \longrightarrow K$ on the first coordinate. For every element e in K there are 14 elements c such that $(e,c) \in J$, so λ is surjective. From Corollary 5.8 and Lemma 2.14 it follows that W_K acts transitively on K. □

Proposition 5.10. *For $n \in \{2,3,5,6,7,8\}$, the group W acts transitively on the set*

$$D_n = \{\{e_1,\dots,e_n,-e_1,\dots,-e_n\} \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}.$$

Proof. For $n = 2,3,5$, this follows from the fact that W acts transitively on the cliques of size n in Γ_0 (Propositions 4.1 and 4.6), and the fact that there is a surjective map from the set of cliques in Γ_0 of size n to D_n. The case $n = 8$ is Corollary 5.6. From Proposition 5.8, it follows that the stabilizer W_K in W of K acts transitively on the set

$$\{(e_1,e_2,-e_1,-e_2) \in K^4 \mid e_1 \cdot e_2 = 0\}$$

Since K consists of eight pairwise orthogonal roots and their inverses, the cliques of six pairwise orthogonal roots and their inverses in K are the complements of the cliques of two orthogonal roots and their inverses in K, so this implies that W_K acts transitively on the set of cliques of six pairwise orthogonal roots and their inverses in K, too. From Corollary 5.6, the statement now follows for $n = 6$. The case $n = 7$ is proved analogously since we showed that W_K acts transitively on K. □

Remark 5.11. There are two orbits under the action of W on the set

$$\{\{e_1,\dots,e_4,-e_1,\dots,-e_4\} \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}.$$

Indeed, this follows from Proposition 4.1 and the fact that there is a surjective map from the set of cliques of size 4 in Γ_0 to this set.

As we mentioned before, the fact that W acts transitively on the set of cliques of size r for $1 \leq r \leq 8$ in Γ_0 is in [DM10]. The following proposition shows how it follows from our results about $\Gamma_{-2,0}$ as well.

Proposition 5.12. *for $n = 6,7,8$, the group W acts transitively on the cliques of size n in Γ_0.*

Proof. We know that W acts transitively on the set

$$D_n = \{\{e_1,\dots,e_n,-e_1,\dots,-e_n\} \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}$$

from Proposition 5.10. Let F_n be the set of cliques of size n in Γ_0. We have an obvious map $f\colon F_n \longrightarrow D_n$ which adds adds the inverses to all roots in an element in F_n. Let $D = \{e_1,\dots e_n,-e_1,\dots,-e_n\}$ be an element in D_n and consider its fiber $f^{-1}(D)$ in F_n. This consists of all cliques $\{\pm e_1,\dots,\pm e_n\}$, where for each root either itself or its inverse is chosen. The stabilizer W_D of D acts on $f^{-1}(D)$. Note that for $i \in \{1,\dots,n\}$, the reflection in the hyperplane orthogonal to e_i switches e_i and $-e_i$ and fixes all other roots in D, hence it is an element in W_D. Therefore, W_D acts transitively on $f^{-1}(D)$, and by Lemma 2.14, W acts transitively on F_n. □

Corollary 5.13. *The table in Appendix A gives the complete list of orbits of the maximal cliques in Γ_0 and $\Gamma_{-2,0}$, as well as a correct representative for each orbit, the size of its stabilizer in W, and the size of its automorphism group.*

Proof. Al maximal cliques in $\Gamma_{-2,0}$ are of size 16 (Lemma 5.4) and there is only one orbit of them, of size 2025 (Corollary 5.6). It is an easy check that the clique in the table is a representative of this orbit. Its stabilizer size is $\frac{|W|}{|2025|} = 344064$. Its automorphism group is isomorphic to $\mu_2^8 \rtimes S_8$ by Lemma 3.29, hence has size $2^8 \cdot 8!$. In Theorem 4.6 we showed that all maximal cliques in Γ_0 have size 8, and that there are 518400 of them. In Proposition 5.12 we showed that W acts transitively on the set of these cliques. Therefore the stabilizer of the clique in the table has size $\frac{|W|}{518400} = 1344$. Its automorphism group is the symmetric group on the 8 vertices. □

We finish this subsection by proving Theorem 1.2 for maximal cliques in $\Gamma_{-2,0}$.

Lemma 5.14. *Let K_1 and K_2 be two maximal cliques in $\Gamma_{-2,0}$, and let $f\colon K_1 \longrightarrow K_2$ be an isomorphism between them. Then f extends to an automorphism of Λ if and only if for every subclique S of four pairwise orthogonal roots in K_1, the image $f(S)$ in K_2 is conjugate to S under the action of W.*

Proof. By Corollary 5.6, the group W acts transitively on the set of maximal cliques in $\Gamma_{-2,0}$. Therefore there is an element α in W such that $\alpha(K_1) = K_2$. So $\alpha^{-1} \circ f$ is an element in the automorphism group $\mathrm{Aut}(K_1)$ of K_1. Of course, f extends to an element in W if and only if $\alpha^{-1} \circ f$ does. Moreover, for every set S of four pairwise orthogonal roots, $f(S)$ and $(\alpha^{-1} \circ f)(S)$ are conjugate. We conclude that we can reduce to the case where $K_1 = K_2$, and f is an element in $\mathrm{Aut}(K_1)$.
By Lemma 5.4, we can choose a subclique $H = \{e_1, \ldots, e_8\}$ of K_1 of eight pairwise orthogonal roots, such that we have $K_1 = \{e_1, \ldots, e_8, -e_1, \ldots, -e_8\}$. Let $\mathrm{Aut}(H)$ be the automorphism group of H as colored graph, and let $(\mathrm{Aut}(K_1))_H$ be the stabilizer of H in $\mathrm{Aut}(K_1)$. Since for every element $e \in K_1$ we have $e \in H$ or $-e \in H$, an element in $\mathrm{Aut}(H)$ determines a unique element in $(\mathrm{Aut}(K_1))_H$, and conversely, every element in $(\mathrm{Aut}(K_1))_H$, when restricted to H, determines a unique element in $\mathrm{Aut}(H)$. So we have an isomorphism $\varphi\colon \mathrm{Aut}(H) \xrightarrow{\sim} (\mathrm{Aut}(K_1))_H$. Let f be an element in $\mathrm{Aut}(K_1)$. Using Lemma 3.29, write $f = a \circ r|_{K_1}$, where a is an element in $\varphi(\mathrm{Aut}(H))$, and r is a composition of reflections r_i in the hyperplanes orthogonal to e_i for certain $i \in \{1, \ldots, 8\}$. By definition, $r|_{K_1}$ extends to the element r in W, and $r(S)$ and S are conjugate for all cliques S of four orthogonal roots, so the statement in the lemma is true for f if and only if it is true for a. Of course, if a extends to an automorphism of Λ, then a and $a(S)$ are conjugate for all subcliques S of K_1 of four orthogonal roots. Conversely, assume that $a(S)$ and S are conjugate for all such S. Then in particular, for every subclique S' of size 4 in H, the sets $a|_H(S')$ and S' are conjugate. From Corollary 4.5 it follows that $a|_H$ extends to an element in W. Write w for an element in W with $w|_H = a|_H$. Then $w|_{K_1}$ and a are both elements in $(\mathrm{Aut}(K_1))_H$, that are identical on H, hence also on K_1. We conclude that $w|_{K_1}$ and a are the same, so a extends to $w \in W$. This finishes the proof. □

5.3. **Cliques in** $\Gamma_{-1,0}$

Consider the following twelve elements in E.

$t_1 = (1,1,0,0,0,0,0,0)$; $t_7 = \left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$

$t_2 = (0,0,1,1,0,0,0,0)$; $t_8 = \left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$

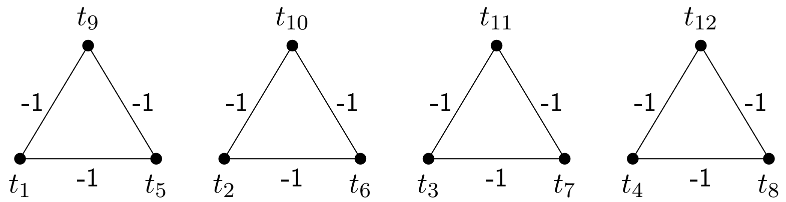
$t_3 = (0,0,0,0,1,1,0,0)$; $t_9 = \left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

$t_4 = (0,0,0,0,0,0,-1,1)$; $t_{10} = \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$

$t_5 = \left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$; $t_{11} = \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

$t_6 = \left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$; $t_{12} = \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$

One can easily check that these twelve elements form a clique in $\Gamma_{-1,0}$, depicted below (where edges of color 0 are not drawn). We call this clique T.



The existence of this clique implies that the maximal size of cliques in $\Gamma_{-1,0}$ is at least twelve. We will show that this is in fact the maximum. Moreover, we will show that all cliques of size twelve in $\Gamma_{-1,0}$ are isomorphic, and that W acts transitively on the set of cliques of size twelve (Propositions 5.20 and 5.21). To describe all maximal cliques of smaller size in $\Gamma_{-1,0}$ and their orbits under the action of W, we use `magma` for part of the computations.

Lemma 5.15. *Take $e_1,e_2,e_3 \in E$ with $e_1\cdot e_2 = e_2\cdot e_3 = e_1\cdot e_3 = -1$. For $e \in E$ with $e \neq e_1,e_2,e_3$, we have $e\cdot e_i \neq 1$ for all $i = 1,2,3$ if and only if $e\cdot e_1 = e\cdot e_2 = e\cdot e_3 = 0$.*

Proof. Take $e_1,e_2,e_3 \in E$ with $e_1\cdot e_2 = e_2\cdot e_3 = e_1\cdot e_3 = -1$. Then we have $\|e_1+e_2+e_3\| = 0$, so $e_1+e_2+e_3 = 0$. For an element $e \in E$ with $e \neq e_1,e_2,e_3$ we have $e\cdot e_i \in \{-2,-1,0,1\}$ for $i = 1,2,3$, so $e\cdot e_i \neq 1$ for $i=1,2,3$ implies $e\cdot e_i \leq 0$ for $i = 1,2,3$. But $e\cdot(e_1+e_2+e_3) = e\cdot 0 = 0$, so we have $e\cdot e_i \neq 1$ for $i = 1,2,3$ if and only if $e\cdot e_i = 0$ for $i=1,2,3$. □

Lemma 5.16. *The maximum size of a clique in $\Gamma_{-1,0}$ that contains $e_1,e_2,e_3 \in E$ with $e_1\cdot e_2 = 0$ and $e_1\cdot e_3 = e_2\cdot e_3 = -1$, is ten.*

Proof. Consider the elements $e_1 = (1,1,0,0,0,0,0,0)$, $e_2 = (0,0,1,1,0,0,0,0)$, and $e_3 = (-1,0,-1,0,0,0,0,0)$. By Lemma 3.32, it is enough to prove that the maximal size of all cliques in $\Gamma_{-1,0}$ containing e_1,e_2,e_3 is ten. Let A be the set

$$\{e \in E \mid \text{for } i \in \{1,2,3\} : e\cdot e_i \in \{-1,0\}\}.$$

For an element $e = (a_1 \dots, a_8)$ in A, we have $a_1 + a_2 \in \{-1, 0\},\ a_3 + a_4 \in \{-1, 0\}$, and $-a_1 - a_3 \in \{-1, 0\}$. This gives the following possibilities for (a_1, a_2, a_3, a_4):

$$\begin{aligned}
(a_1, a_2, a_3, a_4) = &\left(-\tfrac{1}{2}, \pm\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right) && \text{(16 roots)}\\
&\left(\tfrac{1}{2}, -\tfrac{1}{2}, \pm\tfrac{1}{2}, -\tfrac{1}{2}\right) && \text{(16 roots)}\\
&\left(\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right) && \text{(8 roots)}\\
&(0, -1, 0, -1) && \text{(1 roots)}\\
&(0, 0, 1, -1) && \text{(1 root)}\\
&(1, -1, 0, 0) && \text{(1 root)}\\
&(0, -1, 0, 0) && \text{(8 roots)}\\
&(0, 0, 0, -1) && \text{(8 roots)}\\
&(0, 0, 0, 0) && \text{(24 roots)}
\end{aligned}$$

We conclude that the cardinality of A is 83. As it is too tedious to compute the maximal size of the cliques in $\Gamma_{-1,0}$ with only vertices in A by hand, we compute this with `magma`. This number is seven, which implies that the maximal size of a clique in $\Gamma_{-1,0}$ containing $e_1,\ e_2$ and e_3 is ten. □

Lemma 5.17. *The maximum size of a clique in $\Gamma_{-1,0}$ that contains a clique of five pairwise orthogonal vertices is ten.*

Proof. Consider the set

$$V_5 = \{\{e_1, \dots, e_5\} \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}.$$

The group W acts transitively on V_5 by Theorem 4.6, so it suffices to take

$$\begin{aligned}
e_1 &= (1, 1, 0, 0, 0, 0, 0, 0); & e_4 &= (0, 0, 0, 0, 0, 0, 1, 1);\\
e_2 &= (0, 0, 1, 1, 0, 0, 0, 0); & e_5 &= (0, 0, 0, 0, 0, 0, 1, -1),\\
e_3 &= (0, 0, 0, 0, 1, 1, 0, 0); & &
\end{aligned}$$

and show that a clique in $\Gamma_{-1,0}$ containing $e_1, \dots, e_5$ has size at most ten. Let A be the set

$$\{e \in E \mid \text{for } i \in \{1, \dots, 5\} : e \cdot e_i \in \{-1, 0\}\}.$$

For an element $e = (a_1, \dots, a_8) \in A$, we have $a_i + a_{i+1} \in \{-1, 0\}$ for $i \in \{1, 3, 5, 7\}$, and $a_7 - a_8 \in \{-1, 0\}$. If e is of the form $\left(\pm\frac{1}{2}, \dots, \pm\frac{1}{2}\right)$, then $a_7 + a_8,\ a_7 - a_8 \in \{-1, 0\}$ implies that $a_7 = -\frac{1}{2}$. Moreover, for $i \in \{1, 3, 5\}$, we have either $a_i = a_{i+1} = -\frac{1}{2}$ or $a_i = -a_{i+1}$. This gives three possibilities for each tuple (a_i, a_{i+1}) for $i \in \{1, 3, 5\}$, and a_8 is then determined since an even number of the entries of e should be negative. We find $3^3 = 27$ possibilities.

If e has two non-zero entries that are ± 1, then $a_7 + a_8,\ a_7 - a_8 \in \{-1, 0\}$ implies that either $(a_7, a_8) = (-1, 0)$, or $(a_7, a_8) = (0, 0)$. Moreover, for $i \in \{1, 3, 5\}$ we have $\{a_i, a_{i+1}\} = \{-1, 0\}$ or $\{a_i, a_{i+1}\} = \{-1, 1\}$. It is easy to check that this gives 24 possibilities.

We find that the cardinality of A is 51. As it is too tedious to compute the maximal size of the cliques in $\Gamma_{-1,0}$ with all vertices in A by hand, we compute this with `magma`. The maximal size of a clique in $\Gamma_{-1,0}$ with all vertices in A is five, so the maximal size of a clique in $\Gamma_{-1,0}$ containing $e_1, \dots, e_5$ is ten. □

We recall some known Ramsey numbers.

Theorem 5.18. *(Ramsey Numbers). For two integers l, k, let $R(l, k)$ be the least positive integer n such that every undirected graph with n verices contains either a clique of order 4 or an independent set of order 5. Then we have $R(3, 3) = 6$, $R(3, 4) = 9$, and $R(4, 5) = 25$.*

Proof. See [GRS90], Table 4.1 for $R(3, 3)$ and $R(3, 4)$, and [MR95] for $R(4, 5)$. □

Proposition 5.19. *Every clique in $\Gamma_{-1,0}$ of size bigger than ten contains a subclique of size four with only edges of color 0.*

Proof. Let K be a clique in $\Gamma_{-1,0}$ of size bigger than ten. Consider the subgraph K' of K whose vertex set consists of all vertices of K, and whose edge set is obtained by taking only the edges in K of color -1. We consider different cases depending on the number of connected components of K'.
If K' has at least four connected components, then we can take four vertices, each from a different connected component, and these vertices form a clique of size four with only edges of color 0 in K.
Now assume that K' has at most three connected components. We first show that every connected component of K' that contains a clique of size three is a clique of size three in itself. To this end, assume that K' contains a clique of size three, given by $\{e_1, e_2, e_3\}$. By Lemma 3.8, we have $e_1 + e_2 + e_3 = 0$. If e is another vertex of K', then $e \cdot e_i \in \{-1, 0\}$ for $i \in \{1, 2, 3\}$, and $e \cdot (e_1 + e_2 + e_3) = 0$, from which it follows that $e \cdot e_i = 0$ for $i \in \{1, 2, 3\}$. We conclude that the vertices e_1, e_2, e_3 form a connected component of K'. Since there are at most three connected components by assumption, and K' has more than ten vertices, we conclude that not all components contain a clique of size three. Now remove a vertex from every connected component in K' that is a clique of size three (of which there are at most two), then we are left with a subgraph of K' with at least 9 vertices, and no cliques of size three left. Hence by Theorem 5.18, there must be a set of four vertices that are pairwise disjoint in K', meaning that they form a clique with edges of color 0 in K. □

Let V_3, V_4, Z, α, π and Y be as in the diagram above Lemma 3.23.

Proposition 5.20. *The following hold.*

(i) Let $v = (e_1, e_2, e_3, e_4)$ be an element in V_4. Then e_1, e_2, e_3 and e_4 are contained in a clique of size bigger than ten in $\Gamma_{-1,0}$ if and only if v is an element of Y.

(ii) Every maximal clique of size at least eleven in $\Gamma_{-1,0}$ is of the form

$$\left\{ \left\{ \begin{array}{c} e_1, \ldots, e_4, \\ f_1, \ldots, f_4, \\ -e_1 - f_1, \ldots, -e_4 - f_4 \end{array} \right\} \middle| \begin{array}{c} \forall i \neq j : e_i \cdot e_j = f_i \cdot f_j = 0; \\ \forall i : e_i \cdot f_i = -1; \\ \forall i \neq j : e_i \cdot f_j = 0. \end{array} \right\}.$$

(iii) The maximal size of a clique in $\Gamma_{-1,0}$ is twelve, and there are no maximal cliques of size eleven in $\Gamma_{-1,0}$.

(iv) For an element $v \in Y$, there are eight cliques of size twelve in $\Gamma_{-1,0}$ containing the elements of v.

(v) For K a clique of size twelve in $\Gamma_{-1,0}$, we have $|K^4 \cap V_4| = |K^4 \cap Y| = 1944$.

Proof. Let K be a clique of size bigger than ten in $\Gamma_{-1,0}$. By Proposition 5.19, we know that K contains a subclique of size four with only edges of color 0. Let $\{e_1, e_2, e_3, e_4\}$ be such a subclique in K. Let e be another element in K. By Lemmas 5.16 and 5.17, there is exactly one $i \in \{1, 2, 3, 4\}$ such that $e \cdot e_i = -1$, and $e \cdot e_j = 0$ for $i \neq j \in \{1, 2, 3, 4\}$. It follows that $e \cdot (e_1 + e_2 + e_3 + e_4) = -1$, hence $\sum_{i=1}^4 e_i \notin 2\Lambda$. By Proposition 3.28, this implies that (e_1, e_2, e_3, e_4) is an element in Y. Conversely, the tuple (t_1, t_2, t_3, t_4) is an element in Y and it is contained in the clique T (page 36), so by Proposition 3.28, every element in Y is contained in a clique of size twelve in $\Gamma_{-1,0}$. This proves (i).

Recall the clique T defined above Lemma 5.15. We define the following sets for $i \in \{1, 2, 3, 4\}$.

$$F_i = \left\{ e \in E \;\middle|\; \begin{array}{c} e \cdot t_i = -1, \\ e \cdot t_j = 0 \text{ for } j \in \{1,2,3,4\},\ j \neq i \end{array} \right\}.$$

Let K be a clique in $\Gamma_{-1,0}$ of size at least eleven. Such a K exists, since the clique T is an example. By Proposition 5.19, the clique K contains four vertices that form an element of V_4, and by part (i) this is an element of Y. By Proposition 3.28 we can without loss of generality assume that K contains the four vertices t_1, t_2, t_3, t_4. By Lemma 5.16 and Lemma 5.17, for every element t in $K \setminus \{t_1, t_2, t_3, t_4\}$ there is an $i \in \{1, 2, 3, 4\}$ such that $t \cdot t_i = -1$ and $t \cdot t_j = 0$ for $i \neq j \in \{1, 2, 3, 4\}$. Therefore we have

$$K \setminus \{t_1, t_2, t_3, t_4\} = \bigcup_{i \in \{1,2,3,4\}} K \cap F_i.$$

Fix $i \in \{1, 2, 3, 4\}$. For an element $f \in F_i$ we have $f \cdot t_i = -1$, so by Lemma 3.8 there is a unique element $g \in E$ such that $f \cdot g = t_i \cdot g = -1$, given by $g = -t_i - f$. Note that this element is also in F_i, since $(-t_i - f) \cdot t_j = 0$ for $j \in \{1, 2, 3, 4\}$ with $j \neq i$. So for $i \in \{1, 2, 3, 4\}$, the set F_i is the union of different sets $\{f, -t_i - f\}$, and we claim that $K \cap F_i$ is contained in one of these sets. To prove this, fix i and $f \in K \cap F_i$. Assume by contradiction that there is an element $h \in (K \cap F_i) \setminus \{f, -t_i - f\}$. Then h is in F_i, so $h \cdot f \neq -1$ by uniqueness of g. But h, f are both elements in K, so this implies $h \cdot f = 0$. But then we have $h \cdot t_i = f \cdot t_i = -1$ and $h \cdot f = 0$, so by Lemma 5.16, the clique K has size at most ten, which gives a contradiction. So for $i \in \{1, 2, 3, 4\}$, there are $f_i \in F_i$ such that $K \cap F_i \subseteq \{f_i, -t_i - f_i\}$, and we have

$$K \subseteq \bigcup_{i \in \{1,2,3,4\}} \{t_i, f_i, -t_i - f_i\}.$$

Fix such $f_i \in F_i$ for $i \in \{1, 2, 3, 4\}$. We have $f_i \cdot f_j = 0$ for $i \neq j \in \{1, 2, 3, 4\}$, because if this were not the case then K would contain a triple t_i, f_i, f_j with $t_i \cdot f_i = f_i \cdot f_j = -1$, $f_j \cdot t_i = 0$, which contradicts the fact that K has size bigger than ten by Lemma 5.16. Hence $\bigcup_{i \in \{1,2,3,4\}} \{t_i, f_i, -t_i - f_i\}$ forms a clique in $\Gamma_{-1,0}$ of the required form, and if K is maximal, it is equal to this clique. This proves part (ii), and part (iii) follows directly.

We proceed by proving (iv). Note that (t_1, t_2, t_3, t_4) is an element in Y. We count the number of cliques of size twelve in $\Gamma_{-1,0}$ containing $t_1, \ldots, t_4$. By (ii), we know that such a clique is of the form $\bigcup_{i \in \{1,2,3,4\}} \{t_i, f_i, -t_i - f_i\}$, where f_i and $-t_i - f_i$ are elements in F_i for $i \in \{1, 2, 3, 4\}$. By simply considering all elements in E we find

$$F_1 = \left\{ \left(-\tfrac{1}{2}, -\tfrac{1}{2}, a_3, a_4, a_5, a_6, a_7, a_8\right) \;\middle|\; \begin{array}{c} \{a_3, a_4\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \\ \{a_5, a_6\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \\ a_7 = a_8 \end{array} \right\}.$$

Since $|F_1| = 8$, there are four choices for the set $\{f_1, -t_1 - f_1\}$. Fix f_1, and write $f_1 = \left(-\frac{1}{2}, -\frac{1}{2}, a_3, \ldots, a_8\right)$. Then f_2, $-t_2 - f_2$ are elements in F_2 that are orthogonal to f_1 by (ii). Again, by considering all elements in E we find

$$F_2 = \left\{ \left(b_1, b_2, -\tfrac{1}{2}, -\tfrac{1}{2}, b_5, b_6, b_7, b_8\right) \;\middle|\; \begin{array}{c} \{b_1, b_2\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \\ \{b_5, b_6\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \\ b_7 = b_8 \end{array} \right\}.$$

Let $f = (b_1, \ldots, b_8)$ be an element in F_2. Then f is orthogonal to f_1 if and only if $0 = \sum_{i=5}^{8} a_i b_i = 2(a_5 b_5 + a_7 b_7)$, which holds if and only if $\frac{b_5}{b_7} = -\frac{a_7}{a_5}$. This gives two choices for the tuple (b_5, b_7), and together with the two choices for (b_1, b_2) we find four elements in F_2 that are orthogonal to f_1. This gives two choices for the set $\{f_2, -t_2 - f_2\}$. Fix one. Then f_3, $-t_3 - f_3$, and f_4, $-t_4 - f_4$, are elements in F_3 and F_4 respectively, that are orthogonal to f_1 and f_2. It is an easy check that this determines the sets $\{f_3, -t_3 - f_3\}$ and $\{f_4, -t_4 - f_4\}$ uniquely. So for f_1 we had four choices, for f_2 we had two, and the set $\{f_3, -t_3 - f_3, f_4, -t_4 - f_4\}$ is determined after choosing f_1, f_2. We conclude that there are $4 \cdot 2 = 8$ cliques of size twelve in $\Gamma_{-1,0}$ containing $t_1, \ldots, t_4$. By Proposition 3.28, this holds for every element in Y. This proves (iv).

Let K be a clique of size twelve in $\Gamma_{-1,0}$. Using the notation in (ii), write

$$K = \{e_1, \ldots, e_4, f_1, \ldots, f_4, -e_1 - f_1, \ldots, -e_4 - f_4\}.$$

It follows from (ii) that the sets of four pairwise orthogonal roots in K are given by

$$\{\{a_1, a_2, a_3, a_4\} \mid a_i \in \{e_i, f_i, -e_i - f_i\} \text{ for } i \in \{1, 2, 3, 4\}\}.$$

This gives $3^4 = 81$ such sets, and these give rise to $81 \cdot 4! = 1944$ elements in $K^4 \cap V_4$. From (i) it follows that $K^4 \cap V_4 = K^4 \cap Y$. This proves (v). □

Proposition 5.21. *Let $\mathcal{T}$ be the set of all cliques of size twelve in $\Gamma_{-1,0}$, and R an element in $\mathcal{T}$. The following hold.*

(i) We have $|\mathcal{T}| = 179200$, and the group W acts transitively on $\mathcal{T}$.

(ii) The stabilizer W_R in W of R acts transitively on $R^4 \cap Y$.

Proof. Let T be the clique $\{t_1, \ldots, t_{12}\}$, as defined above Lemma 5.15. Define the set

$$S = \{((e_1, e_2, e_3, e_4), K) \in Y \times \mathcal{T} \mid e_1, \ldots, e_4 \in K\}.$$

We have projections $\lambda\colon S \longrightarrow Y$ and $\mu\colon S \longrightarrow \mathcal{T}$.

From the previous proposition we know that the fibers of λ have cardinality 8, and the fibers of μ have cardinality 1944. Therefore we have $|S| = |Y| \cdot 8 = 348364800$ (Proposition 3.28), and $|\mathcal{T}| = \frac{|S|}{1944} = 179200$. We will show that W acts transitively on S, which implies that it acts transitively on $\mathcal{T}$ by the projection μ. Consider the clique $T \in \mathcal{T}$, and set $y = (t_1, t_2, t_3, t_4) \in T^4 \cap Y$. Then (y, T) is in the fiber of λ above y. The stabilizer W_y in W of y acts on this fiber. We show that this action is transitive, that is, that the orbit $W_y T$ is equal to the whole fiber. We have $|W_yT| = \frac{|W_y|}{|W_{y,T}|}$, and $|W_y| = \frac{|W|}{|Wy|} = \frac{|W|}{|Y|} = 16$. Note that t_1, t_2, t_3, t_4 are all orthogonal to the four roots

$$e_1 = (1, -1, 0, 0, 0, 0, 0, 0), \qquad e_2 = (0, 0, 1, -1, 0, 0, 0, 0),$$
$$e_3 = (0, 0, 0, 0, 1, -1, 0, 0), \qquad e_4 = (0, 0, 0, 0, 0, 0, 1, 1).$$

Therefore, for $i \in \{1, 2, 3, 4\}$, the reflection r_i in the hyperplane orthogonal to e_i is contained in the stabilizer W_y. Since the subgroup generated by these four reflections has cardinality 16, we conclude that this is the whole group W_y. We can now compute that for every element r in W_y we have $rT \neq T$, except for the identity and the composition of all four reflections r_1, r_2, r_3, r_4. So $|W_{y,T}| = 2$, and we have $|W_yT| = \frac{|W_y|}{|W_{y,T}|} = \frac{16}{2} = 8$. Since the fiber of λ above y has cardinality 8, we conclude that W_y acts transitively on this fiber. Since W acts transitively on Y, we conclude from Lemma 2.14 that W acts transitively on S. Finally, from the surjective projection μ and Lemma 2.14, it follows that W acts transitively on $\mathcal{T}$. This proves (i). Since W acts transitively on S, the stabilizer W_R in W of the clique R acts transitively on the fiber $\mu^{-1}(R)$. Since there is a bijection $\mu^{-1}(R) \longrightarrow R^4 \cap Y$ given by the projection λ, the group W acts transitively on $R^4 \cap Y$ by Lemma 2.14. This proves (ii). □

Corollary 5.22. *let R be a clique of size twelve in $\Gamma_{-1,0}$. Let W_R be its stabilizer in W. Then W_R acts transitively on R.*

Proof. We have a surjective map $R^4 \cap Y \longrightarrow R$ projecting on the first coordinate, so this follows from the previous proposition and Lemma 2.14. □

Now that we described all the largest cliques (with respect to number of vertices) in $\Gamma_{-1,0}$, we continue to describe all other maximal cliques. Since the size of the stabilizer of a clique is the same for every two cliques that are in the same orbit, we make the following definition.

Definition 5.23. *The stabilizer size of an orbit is the size of the stabilizer of any of the elements in the orbit.*

As one can see in the table in Appendix A, for a set c that contains 0 in combination with either -1 or 1, there are many maximal cliques in Γ_c with small stabilizer sizes, which means large orbits. This means that, even though we use `magma` to find all cliques and orbits, computations can become very large and time consuming. Therefore we use the following lemma throughout.

Lemma 5.24. *Let H be a finite group acting on a finite set X and consider its induced action on the power set of X. Let A and S be subsets of X and let m denote the number of H-conjugates of A that are contained in S. Then the number of H-conjugates of S that contain A equals*

$$\frac{m \cdot |H_A|}{|H_S|},$$

where H_A and H_S denote the stabiliser subgroups of A and S, respectively.

Proof. Let Z denote the H-subset of the product $HA \times HS$ consisting of all pairs (B, T) with $B \in HA$ and $T \in HS$ satisfying $B \subset T$. The group H acts transitively on the codomains of the projection maps $\pi\colon Z \to HA$ and $\rho\colon Z \to HS$. This implies that all fibers of π have the same size, say r, as the fiber above A, which is the number of H-conjugates of S that contain A, that is, the number that we are looking for. All fibers of ρ have the same size as the fiber above S, which equals m. Hence, we can express the size of Z as both $|HA| \cdot r$ and $|HS| \cdot m$. Since the orbits HA and HS have size $|H|/|H_A|$ and $|H|/|H_S|$, respectively, we find

$$r = \frac{m \cdot |HS|}{|HA|} = \frac{m \cdot |H_A|}{|H_S|}. \qquad \square$$

Note that for $A = \emptyset$, we recover the well-known fact that the length of the orbit of S equals the index $[H : H_S]$.
The following proposition describes all maximal cliques and their orbits in $\Gamma_{-1,0}$.

Proposition 5.25. *For two maximal cliques K_1 and K_2 of the same size in $\Gamma_{-1,0}$, the following are equivalent.*

(i) K_1 and K_2 are conjugate under the action of W.

(ii) K_1 and K_2 are isomorphic.

(iii) K_1 and K_2 have the same stabilizer size.

(iv) The automorphism groups of K_1 and K_2 have the same cardinality, and, if this cardinality is 16 and K_1 and K_2 have size 9, then K_1 and K_2 both contain a monochromatic clique of size 7 and color 0, or they both do not.

Moreover, the table in Appendix A gives a complete list of representatives of the orbits of the maximal cliques in $\Gamma_{-1,0}$, as well as for each representative its stabilizer size and the size of its automorphism group.

Proof. The implications (i)⇒(ii), (i)⇒(iii), (i) ⇒(iv), and (ii)⇒(iv) are immediate. We will show (iii)⇒(i) and (iv)⇒(i), which together with the immediate implications prove all equivalences. To this end, we first show that the table is complete and correct as stated. From Propositions 5.20 and 5.21 we know that the maximal size of all cliques in $\Gamma_{-1,0}$ is twelve, that there are 179200 cliques of size twelve, and that these cliques form one orbit under the action of W, proving the equivalences for K_1, K_2 of size at least 12. The clique of size 12 in the table is the clique T that is defined above Lemma 5.15. The size of its stabilizer in W is $\frac{|W|}{179200} = 3888$. From the description of T we see that its automorphism group is isomorphic to the semidirect product $S_3^4 \rtimes S_4$, where S_4 works on S_3^4 by permuting the four coordinates. This group has order $6^4 \cdot 24 = 31104$.
To find maximal cliques in $\Gamma_{-1,0}$ of size smaller than 12, note that there are no maximal cliques in $\Gamma_{-1,0}$ of size 11 by Proposition 5.20, so we only have to look at the cliques of size at most ten. To make computations easier, we first show that every maximal clique in $\Gamma_{-1,0}$ contains at least one edge of color 0. We know that the only maximal cliques in Γ_{-1} are the cliques of size three. Set $e_1 = (1,1,0,0,0,0,0,0,0,0)$, $e_2 = (-1,0,1,0,0,0,0,0)$, and $e_3 = (0,-1,-1,0,0,0,0,0)$, then $\{e_1, e_2, e_3\}$ is a maximal clique in Γ_{-1}. Note that for $e_4 = (0,0,0,0,0,0,1,1)$, the set $\{e_1, e_2, e_3, e_4\}$ forms a clique in $\Gamma_{-1,0}$, hence $\{e_1, e_2, e_3\}$ is not a maximal clique in $\Gamma_{-1,0}$. Since W acts transitively on the set of maximal cliques in Γ_{-1} (Corollary 3.9), it follows that all maximal cliques in Γ_{-1} are not maximal in $\Gamma_{-1,0}$. Thus we can assume that the maximal cliques in $\Gamma_{-1,0}$ contain at least one pair of orthogonal roots. Fix the roots $c_1 = (1,1,0,0,0,0,0,0)$, $c_2 = (0,0,1,1,0,0,0,0)$. Since W acts transitively on the pairs of orthogonal roots, every maximal clique in

$\Gamma_{-1,0}$ is conjugate to a clique containing c_1, c_2, so by considering only the maximal cliques in $\Gamma_{-1,0}$ that contain c_1 and c_2, we find representatives for all orbits of the maximal cliques in $\Gamma_{-1,0}$ under the action of W. This reduces computations, since there are only 136 roots that have dot product -1 or 0 with both c_1 and c_2, which is quickly computed with `magma`, as well as the number of maximal cliques containing c_1, c_2. We find the following.

r	Number of maximal cliques of size r in $\Gamma_{-1,0}$ containing c_1 and c_2
≤ 7	0
8	261600
9	2779392
10	228408

We now turn to the table in the appendix. One can easily check with `magma` that the sets in the table for $\Gamma_{-1,0}$ are indeed maximal cliques in $\Gamma_{-1,0}$; in Remark 5.26. For each of these cliques we compute the automorphism groups with `magma`. We see that apart from the cliques

$$L_1 = \{19, 41, 48, 50, 65, 150, 172, 214, 240\}$$

and

$$L_2 = \{41, 48, 50, 55, 65, 78, 178, 214, 240\}$$

of size 9, which both have an automorphism group of size 16, every two cliques of the same size in the table have a different automorphism group. One can check that L_2 contains a subclique with only edges of color zero of size 7, and L_1 does not, so L_1 and L_2 are not isomorphic. This shows that any two cliques of the same size in the table are not isomorphic, and therefore not conjugate.
We claim that every maximal clique in $\Gamma_{-1,0}$ is conjugate to one of these cliques in the table. To this end, set $A = \{c_1, c_2\}$, and let W_A be the stabilizer of A in W. From Proposition 4.1 it follows that $|W_A| = \frac{|W|}{|WA|} = \frac{|W|}{15120} = 46080$. We now show how to proceed for the cliques of size 8, the proof for sizes 9 and 10 goes completely analogous. For each of the five cliques of size 8 in the table we compute the size of its stabilizer (144,128,16,14, and 8) and the number of conjugates of A contained in it (21,20,20,21, and 21, respectively), with `magma`. Lemma 5.24 now gives us the number of conjugates of each clique that contain A. This sums up to the number 261600 we find in the table above, proving our claim.
We have showed that the table in the appendix gives exactly one representative for each orbit of the maximal cliques in $\Gamma_{-1,0}$, so K_1 and K_2 are both conjugate to an element in the table. If either (iii) or (iv) holds, then by looking at the table we see that this implies that K_1 and K_2 are conjugate to the same clique in the table, and in particular, they are conjugate to each other, implying (i). This finishes the proof. □

Remark 5.26. In the proof of Proposition 5.25 we found 261600 cliques of size 8 in $\Gamma_{-1,0}$ containig $c_1 = (1, 1, 0, 0, 0, 0, 0, 0)$ and $c_2 = (0, 0, 1, 1, 0, 0, 0, 0)$. One can check for any two of them whether they are conjugate with `magma`, but this takes a very long time. To reduce computations, we first sort the cliques by size of their stabilizer. We then go through each set of cliques with the same stabilizer size by taking one clique, and removing all cliques that are conjugate to it from the set.

5.4. Maximal cliques of other colors

In this subsection we prove Theorem 1.1 and 1.2 for all maximal cliques in Γ_c with $c \in \{\{-1,1\},\{-2,-1,1\},\{0,1\},\{-2,-1,0\},\{-2,0,1\}\}$. We make use of `magma` in all cases. The following lemma deals with the cases for which this is straightforward.

Lemma 5.27. *For $c \in \{\{-1,1\},\{-2,-1,1\}\}$, and for two maximal cliques K_1 and K_2 of the same size in Γ_c, the following are equivalent.*

(i) K_1 and K_2 are conjugate under the action of W.

(ii) K_1 and K_2 are isomorphic.

(iii) K_1 and K_2 have the same stabilizer size.

(iv) The automorphism groups of K_1 and K_2 have the same cardinality.

Moreover, for $c \in \{\{-1,1\},\{-2,-1,1\}\}$, the table in Appendix A gives a complete list of representatives of the orbits of maximal cliques in Γ_c, as well as for each representative its stabilizer size and the size of its automorphism group.

Proof. In these two graphs there are not so many maximal cliques, and we can ask `magma` to compute them, compute the orbits under the action of W, and a representative of each orbit directly. The results are in the table. The size of the stabilizers is found by dividing the order of W by the size of the orbit. The automorphism group of the cliques is also easily found with `magma`. Since cliques of the same size in the table have automorphism groups of different size, they are not isomorphic. The equivalence of the statements (i), (ii), (iii), and (iv) now follows from the table. □

Corollary 5.28. *For $c \in \{\{-1,1\},\{-2,-1,1\}\}$, let K_1 and K_2 be two maximal cliques in Γ_c, and $f\colon K_1 \longrightarrow K_2$ an isomorphism between them. Then f extends to an automorphism of Λ.*

Proof. Since K_1 and K_2 are isomorphic, from Lemma 5.27 it follows that they are both conjugate to the same clique in the table in de appendix; call this clique H. Then there are elements α, β in W such that $\alpha(K_1) = \beta(K_2) = H$. So $\beta \circ f \circ \alpha^{-1}$ is an element in the automorphism group $\mathrm{Aut}(H)$ of H. Of course, f extends to an element in W if and only if $\beta \circ f \circ \alpha^{-1}$ does. We conclude that we can reduce to the case where $K_1 = K_2 = H$, and f is an element in $\mathrm{Aut}(H)$.
For each clique H in the table, we construct in `magma` the map $W_H \longrightarrow \mathrm{Aut}(H)$ from the stabilizer W_H to the automorphism group $\mathrm{Aut}(H)$ given by restriction. For all these cliques, this is a surjective map. It follows that every element in $\mathrm{Aut}(H)$ extends to an element in W. □

The final three cases are much more work, because of the large numbers of maximal cliques and their sizes. The most extreme case is that of maximal cliques of size 29 in $\Gamma_{0,1}$ and $\Gamma_{-2,0,1}$; we treat this separately in Section 5.4.1.

Proposition 5.29. *For two maximal cliques K_1 and K_2 of the same size in $\Gamma_{-2,-1,0}$, the following are equivalent.*

(i) K_1 and K_2 are conjugate under the action of W.

(ii) K_1 and K_2 are isomorphic.

(iii) K_1 and K_2 have the same stabilizer size, and, if the stabilizer size is 32 and K_1 and K_2 have size 10, then K_1 and K_2 both contain a pair of inverse roots, or they both do not.

(iv) The automorphism groups of K_1 and K_2 have the same cardinality, and, if this cardinality is 80 and K_1 and K_2 have size 9, or this cardinality is 64 and K_1 and K_2 have size 10, then K_1 and K_2 both contain a pair of inverse roots, or they both do not.

(v) K_1 and K_2 have the same stabilizer size and their automorphism groups have the same cardinality.

Moreover, the table in Appendix A gives a complete list of representatives of the orbits of maximal cliques in $\Gamma_{-2,-1,0}$, as well as for each representative its stabilizer size and the size of its automorphism group.

Proof. This proof follows the same steps as the proof of Proposition 5.25. See also Remark 5.26 on how we found the representatives of each orbit that are written in the table.
Cliques in $\Gamma_{-2,-1,0}$ without an edge of color 0 are monochromatic and not maximal in $\Gamma_{-2,-1,0}$ (follows from the results in $\Gamma_{-2,-1}, \Gamma_{-2,0}, \Gamma_{-1,0}$). Therefore, to find the maximal cliques in $\Gamma_{-2,-1,0}$, we only consider cliques that contain two orthogonal roots, and we can choose these arbitrarily since W acts transitively on the set of pairs of orthogonal roots. Define the following roots.

$$e_1 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right), \; e_2 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right).$$

We find the following.

r	Number of maximal cliques of size r in $\Gamma_{-2,-1,0}$ containing e_1 and e_2
≤ 7	0
8	192480
9	1961088
10	743536
11	111680
12	8290
13	2100
14-15	0
16	15
≥ 17	0

We turn to the table in the appendix. One can check that all the sets in the table for $\Gamma_{-2,-1,0}$ are indeed maximal cliques in $\Gamma_{-2,-1,0}$. For each of these cliques we compute the automorphism group with `magma`. As one can see in the table, except from two cliques

$$L_1 = \{1, 8, 26, 47, 51, 86, 121, 128, 228\}, \quad L_2 = \{1, 8, 26, 47, 51, 86, 124, 125, 228\}$$

of size 9 that both have an automorphism group of size 80, and two cliques

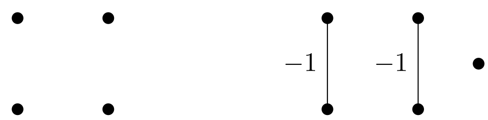
$$M_1 = \{1, 8, 26, 31, 43, 46, 84, 98, 103, 125\},$$
$$M_2 = \{1, 8, 26, 31, 43, 46, 84, 101, 226, 238\}$$

of size 10 that both have an automorphism group of size 64, any two cliques of the same size have different automorphism groups and are therefore not isomorphic. Moreover, L_1 contains the roots 1 and 128, which are each other's inverse, whereas L_2 contains no pairs of inverse roots. And M_1 contains the roots 26 and 103, which are each other's inverse, and M_2 contains no pairs of inverse roots. So also L_1, L_2, M_1 and M_2 are pairwise not isomorphic. We conclude that any two of the cliques in the table are not isomorphic, hence not conjugate.

For each size r in the table above, as we do in the proof of Proposition 5.25, we compute with Lemma 5.24 and `magma` the number of maximal cliques of size r containing e_1 and e_2 that are conjugate to one of the cliques in the table in the appendix. This gives exactly the number of maximal cliques of size r containing e_1 and e_2 in the table above. So every maximal clique in $\Gamma_{-2,0,1}$ containing e_1 and e_2 is conjugate to a clique in the table in the appendix, hence the same holds for every maximal clique in $\Gamma_{-2,0,1}$. We conclude that the table in the appendix gives a unique representative for each orbit of the set of maximal cliques under the action of W. Finally, for each clique in the table, we compute the size of its stabilizer in W. We see that except for $N_1 = \{1, 8, 26, 31, 43, 86, 106, 115, 224, 234\}$ and $N_2 = \{1, 8, 26, 31, 43, 46, 84, 101, 226, 238\}$, two cliques of the same size in the table have different stabilizer sizes. In N_1, we have roots 43 and 86, and these are each other's inverse; in N_2, there are no two roots that are each other's inverse. Finally, N_1 and N_2 have different automorphism groups.

The equivalence of statements (i) - (v) follows in a similar way as in the proof of Proposition 5.25. The implications (i)⇒(ii), (i)⇒(iii), (i)⇒(iv), (i)⇒(v) and (ii)⇒(iv) are immediate. Since both K_1 and K_2 are conjugate to one of the cliques in the table, if any of (iii) - (v) are true, by looking at the table we see that this implies that K_1 and K_2 are conjugate to the same clique in the table, and in particular, they are conjugate to each other, implying (i). This proves that all 5 statements are equivalent. □

We can now prove Theorem 1.2 for maximal cliques in $\Gamma_{-1,0}$ and $\Gamma_{-2,-1,0}$; the statement is the same for these two graphs. Recall the following graphs that are defined in the introduction, where any two disjoint vertices have an edge of color 0 between them.



A $\qquad\qquad\qquad$ C_{-1}

Lemma 5.30. *Let K_1 and K_2 be two maximal cliques, both in $\Gamma_{-1,0}$ or both in $\Gamma_{-2,-1,0}$, and let $f\colon K_1 \longrightarrow K_2$ be an isomorphism between them. The following hold.*

(i) The map f extends to an automorphism of Λ if and only if for every ordered sequence $S = (e_1, \ldots, e_r)$ of distinct roots in K_1 such that the colored graph on them is isomorphic to A or C_{-1}, its image $f(S) = (f(e_1), \ldots, f(e_r))$ is conjugate to S under the action of W;

(ii) If $S = (e_1, \ldots, e_5)$ is a sequence of distinct roots in K_1 such that the colored graph on them is isomorphic to C_{-1} with $e_1 \cdot e_4 = e_2 \cdot e_5 = -1$, then S and $f(S)$

are conjugate under the action of W *if and only if both* $e = e_1 + e_2 + e_3 - e_4 - e_5$ *and* $f(e)$ *are in the set* $\{2f_1 + f_2 \mid f_1, f_2 \in E\}$*, or neither are.*

Proof. Since K_1 and K_2 are isomorphic, from Propositions 5.25, and 5.29 it follows that they are both conjugate to the same clique in the table in de appendix; call this clique H. Then there are elements α, β in W such that $\alpha(K_1) = \beta(K_2) = H$, so $\beta \circ f \circ \alpha^{-1}$ is an element in the automorphism group $\mathrm{Aut}(H)$ of H. Of course, f extends to an element in W if and only if $\beta \circ f \circ \alpha^{-1}$ does. Moreover, for every sequence S as in the statement, $f(S)$ and $(\beta \circ f \circ \alpha^{-1})(S)$ are conjugate. We conclude that we can reduce to the case where $K_1 = K_2 = H$, and f is an element in $\mathrm{Aut}(H)$. Let $g \colon W_H \longrightarrow \mathrm{Aut}(H)$ be the map from the stabilizer of H to the automorphism group that restricts elements in W_H to H, and T_H a set of representatives of the classes in the cokernel of g. Since f is a composition of (restrictions of) elements in W_H with an element in T_H, we can reduce further to the case where f is an element in T_H.

For each of the 56 cliques H in the table at $\Gamma_{-1,0}$ and $\Gamma_{-2,-1,0}$, we compute the map $g \colon W_H \longrightarrow \mathrm{Aut}(H)$ with `magma`. In all cases, this map is injective. This means that for all cliques with $|W_H| = |\mathrm{Aut}(H)|$, every element in the automorphism group of H extends to a unique automorphism of Λ. We see in the list that this holds for the first five cliques and the 11^{th}, 12^{th}, 15^{th}, and 16^{th} clique in $\Gamma_{-1,0}$, and the first five cliques and the 8^{th}, 10^{th}, 11^{th}, 13^{th}, 17^{th}, 20^{th}, 23^{rd}, and 24^{th} clique in $\Gamma_{-2,-1,0}$.

For each clique H of the remaining 34 cliques, we compute the following with the function `CokernelClassesTypeCminus1` ([Win]). First, we create a set T_H of representatives of the classes of the cokernel of the map from W_H to $\mathrm{Aut}(H)$. We then check for each t in T_H, and for all sequences $S = (e_1, e_2, e_3, e_4, e_5)$ of distinct roots in H such that the colored graph on S is isomorphic to C_{-1} with $e_1 \cdot e_4 = e_2 \cdot e_5 = -1$, whether S and $t(S)$ are not conjugate. For all t and S for which this is the case, we verify that either $e = e_1 + e_2 + e_3 - e_4 - e_5$ is in the set $F = \{2f_1 + f_2 \mid f_1, f_2 \in E\}$ and $t(e)$ is not, or vice versa. This proves part (ii).

For H equal to the $7^{\text{th}} - 10^{\text{th}}$,13^{th} ,14^{th}, and $18^{\text{th}} - 23^{\text{rd}}$ clique in $\Gamma_{-1,0}$ and the 7^{th}, 9^{th}, 12^{th}, 14^{th}, 16^{th} ,18^{th} ,19^{th} ,21^{st} ,22^{nd} ,$25^{\text{th}} - 29^{\text{th}}$, and 31^{st} clique in $\Gamma_{-2,-1,0}$, the check we just described gives us for all t in T_H a sequence S with distinct roots in H and graph isomorphic to C_{-1}, such that S and $t(S)$ are not conjugate. For the remaining 7 cliques in the table, we do an almost analogous check with the function `CokernelClassesTypeA` in `magma` ([Win]), where S is now a clique whose graph is isomorphic to A. For all 7 cliques H, for all elements in T_H, there exists such an S with S not conjugate to $t(S)$. This finishes the proof of (i). ☐

Proposition 5.31. *For* $c \in \{\{0,1\}, \{-2,0,1\}\}$*, and* K_1*,* K_2 *two maximal cliques of the same size* $r \neq 29$ *in* Γ_c*, the following are equivalent.*

(i) K_1 *and* K_2 *are conjugent under the action of* W*.*

(ii) K_1 *and* K_2 *are isomorphic.*

(iii) K_1 *and* K_2 *have the same stabilizer size, and they contain the same number of pairs of orthogonal roots.*

(iv) The automorphism groups of K_1 and K_2 have the same cardinality, and K_1 and K_2 contain the same number of pairs of orthogonal roots.

Moreover, the table in Appendix A gives a complete list of representatives of the orbits of maximal cliques in Γ_c, as well as for each representative its stabilizer size and the size of its automorphism group.

Proof. We show that the table is correct and complete for each c. The steps in the proof are the same as in the proofs of Propositions 5.25 and 5.29, and the equivalence of statements (i) - (iv) follows in the same way as in these propositions. See also Remark 5.26 on how we found the representatives of each orbit that are written in the table.
• $c = \{0, 1\}$
We know that the maximal cliques in Γ_1 form two orbits; one with cliques of size 7 and one with the cliques of size 8 (Proposition 4.7). Note that the clique of size 7 in Γ_1 in the table is contained in the clique of size 22 in $\Gamma_{0,1}$, and the clique of size 8 in Γ_1 is contained in the clique of size 33 in $\Gamma_{0,1}$. This means that there are no maximal cliques with only edges of color 1 in $\Gamma_{0,1}$. We fix two orthogonal roots $e_1 = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$, $e_2 = (-1, 0, 0, 0, -1, 0, 0, 0)$. With `magma` we compute that there are only 136 roots that have dot product 0 or 1 with e_1 and e_2, and we find the following.

r	Number of maximal cliques of size r in $\Gamma_{0,1}$ containing e_1 and e_2
≤ 21	0
22	3120
23-27	0
28	21120
30	16263276
31	2792800
32	655680
33	105120
34	18800
35	0
36	304
≥ 37	0

For each set K in the table in Appendix A, one can check that it is indeed a maximal clique in $\Gamma_{0,1}$. We compute the automorphism groups of all cliques. As we see in the table, for all sizes except 30, two cliques of the same size have a different automorphism group, so they are not isomorphic, hence not conjugate. For size 30, all cliques whose automorphism groups have the same cardinality have a different number of pairs of orthogonal roots that they contain; for example, the cliques of size 30 with stabilizer size 48 contain (in order of appearence in the table) 171, 179, 180, 183, 198 subsets of two orthogonal roots. This shows that no two cliques in the table are isomorphic, hence not conjugate. Moreover, using the stabilizer size and the number of subsets of orthogonal roots of each clique K in the list, we can find the number of conjugates of K that contain $\{e_1, e_2\}$ with Lemma 5.24. Adding all these numbers up we recover the numbers in the table above, which shows that every maximal clique in $\Gamma_{-1,0}$ of size unequal to 29 is conjugate to one of the cliques in the list. We conclude that the table in Appendix A is complete. Finally, we see that for each clique in the table, the stabilizer size and the cardinality of the automorphism group is the same. Therefore, by what we showed above, different cliques of the same size and with the same stabilizer size in the table have a different number of subsets of two orthogonal roots.

• $c = \{-2, 0, 1\}$

We start with cliques in $\Gamma_{-2,0,1}$ containing an edge of color -2. We fix a root e and compute the maximal cliques in $\Gamma_{-2,0,1}$ containing e and $-e$. We find the following.

r	Number of maximal cliques of size r in $\Gamma_{-2,0,1}$ containing e and $-e$
≤ 12	0
13	370440
14	250236
15	0
16	77895
17-18	0
19	7019208
20	861840
21	120960
22	44352
23	0
24	4032
25-28	0
≥ 30	0

Since there are no maximal cliques of size bigger than 29 containing an edge of color -2, we conclude that all the maximal cliques in $\Gamma_{0,1}$ of size at least 29 are also maximal cliques in $\Gamma_{-2,0,1}$. This leaves us with the maximal cliques in $\Gamma_{0,1}$ of size 22 and 28. Looking at the table in the appendix, we see that for both sizes there is only one orbit, and it is an easy check that for the listed representatives L_{22} of size 22 and L_{28} of size 28 of both these orbits, there are no roots that can be added to extend the clique in $\Gamma_{-2,0,1}$. Therefore L_{22} and L_{28} are still maximal in $\Gamma_{-2,0,1}$. We now turn to the cliques in $\Gamma_{-2,0,1}$ in the table. First of all, one can check easily with `magma` that these are indeed maximal cliques in $\Gamma_{-2,0,1}$. For K_1 and K_2 of size 28 or ≥ 30, everything is exactly the same as for $\Gamma_{0,1}$, and we showed that the proposition holds in these cases. For the other cliques, we see that for all sizes except 13, 19, and 20, two different cliques of the same size have different automorphism groups. For sizes 13, 19, and 20, we compute, completely analogously to what we did for $c = \{0, 1\}$, that the number of subsets of two orthogonal roots in two different cliques whose automorphism groups have the same cardinality is different. For example, the cliques of size 19 whose automorphism group has size 96, contain (in order of appearence in the table) 91, 95, 94, 98, 103 subsets of two orthogonal roots. This proves that all the cliques in the table are pairwise not isomorphic, hence not conjugate. Again using Lemma 5.24, we can check that every maximal clique in $\Gamma_{-2,0,1}$ that is conjugate to one of the cliques in the table, showing that the table is complete. Finally, except for the cliques

$$L_1 = \{1, 8, 12, 14, 15, 20, 22, 23, 36, 38, 39, 128, 136, 137, 138, 139, 149, 160, 169\}$$

and

$$L_2 = \{1, 8, 12, 14, 50, 68, 70, 74, 128, 136, 137, 154, 169, 170, 176, 177, 181, 182, 215\}$$

of size 19, any two different cliques of the same size that have the same stabilizer size have the same cardinality of their automorphism groups as well. We already showed that this means that they contain a different number of pairs of orthogonal roots. We compute that L_1 contains 109 such pairs, and L_2 contains 79. Therefore we can conclude that different cliques of the same size and with the same stabilizer size in the table have a different number of subsets of two orthogonal roots. □

5.4.1. **Cliques of size 29 in $\Gamma_{0,1}$ and $\Gamma_{-2,0,1}$**

Cliques of size 29 in $\Gamma_{0,1}$

The graph $\Gamma_{0,1}$ contains a surprisingly large number of maximal cliques of size 29, so we will treat this case separately in this section. As before, we say that the stabiliser size of an orbit is the size of the stabiliser of any of the elements in the orbit (Definition 5.23).

Proposition 5.32. *In the graph $\Gamma_{0,1}$ there are 62825152320 maximal cliques of size 29. They form 432 orbits under the automorphism group W. The multiset of their stabiliser sizes is*

$$\{1^{(8)}, 2^{(81)}, 4^{(107)}, 6^{(5)}, 8^{(50)}, 10, 12^{(41)}, 14^{(2)}, 16^{(28)}, 18^{(2)}, 20^{(5)}, 24^{(28)}, 32^{(4)}, 36, 48^{(21)}, 60, 64^{(2)}, 72^{(7)}, 96^{(3)}, 120, 128^{(2)}, 144^{(4)}, 192^{(7)}, 240^{(6)}, 360, 384^{(3)}, 432^{(2)}, 720^{(2)}, 1152^{(2)}, 1440, 1920, 40320, 51840, 103680\},$$

where the superscripts indicate the multiplicity of the elements in the multiset. For two maximal cliques K_1 and K_2 of size 29 in $\Gamma_{0,1}$, the following are equivalent.

(i) K_1 and K_2 are conjugate under the action of W.

(ii) K_1 and K_2 are isomorphic.

(iii) K_1 and K_2 have the same stabilizer size, and the same number of maximal monochromatic subcliques of color 1 of size r, for all $r \in \{1, \dots, 8\}$.

(iv) The automorphism groups of K_1 and K_2 have the same cardinality, and K_1 and K_2 have the same number of maximal monochromatic subcliques of color 1 of size r, for all $r \in \{1, \dots, 8\}$.

Moreover, the table in Appendix B gives a complete list of representatives of the orbits of maximal cliques of size 29 in $\Gamma_{0,1}$.

The number of cliques mentioned in Proposition 5.32 is too large to fit in most computers' memory: even if we were to use only 30 bytes per clique to store the vertices in the clique, then all cliques together would still require close to two terrabytes of storage. Instead of doing this, we will use the fact that each 29-clique contains a monochromatic 5-clique of color 0 or a monochromatic 4-clique of color 1.

Proof. The Ramsey number $R(4, 5)$ equals 25 (Theorem 5.18). This implies that a 29-clique in $\Gamma_{0,1}$ contains a 5-clique of edges of color 1 or a 4-clique of edges of

color 0. Under the action of the automorphism group W there is only one orbit of 5-cliques with only edges of color 1 (see Proposition 2.12; we call these cliques of type $K_5(1)$), and there are two orbits of 4-cliques with pairwise orthogonal roots (See Proposition 4.1; we call the 4-cliques of which the sum is a double root of type $K_4^a(0)$ and those of which the sum is not a double of type $K_4^b(0)$). Therefore, if we fix a representative clique for each of these three orbits, then each 29-clique is conjugate to a 29-clique that contains one of our three cliques of size 4 or 5.
We pick the clique $A = \{1, 2, 129, 130, 131\}$ of type $K_5(1)$. There are 109 other vertices that are connected with color 0 or 1 to each of the 5 vertices of A. With `magma`, we count that the graph on these 109 vertices with only edges of color 0 or 1 has exactly $n_1 = 127168449$ maximal cliques of size 24. After adding to each the vertices of A, this yields n_1 maximal 29-cliques that contain A in the graph $\Gamma_{0,1}$. Similarly, for the cliques $B_1 = \{1, 8, 26, 31\}$ and $B_2 = \{1, 8, 26, 43\}$ of type $K_4^a(0)$ and $K_4^b(0)$, respectively, we count with `magma` that there are $n_2 = 16685128$ maximal 29-cliques in $\Gamma_{0,1}$ that contain B_1, and $n_3 = 504$ maximal 29-cliques that contain B_2.
One can easily verify with `magma` that the 432 cliques of size 29 in the table in Appendix B are maximal cliques in $\Gamma_{0,1}$. For each clique K of size 29, for each integer $1 \leq r \leq 8$, we can consider the number χ_r of maximal monochromatic subcliques of K of color 1 of size r. These eight invariants together pin down 430 out of the 432 cliques in the table. Only the sequence $(\chi_1, \chi_2, \ldots, \chi_8) = (0, 0, 0, 0, 0, 4, 138, 17)$ occurs twice: for the 67-th and 299-th cliques in the table. These two cliques have 16 and 18 subcliques of type $K_4^a(0)$, respectively, so they are not isomorphic. We conclude that any two cliques in the table are not isomorphic, hence not conjugate. So there are at least 432 orbits of maximal 29-cliques. We know that there are 483840 cliques of size 5 in Γ_1 from Corollary 2.7, so the stabiliser of A has size $\frac{|W|}{|WA|} = \frac{|W|}{483840} = 1440$. The table also lists for each clique c the number of subcliques of type $K_5(1)$, as well as the stabiliser size, so we can use Lemma 5.24 to calculate the number of conjugates of c that contain A. Summing over all these 432 cliques, we obtain exactly the number n_1, so we conclude that all n_1 maximal 29-cliques in $\Gamma_{0,1}$ that contain A are accounted for in these 432 orbits. Similarly, the stabilisers of B_1 and B_2 have sizes 4608 and 384, respectively. The table lists the number of subcliques of type $K_4^a(0)$ and $K_4^b(0)$ for every given clique c, so we can use Lemma 5.24 again to calculate the number of conjugates of c that contain B_i for $i = 1, 2$. Summing over all 432 cliques, we find again that all maximal 29-cliques containing B_1 or B_2 are accounted for in these 432 orbits.
We conclude that there are 432 orbits of 29-cliques in $\Gamma_{0,1}$, as claimed, and since no two cliques in the table are isomorphic, this proves (i) $\Leftrightarrow$ (ii). The multiset of stabiliser sizes follows from the table. The length of the orbit of any clique c is $\frac{|W|}{|W_c|}$. Summing over all 432 cliques in the table, we find that the total number of 29-cliques is also as claimed. Finally, as we saw before, the invariant χ_r is different for all cliques except for the 67-th and 299-th cliques in the table. These two cliques have stabiliser size 4 and 8, respectively, so the stabiliser size, together with the χ_r form a set of invariants that uniquely determine each of the 432 orbits of maximal 29-cliques. This proves (i) $\Leftrightarrow$ (iii). The stabiliser of a clique maps to the automorphism group of this clique as a colored graph. In all 432 cases, the clique generates a full rank sublattice of our lattice, so this map is injective. It turns out that in all cases, it is in fact a bijection. This proves (iii) $\Leftrightarrow$ (iv). □

Corollary 5.33. *Let K_1 and K_2 be two maximal cliques in $\Gamma_{0,1}$, and $f\colon K_1 \longrightarrow K_2$ an isomorphism between them. Then f extends to a unique automorphism of Λ.*

Proof. Since K_1 and K_2 are isomorphic, from Propositions 5.31 and 5.32 it follows that they are conjugate to each other; this means that they are both conjugate to the same clique in the tables in de appendix; call this clique H. Then there are elements α, β in W such that $\alpha(K_1) = \beta(K_2) = H$, so $\beta \circ f \circ \alpha^{-1}$ is an element in the automorphism group $\mathrm{Aut}(H)$ of H. Of course, f extends to an element in W if and only if $\beta \circ f \circ \alpha^{-1}$ does. We conclude that we can reduce to the case where $K_1 = K_2 = H$, and f is an element in $\mathrm{Aut}(H)$.
In Proposition 5.31 we computed the stabilizers and automorphism groups of all cliques in $\Gamma_{0,1}$ of size unequal to 29, and in Proposition 5.32 we did the same for cliques of 29. In `magma` we construct for each clique in the table the map between the stabilizer and the automorphism group that is given by restriction. In all cases, this is an isomorphism. We conclude that all automorphisms of the cliques in the table extend to an element in W. □

The table in Appendix B contains the results of the previous proposition, with a representative of each orbit. The notation in the table means the following.

- K: a clique in $\Gamma_{0,1}$; we denote vertices by their index as in the notation above Remark 5.1.
- $|W_K|$: the size of the stabilizer of clique K in the group W.
- $\#K_5(1)$: the number of cliques of size 5 with only edges of color 0 in K.
- $\#K_4^a(1)$: the number of cliques in K of four roots that sum up to a double root in Λ, with only edges of color 1.
- $\#K_4^b(1)$: the number of cliques in K of four roots that do not sum up to a double root in Λ, with only edges of color 1.

Remark 5.34. In the proof of Proposition 5.32, we found more than 127 million cliques of size 29 that contain $A = \{1, 2, 129, 130, 131\}$. To find that they represent exactly 432 different orbits, one might naively try to just verify for each pair whether they are conjugate. This takes too much time; as described in Remark 5.26, we divided the big set into smaller sets according to the stabilizer sizes.

Cliques of size 29 in $\Gamma_{-2,0,1}$

It is an easy check that all 432 cliques of size 29 in $\Gamma_{0,1}$ in the table are maximal in $\Gamma_{-2,0,1}$ as well. We conclude that the orbits of maximal cliques of size 29 in $\Gamma_{-2,0,1}$ are exactly the 432 that we found in $\Gamma_{0,1}$, and the orbits of maximal cliques of size 29 that contain an edge of color -2.
As we did in Proposition 5.31, we fix a root e and compute all maximal cliques of size 29 in $\Gamma_{-2,0,1}$ that contain e and $-e$ with `magma`. There are 56 of these, and they form one orbit under the action of the stabilizer W_e of e. Since W acts transitively on pairs of inverse roots, we conclude that all maximal cliques of size 29 in $\Gamma_{-2,0,1}$ that contain an edge of color -2 are in the same orbit; call this orbit A. One can easily check with `magma` that the clique of size 29 that is written in the table for $\Gamma_{-2,0,1}$ is maximal, and moreover, it contains the roots 1 and 128, that are each other's inverse. We conclude that it is a representative of A. The stabilizer and automorphism group are computed with `magma`.

We finish with the proof of Theorem 1.2 for maximal cliques in $\Gamma_{-2,0,1}$. This is very similar to the proof of Lemma 5.30. Recall the graphs A, C_1, D, and F as defined before Theorem 1.2.

Lemma 5.35. *Let K_1 and K_2 be two maximal cliques in $\Gamma_{-2,0,1}$, and $f\colon K_1 \longrightarrow K_2$ an isomorphism between them. The following hold.*

(i) The map f extends to an automorphism of Λ if and only if for every subclique $S = \{e_1, \dots, e_r\}$ of K_1 that is isomorphic to A, C_1, D, or F, its image $f(S)$ in K_2 is conjugate to S under the action of W.

Let S be a subclique of K_1.

(ii) If S is isomorphic to C_1, then f and $f(S)$ are conjugate if and only if both $\sum_{i=1}^{5} e_i$ and $\sum_{i=1}^{5} f(e_i)$ are in the set $\{2f_1 + f_2 \mid f_1, f_2 \in E\}$, or neither are.

(iii) If S is isomorphic to D, then f and $f(S)$ are conjugate if and only if both $\sum_{i=1}^{5} e_i$ and $\sum_{i=1}^{5} f(e_i)$ are in the set $\{2f_1 + 2f_2 \mid f_1, f_2 \in E\}$, or neither are.

(iv) If S is isomorphic to F, then f and $f(S)$ are conjugate if and only if both $\sum_{i=1}^{5} e_i$ and $\sum_{i=1}^{6} f(e_i)$ are in 2Λ, or neither are.

Proof. This proof is very similar to the proof of Lemma 5.30, so will sketch what we did and refer to the other proof for details.

We reduce again to the case $K_1 = K_2 = H$, with H one of the 54 cliques in the list for $\Gamma_{-2,0,1}$ in the appendix, and f a representative of a class of the cokernel of the map $g\colon W_H \longrightarrow \mathrm{Aut}(H)$, where W_H is the stabilizer of H in W, and $\mathrm{Aut}(H)$ is the automorphism group of H.

For each clique H of those 54 in the table, we check with `magma` that the map $g\colon W_H \longrightarrow \mathrm{Aut}_H$ is injective; for the 13^{th}, 15^{th}, and $17^{\text{th}} - 54^{\text{th}}$ clique it is an isomorphism. It follows that for those cliques, every automorphism extends to an element in W, so we are done. Here we refer to Lemma 5.33 for the cliques that are the same as in $\Gamma_{0,1}$.

For each clique H of the remaining 14 cliques in the list, we do the following in `magma` with the functions `CokernelClassesTypeF`, `CokernelClassesTypeD`, and `CokernelClassesTypeC1` ([Win]). We construct a set T_H of representatives of the classes of the cokernel of the map from W_H to $\mathrm{Aut}(H)$. We then check for each t in T_H, and for all subcliqes $S = \{e_1, \dots, e_r\}$ of H that are isomorphic to F (or D, or C_1, respectively), whether S and $t(S)$ are not conjugate. For all t and S for which this is the case, we verify that $\sum_{i=1}^{r} e_i$ is in 2Λ (or in the set $\{2f_1 + 2f_2 \mid f_1, f_2 \in E\}$, or in the set $\{2f_1 + f_2 \mid f_1, f_2 \in E\}$, respectively), and $\sum_{i=1}^{r} t(e_i)$ is not, or vice versa. This proves (ii), (iii), and (iv).

For H equal to the 4^{th}, 8^{th}, 10^{th}, 14^{th}, and 16^{th} clique, for each non-trivial element t in T_H there is a subclique S of H that is isomorphic to F, and such that t and $t(S)$ are not conjugate. Similarly, for each clique H of the remaining 9 cliques in the list, for each non-trivial element t of T_H, there is a subclique S of H that is isomorphic to either C_1, D, or A, and such that S and $t(S)$ are not conjugate. This finishes the proof of (i). □

6. Proof of main theorems

We now put together all the results that form the proofs of Theorem 1.1 and Theorem 1.2, which are both stated in the Introduction.

Proof of Theorem 1.1. Part (i) is Proposition 4.1 (iii), and part (ii) is Proposition 4.7 (ii). We proceed with (iii). Of course, if K_1 and K_2 are conjugate under the action of W, they are isomorphic as colored graphs, since W respects the dot product. Now assume that K_1 and K_2 are isomorphic as colored graphs. We will show that they are conjugate under the action of W. First of all, by Lemma 2.13, we can assume that there is a type I, II, III, or IV, that both K_1 and K_2 belong to. Therefore we continue to prove the result per type.
For type I, the results for colors -2 and -1 are at the beginning of Section 4; the results for color 0 are in Propositions 4.1, 4.6 (iii), and 5.12, and the results for color 1 are in Proposition 2.12.
For type II, from Proposition 2.5 we know what the cliques look like, and the results are then in Proposition 2.12 and Corollary 3.16.
For type III, the results follow from Propositions 3.1 and 3.2.
Finally, for type IV, the results follow from Propositions 5.3, 5.25, Lemma 5.27, Propositions 5.29and 5.31, and Section 5.4.1.

Proof of Theorem 1.2. By Lemma 2.13, we can assume that there is a type I, II, III, or IV, that both K_1 and K_2 belong to. Therefore we continue to prove (i) per type. First of all, if K_1 and K_2 are of type III, then f always extends; this is shown in Corollary 3.33.
If K_1 and K_2 are of type I, they are monochromatic. If they have color -2 or -1, then they are of type III (See Section 4). For color 0 the proof is in Corollary 4.5, and for color 1 in Corollary 4.9.
For type II, by Proposition 2.5, K_1 and K_2 are either monochromatic of color 0, hence of type I, or they are both sets of the vertices of a 7-crosspolytope, in which case the statement is in Corollary 3.31.
If K_1 and K_2 are of type IV, they are maximal cliques in a graph Γ_c, where there are 14 different possibilities for c. For $c \in \{\{-2\},\{-1\},\{0\},\{1\}\}$, the cliques K_1 and K_2 are of type I, which we already covered (note that for K_1 and K_2 *maximal* in Γ_1, there is always an automorphism extending f!). For c in $\{\{-2,-1\},\{-2,1\}\}$, the cliques K_1 and K_2 are of type I as well (Lemma 5.2). For $c = \{-2,0\}$, the proof is in Lemma 5.14. For $c \in \{\{-1,1\},\{-2,-1,1\}\}$, an isomorphism of maximal cliques always extends, see Corollary 5.28. The same holds for $c = \{0,1\}$, see Corollary 5.33. For $c \in \{\{-1,0\},\{-2,-1,0\}\}$, the statement is in Lemma 5.30.
For $c = \{-2,0,1\}$ the statement is Lemma 5.35.
Finally, for $c = \{-2,-1,0,1\}$ there is one clique, which is Γ itself, and every automorphism of Γ is an element in W. This finishes (i).
Part (ii) follows from Propositions 3.28 and 4.1 for type A, and it follows from Propositions 2.12 and 4.7 for type B. Finally, part (iii) is in Lemma 5.30, and part (iv) is in Lemma 5.35.

Remark 6.1. From Theorem 1.2 it follows that for an isomorphism f of two cliques K_1 and K_2 of types I, II, III, or IV, one can determine whether f extends to an automorphism of Λ by checking for all subcliques of K_1 of the form A, B, C_α, D, or F, if f restricted to an associated ordered sequence extends. However, one never has to check all subcliques of those six forms. The following table shows for each type of K_1 and K_2 which subcliques are sufficient to check.

Type	Subtype	All isomorphisms extend	A	B	C_{-1}	C_1	D	F
I	Γ_{-2}	x						
I	Γ_{-1}	x						
I	Γ_0		x					
I	Γ_1			x				
II	k-simplex, $k \leq 7$		x					
II	7-crosspolyopte			x				
III	all	x						
IV	Γ_{-2}	x						
IV	Γ_{-1}	x						
IV	Γ_0		x					
IV	Γ_1	x						
IV	$\Gamma_{-2,-1}$	x						
IV	$\Gamma_{-2,0}$		x					
IV	$\Gamma_{-2,1}$	x						
IV	$\Gamma_{-1,0}$		x		x			
IV	$\Gamma_{-1,1}$	x						
IV	$\Gamma_{0,1}$	x						
IV	$\Gamma_{-2,-1,0}$		x		x			
IV	$\Gamma_{-2,-1,1}$	x						
IV	$\Gamma_{-2,0,1}$		x			x	x	x
IV	$\Gamma_{-2,-1,0,1}$	x						

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Mathematisch Instituut, Niels Bohrweg 1, 2333 CA Leiden, Netherlands
E-mail address: `r.l.winter@math.leidenuniv.nl`

Mathematisch Instituut, Niels Bohrweg 1, 2333 CA Leiden, Netherlands
E-mail address: `rvl@math.leidenuniv.nl`

Appendices

A. Table results Section 5

See Section 5, above Remark 5.1, for an explanation of this table.

Graph	$\lvert K\rvert$	$\#O$	$\lvert W_K\rvert$	$\lvert\mathrm{Aut}(K)\rvert$	K
Γ_{-2}	2	1	5806080	2	$\{1, 128\}$
Γ_{-1}	3	1	311040	6	$\{1, 32, 240\}$
Γ_{0}	8	1	1344	40320	$\{1, 8, 26, 31, 43, 46, 52, 53\}$
Γ_{1}	7	1	10080	5040	$\{1, 2, 3, 5, 129, 130, 131\}$
	8	1	40320	40320	$\{1, 2, 3, 4, 129, 130, 131, 132\}$
$\Gamma_{-2,-1}$	2	1	5806080	2	$\{1, 128\}$
	3	1	311040	6	$\{1, 32, 240\}$
$\Gamma_{-2,0}$	16	1	344064	10321920	$\{25, 32, 51, 54, 75, 78, 97, 104, 130, 144, 177, 181, 188, 192, 225, 239\}$
$\Gamma_{-2,1}$	2	1	5806080	2	$\{1, 128\}$
	7	1	10080	5040	$\{1, 2, 3, 5, 129, 130, 131\}$
	8	1	40320	40320	$\{1, 2, 3, 4, 129, 130, 131, 132\}$
$\Gamma_{-1,0}$	8	5	144	144	$\{41, 48, 50, 78, 144, 187, 214, 240\}$
			128	128	$\{12, 17, 41, 71, 170, 193, 214, 240\}$
			16	16	$\{12, 17, 40, 41, 71, 86, 214, 240\}$
			14	14	$\{12, 23, 41, 50, 70, 168, 214, 240\}$
			8	8	$\{7, 41, 48, 50, 75, 86, 214, 240\}$
	9	11	64	192	$\{3, 6, 41, 48, 50, 55, 214, 227, 240\}$
			48	96	$\{19, 41, 48, 50, 75, 146, 193, 214, 240\}$
			40	80	$\{12, 23, 41, 50, 67, 86, 163, 214, 240\}$
			30	60	$\{12, 23, 40, 41, 50, 65, 86, 214, 240\}$
			24	48	$\{19, 41, 48, 50, 70, 75, 193, 214, 240\}$
			18	18	$\{12, 23, 41, 50, 163, 168, 214, 227, 240\}$
			16	16	$\{19, 41, 48, 50, 65, 150, 172, 214, 240\}$
			8	16	$\{41, 48, 50, 55, 65, 78, 178, 214, 240\}$
			4	8	$\{3, 41, 48, 50, 55, 66, 152, 214, 240\}$
			2	2	$\{3, 41, 48, 50, 55, 72, 77, 214, 240\}$
			1	1	$\{7, 41, 48, 50, 68, 78, 85, 214, 240\}$
	10	6	192	1152	$\{41, 48, 50, 55, 66, 152, 178, 184, 214, 240\}$
			128	256	$\{3, 6, 41, 48, 50, 55, 76, 77, 214, 240\}$

Graph	$\lvert K \rvert$	$\#O$	$\lvert W_K \rvert$	$\lvert \mathrm{Aut}(K) \rvert$	K
			100	200	$\{12, 23, 40, 41, 50, 67, 77, 86, 214, 240\}$
			36	72	$\{6, 19, 41, 48, 50, 65, 76, 192, 214, 240\}$
			32	64	$\{3, 6, 41, 48, 50, 55, 76, 85, 214, 240\}$
			18	36	$\{19, 41, 48, 50, 65, 76, 86, 192, 214, 240\}$
	12	1	3888	31104	$\{11, 22, 36, 46, 49, 69, 74, 84, 184, 196, 214, 240\}$
$\Gamma_{-1,1}$	3	2	311040	6	$\{55, 80, 173\}$
			103680	2	$\{84, 88, 194\}$
	7	4	10080	5040	$\{118, 126, 191, 195, 213, 224, 237\}$
			1440	720	$\{8, 24, 32, 113, 129, 138, 151\}$
			480	240	$\{42, 72, 103, 120, 136, 193, 237\}$
			288	144	$\{37, 39, 53, 74, 167, 235, 238\}$
	8	5	40320	40320	$\{33, 41, 49, 57, 132, 133, 134, 142\}$
			5040	5040	$\{6, 7, 21, 24, 135, 148, 193, 201\}$
			1440	1440	$\{24, 32, 99, 107, 139, 152, 195, 213\}$
			1152	1152	$\{34, 63, 111, 114, 180, 182, 196, 203\}$
			720	720	$\{33, 91, 98, 101, 148, 151, 153, 154\}$
$\Gamma_{0,1}$	22	1	1344	1344	$\{1, 2, 3, 5, 9, 12, 17, 29, 33, 38, 51, 129, 130, 131, 132, 133, 134, 135, 136, 144, 158, 173\}$
	28	1	336	336	$\{1, 2, 3, 5, 7, 9, 13, 14, 15, 17, 25, 29, 33, 43, 45, 53, 129, 130, 131, 132, 133, 134, 136, 137, 139, 140, 149, 157\}$
	29	432			separate section
	30	25	3840	3840	$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 17, 18, 20, 22, 26, 33, 129, 130, 131, 132, 133, 134, 136, 137, 143, 144, 145, 146, 149\}$
			1152	1152	$\{1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 15, 17, 19, 21, 23, 26, 33, 129, 130, 131, 132, 133, 134, 136, 137, 140, 141, 145, 146, 149\}$
			720	720	$\{1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 15, 17, 19, 21, 23, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 143, 144, 147\}$
			192	192	$\{1, 2, 3, 5, 7, 8, 9, 13, 14, 17, 22, 29, 33, 37, 39, 53, 65, 129, 130, 131, 132, 133, 134, 136, 144, 147, 157, 160, 166, 184\}$
			72	72	$\{1, 2, 3, 5, 6, 8, 9, 13, 14, 15, 17, 22, 29, 33, 37, 39, 53, 65, 129, 130, 131, 132, 133, 134, 136, 144, 147, 157, 160, 181\}$
			64	64	$\{1, 2, 3, 5, 6, 7, 8, 9, 14, 17, 22, 29, 33, 129, 130, 131, 132, 133, 134, 136, 139, 143, 144, 146, 147, 149, 157, 160, 166, 169\}$
			48	48	$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 17, 18, 19, 21, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 143, 144, 145, 146, 147\}$
			48	48	$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 17, 18, 19, 20, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 143, 144, 147\}$
			48	48	$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 15, 17, 19, 21, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 143, 144, 147\}$
			48	48	$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 14, 15, 17, 33, 36, 37, 42, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 140, 143, 158\}$
			48	48	$\{1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 17, 18, 19, 21, 23, 26, 33, 129, 130, 131, 132, 133, 134, 136, 138, 139, 140, 143, 147, 149\}$
			32	32	$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 17, 18, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 143, 144, 145, 146, 147, 156, 157, 165, 166\}$
			24	24	$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 17, 18, 19, 21, 26, 33, 129, 130, 131, 132, 133, 134, 136, 137, 138, 139, 143, 144, 149\}$
			16	16	$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 15, 17, 19, 20, 21, 26, 33, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 143, 144, 147\}$
			16	16	$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 15, 17, 33, 35, 36, 37, 38, 42, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 140, 143, 158\}$

Graph	$\|K\|$	$\#O$	$\|W_K\|$	$\|\mathrm{Aut}(K)\|$	K
			12	12	$\{1,2,3,4,5,6,8,9,10,12,14,17,18,20,22,26,33,129,130,131,132,133,134,135,136,138,143,144,145,147\}$
			12	12	$\{1,2,3,4,5,6,8,9,10,12,14,17,18,20,22,26,33,129,130,131,132,133,134,135,136,138,139,143,144,147\}$
			12	12	$\{1,2,3,4,5,6,8,9,10,15,17,33,35,36,37,38,42,129,130,131,132,133,134,135,136,138,140,143,156,158\}$
			10	10	$\{1,2,3,4,5,6,8,9,10,12,17,18,22,26,33,36,38,65,129,130,131,132,133,134,136,143,144,149,157,165\}$
			8	8	$\{1,2,3,4,5,6,8,9,10,12,13,17,18,19,22,26,33,129,130,131,132,133,134,135,136,138,143,144,145,147\}$
			8	8	$\{1,2,3,4,5,6,8,9,10,11,15,17,33,35,36,37,42,129,130,131,132,133,134,135,136,139,140,143,157,158\}$
			4	4	$\{1,2,3,5,7,8,9,10,11,13,17,18,19,21,27,29,33,129,130,131,132,133,134,135,136,139,140,143,146,147\}$
			4	4	$\{1,2,3,5,7,9,10,11,13,17,18,19,21,33,37,41,65,129,130,131,132,133,134,136,143,146,147,155,174,179\}$
			4	4	$\{1,2,3,4,5,7,8,9,10,11,13,17,18,19,21,27,33,129,130,131,132,133,134,135,136,139,140,143,146,147\}$
			4	4	$\{1,2,3,5,7,9,10,11,13,17,18,19,33,37,41,65,129,130,131,132,133,134,136,143,146,147,155,156,174,179\}$
	31	7	480	480	$\{1,3,5,9,11,17,19,25,27,33,65,67,73,75,81,83,89,129,132,134,145,147,158,167,174,179,183,184,187,199,208\}$
			120	120	$\{1,3,5,9,11,15,17,19,25,27,65,67,73,75,83,89,129,132,134,143,144,145,147,148,153,156,158,174,179,183,187\}$
			72	72	$\{1,3,5,9,11,17,18,19,21,23,25,27,33,49,51,81,89,129,132,133,134,141,144,145,147,165,167,174,179,183,208\}$
			48	48	$\{1,3,5,9,11,17,19,21,23,25,27,49,65,81,89,129,132,133,134,144,145,146,147,148,158,165,167,174,179,183,208\}$
			16	16	$\{1,3,5,9,11,13,15,17,19,21,25,27,31,41,67,73,89,129,132,134,145,147,158,167,174,179,183,184,187,199,208\}$
			12	12	$\{1,3,5,9,11,13,15,17,19,21,25,27,31,41,67,73,89,129,132,134,145,146,147,148,158,167,174,179,183,199,208\}$
			8	8	$\{1,3,5,9,11,13,15,17,19,21,25,27,31,41,67,73,89,129,132,134,145,147,148,158,167,174,179,183,187,199,208\}$
	32	3	144	144	$\{2,3,5,9,12,17,20,26,27,29,65,74,75,82,83,89,129,132,136,143,145,146,147,149,153,160,173,174,176,181,184,201\}$
			48	48	$\{1,2,3,4,5,7,9,10,11,12,13,15,16,35,41,43,75,129,130,132,134,136,143,145,147,155,156,157,158,160,162,164\}$
			24	24	$\{1,3,5,9,11,17,18,19,20,21,23,25,26,27,29,33,49,57,129,131,132,133,134,135,136,140,141,145,146,147,174,208\}$
	33	1	96	96	$\{1,2,3,4,5,6,7,8,9,10,11,12,14,17,18,20,33,34,129,130,131,132,133,134,135,136,137,143,144,145,146,155,156\}$
	34	2	2880	2880	$\{1,2,5,6,8,14,17,18,22,33,34,36,37,38,42,50,53,54,129,130,131,132,133,136,137,139,142,155,157,166,169,170,181,182\}$
			720	720	$\{1,2,3,4,5,6,8,9,12,14,15,17,20,22,23,26,33,36,38,129,130,131,132,133,134,136,137,138,139,143,144,149,160,169\}$
	36	1	40320	40320	$\{1,2,3,4,5,6,7,8,9,10,11,12,17,18,19,20,33,34,35,36,129,130,131,132,133,134,135,136,137,138,143,144,145,155,156,165\}$
$\Gamma_{-2,-1,0}$	8	4	144	144	$\{1,8,26,31,43,54,227,240\}$
			128	128	$\{1,8,26,47,83,102,226,238\}$
			16	16	$\{1,8,26,47,83,110,226,233\}$
			8	8	$\{1,8,26,31,43,54,228,239\}$
	9	9	80	80	$\{1,8,26,47,51,86,121,128,228\}$
			64	192	$\{1,8,26,31,43,46,84,85,240\}$
			40	80	$\{1,8,26,47,51,86,124,125,228\}$
			28	28	$\{1,8,26,47,51,86,110,121,236\}$
			24	48	$\{1,8,26,31,43,54,100,125,227\}$
			18	18	$\{1,8,26,47,51,86,110,124,232\}$
			12	12	$\{1,8,26,31,43,86,106,115,125\}$

Graph	$\|K\|$	$\#O$	$\|W_K\|$	$\|\mathrm{Aut}(K)\|$	K
			4	8	$\{1, 8, 26, 31, 43, 46, 84, 113, 237\}$
			1	1	$\{1, 8, 26, 31, 43, 54, 100, 113, 238\}$
	10	10	288	576	$\{1, 8, 26, 47, 51, 77, 121, 128, 185, 229\}$
			192	1152	$\{1, 8, 26, 31, 43, 46, 52, 53, 227, 240\}$
			100	200	$\{1, 8, 26, 47, 51, 86, 91, 125, 222, 228\}$
			64	64	$\{1, 8, 26, 31, 43, 46, 84, 98, 103, 125\}$
			60	120	$\{1, 8, 26, 47, 51, 86, 91, 128, 218, 228\}$
			48	96	$\{1, 8, 26, 31, 43, 86, 101, 106, 115, 128\}$
			32	32	$\{1, 8, 26, 31, 43, 86, 106, 115, 224, 234\}$
			32	64	$\{1, 8, 26, 31, 43, 46, 84, 101, 226, 238\}$
			18	36	$\{1, 8, 26, 31, 43, 54, 100, 109, 119, 227\}$
			4	4	$\{1, 8, 26, 31, 43, 46, 84, 98, 117, 238\}$
	11	5	2304	2304	$\{1, 8, 26, 31, 43, 54, 98, 103, 121, 128, 227\}$
			648	2592	$\{1, 8, 26, 47, 51, 77, 108, 121, 185, 213, 229\}$
			192	384	$\{1, 8, 26, 31, 43, 54, 100, 101, 121, 128, 227\}$
			32	64	$\{1, 8, 26, 31, 43, 46, 52, 85, 98, 103, 238\}$
			72	144	$\{1, 8, 26, 31, 43, 54, 100, 109, 113, 128, 227\}$
	12	3	3888	31104	$\{1, 8, 26, 47, 51, 77, 91, 108, 185, 213, 218, 229\}$
			1024	3072	$\{1, 8, 26, 31, 43, 46, 84, 85, 98, 103, 121, 128\}$
			512	1024	$\{1, 8, 26, 31, 43, 46, 84, 85, 98, 103, 226, 238\}$
	13	1	1536	9216	$\{1, 8, 26, 31, 43, 46, 52, 53, 76, 77, 83, 86, 240\}$
	16	1	344064	10321920	$\{1, 8, 26, 31, 43, 46, 52, 53, 76, 77, 83, 86, 98, 103, 121, 128\}$
$\Gamma_{-2,-1,1}$	6	1	622080	12	$\{24, 33, 96, 105, 131, 238\}$
	14	1	20160	10080	$\{11, 41, 88, 118, 135, 159, 169, 175, 183, 186, 194, 200, 210, 234\}$
	16	1	80640	80640	$\{11, 22, 43, 54, 75, 86, 107, 118, 156, 172, 174, 178, 191, 195, 197, 213\}$
$\Gamma_{-2,0,1}$	13	7	1536	9216	$\{1, 8, 31, 40, 71, 98, 128, 136, 150, 164, 178, 191, 205\}$
			768	4608	$\{1, 8, 31, 40, 98, 128, 136, 137, 150, 164, 178, 191, 205\}$
			512	1024	$\{1, 8, 31, 98, 128, 136, 137, 149, 150, 162, 163, 171, 172\}$
			384	768	$\{1, 8, 12, 14, 15, 128, 136, 137, 138, 139, 154, 169, 215\}$
			384	768	$\{1, 8, 12, 38, 47, 82, 128, 136, 137, 152, 160, 161, 171\}$
			192	384	$\{1, 8, 31, 39, 45, 51, 98, 128, 136, 137, 149, 162, 172\}$
			64	128	$\{1, 8, 12, 23, 38, 47, 82, 128, 136, 137, 152, 160, 171\}$
	14	4	15360	30720	$\{1, 27, 43, 51, 57, 59, 128, 136, 138, 140, 141, 142, 177, 192\}$
			3072	18432	$\{1, 12, 23, 30, 45, 47, 77, 79, 99, 106, 117, 128, 163, 199\}$
			192	384	$\{1, 29, 30, 47, 54, 78, 82, 93, 117, 128, 152, 191, 198, 209\}$

Graph	$\lvert K\rvert$	$\#O$	$\lvert W_K\rvert$	$\lvert\mathrm{Aut}(K)\rvert$	K
			64	128	$\{1, 46, 70, 72, 79, 83, 100, 101, 103, 128, 160, 172, 228, 231\}$
	16	3	344064	10321920	$\{1, 22, 46, 57, 72, 83, 107, 128, 138, 153, 160, 177, 192, 209, 216, 231\}$
			336	336	$\{1, 39, 40, 43, 51, 53, 55, 115, 128, 141, 142, 169, 192, 216, 218, 219\}$
			192	384	$\{1, 16, 38, 42, 68, 70, 74, 77, 102, 106, 113, 128, 160, 182, 215, 228\}$
	19	29	2880	2880	$\{1, 8, 12, 14, 15, 20, 22, 23, 36, 38, 39, 128, 136, 137, 138, 139, 149, 160, 169\}$
			2880	5760	$\{1, 8, 12, 14, 50, 68, 70, 74, 128, 136, 137, 154, 169, 170, 176, 177, 181, 182, 215\}$
			2304	2304	$\{1, 8, 12, 14, 23, 24, 39, 40, 68, 70, 128, 136, 137, 151, 152, 162, 163, 169, 170\}$
			1440	1440	$\{1, 8, 12, 15, 16, 20, 23, 24, 36, 39, 40, 70, 128, 136, 137, 139, 151, 162, 171\}$
			384	384	$\{1, 8, 12, 15, 23, 24, 38, 40, 68, 70, 128, 136, 137, 150, 152, 162, 163, 169, 171\}$
			144	144	$\{1, 8, 12, 14, 23, 24, 39, 40, 68, 128, 136, 137, 138, 151, 152, 162, 163, 169, 170\}$
			120	120	$\{1, 8, 12, 14, 20, 24, 36, 39, 128, 136, 137, 138, 139, 149, 151, 161, 162, 169, 170\}$
			96	96	$\{1, 8, 12, 14, 15, 22, 26, 38, 45, 128, 136, 137, 138, 139, 149, 152, 160, 161, 169\}$
			96	96	$\{1, 8, 12, 14, 15, 24, 40, 128, 136, 137, 138, 139, 150, 151, 152, 161, 162, 163, 169\}$
			96	96	$\{1, 8, 12, 14, 20, 23, 38, 39, 128, 136, 137, 138, 139, 149, 152, 160, 162, 169, 170\}$
			96	96	$\{1, 8, 12, 14, 39, 40, 128, 136, 137, 138, 139, 151, 152, 160, 161, 162, 163, 169, 170\}$
			96	96	$\{1, 8, 12, 15, 16, 22, 23, 24, 36, 39, 40, 68, 70, 128, 136, 137, 151, 163, 171\}$
			72	72	$\{1, 8, 12, 14, 15, 22, 24, 38, 40, 128, 136, 137, 138, 139, 150, 152, 161, 163, 169\}$
			48	48	$\{1, 8, 12, 14, 20, 22, 39, 40, 128, 136, 137, 138, 139, 151, 152, 160, 161, 169, 170\}$
			32	32	$\{1, 8, 12, 14, 15, 20, 22, 23, 36, 38, 40, 128, 136, 137, 138, 139, 150, 160, 169\}$
			32	32	$\{1, 8, 12, 14, 23, 39, 128, 136, 137, 138, 139, 149, 151, 152, 160, 162, 163, 169, 170\}$
			32	32	$\{1, 8, 12, 14, 23, 24, 39, 68, 128, 136, 137, 138, 149, 151, 152, 162, 163, 169, 170\}$
			24	24	$\{1, 8, 12, 14, 15, 23, 36, 38, 40, 128, 136, 137, 138, 139, 150, 160, 162, 163, 169\}$
			24	24	$\{1, 8, 12, 14, 15, 24, 38, 40, 128, 136, 137, 138, 139, 150, 152, 161, 162, 163, 169\}$
			24	24	$\{1, 8, 12, 14, 23, 24, 38, 39, 68, 128, 136, 137, 138, 149, 152, 162, 163, 169, 170\}$
			20	20	$\{1, 8, 12, 14, 20, 24, 39, 128, 136, 137, 138, 139, 149, 151, 152, 161, 162, 169, 170\}$
			16	16	$\{1, 8, 12, 14, 15, 20, 23, 38, 40, 128, 136, 137, 138, 139, 150, 152, 160, 162, 169\}$
			16	16	$\{1, 8, 12, 14, 23, 40, 128, 136, 137, 138, 139, 150, 151, 152, 160, 162, 163, 169, 170\}$
			12	12	$\{1, 8, 12, 14, 15, 22, 23, 38, 40, 128, 136, 137, 138, 139, 150, 152, 160, 163, 169\}$
			8	8	$\{1, 8, 12, 14, 15, 20, 23, 36, 38, 40, 128, 136, 137, 138, 139, 150, 160, 162, 169\}$
			8	8	$\{1, 8, 12, 14, 15, 23, 38, 40, 128, 136, 137, 138, 139, 150, 152, 160, 162, 163, 169\}$
			8	8	$\{1, 8, 12, 14, 15, 23, 24, 38, 40, 128, 136, 137, 138, 139, 150, 152, 162, 163, 169\}$
			8	8	$\{1, 8, 12, 14, 20, 24, 38, 39, 128, 136, 137, 138, 139, 149, 152, 161, 162, 169, 170\}$
			8	8	$\{1, 8, 12, 14, 20, 39, 40, 128, 136, 137, 138, 139, 151, 152, 160, 161, 162, 169, 170\}$
	20	5	192	192	$\{1, 8, 12, 14, 15, 16, 20, 22, 23, 36, 38, 39, 128, 136, 137, 138, 139, 140, 149, 160\}$
			128	128	$\{1, 8, 12, 14, 15, 16, 20, 23, 38, 40, 128, 136, 137, 138, 139, 140, 150, 152, 160, 162\}$

Graph	$\lvert K\rvert$	$\#O$	$\lvert W_K\rvert$	$\lvert\mathrm{Aut}(K)\rvert$	K
			96	96	$\{1, 8, 12, 14, 15, 16, 20, 38, 39, 40, 128, 136, 137, 138, 139, 140, 152, 160, 161, 162\}$
			16	16	$\{1, 8, 12, 14, 15, 16, 20, 22, 38, 39, 40, 128, 136, 137, 138, 139, 140, 152, 160, 161\}$
			16	16	$\{1, 8, 12, 14, 15, 16, 20, 22, 38, 39, 128, 136, 137, 138, 139, 140, 149, 152, 160, 161\}$
	21	1	48	48	$\{1, 8, 12, 14, 15, 16, 22, 38, 39, 45, 128, 136, 137, 138, 139, 140, 149, 152, 160, 161, 163\}$
	22	3	1440	1440	$\{1, 26, 30, 42, 46, 58, 74, 78, 90, 106, 114, 121, 128, 177, 182, 185, 197, 198, 200, 207, 218, 231\}$
			1344	1344	$\{1, 2, 3, 5, 9, 12, 17, 29, 33, 38, 51, 129, 130, 131, 132, 133, 134, 135, 136, 144, 158, 173\}$
			144	144	$\{1, 12, 26, 28, 36, 42, 43, 44, 50, 52, 57, 58, 59, 106, 128, 176, 177, 178, 197, 206, 217, 230\}$
	24	1	1440	1440	$\{1, 8, 12, 14, 15, 16, 38, 39, 40, 45, 46, 47, 128, 136, 137, 138, 139, 140, 152, 160, 161, 162, 163, 164\}$
	28			**as in 0,1**	
	29	433	103680	103680	$\{1, 8, 12, 14, 15, 16, 36, 38, 39, 40, 42, 43, 44, 45, 46, 47, 128, 136, 137, 138, 139, 140, 142, 160, 161, 162, 163, 164, 215\}$
				rest	**as in 0,1**
	30			**as in 0,1**	
	31			**as in 0,1**	
	32			**as in 0,1**	
	33			**as in 0,1**	
	34			**as in 0,1**	
	36			**as in 0,1**	
Γ	240	1	$\lvert W\rvert$	$\lvert W\rvert$	Γ

B. Table cliques of size 29 in $\Gamma_{0,1}$

See Section 5.4.1 for an explanation of this table.

Nr.	$\|W_K\|$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
1	12	1176	36	0	{ 1, 2, 3, 4, 5, 9, 17, 19, 34, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 148, 155, 156, 158, 165, 166, 167, 173, 174 }
2	48	2352	8	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 17, 18, 20, 22, 26, 129, 130, 131, 132, 133, 134, 137, 138, 143, 144, 145, 148, 149 }
3	12	2008	12	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 10, 17, 34, 42, 66, 129, 130, 131, 132, 133, 137, 143, 145, 146, 150, 155, 156, 157, 160, 165, 166 }
4	8	1256	32	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 14, 17, 34, 42, 50, 65, 66, 129, 130, 131, 132, 133, 145, 146, 150, 155, 156, 157, 160, 169, 173 }
5	96	1288	36	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 19, 21, 25, 35, 37, 41, 65, 129, 130, 131, 132, 133, 134, 143, 147, 148, 158, 165, 166, 173 }
6	16	1256	28	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 25, 37, 41, 129, 130, 131, 132, 133, 134, 135, 143, 146, 147, 148, 156, 158, 165, 166, 167, 173 }
7	1	800	45	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 13, 14, 16, 34, 37, 38, 41, 43, 129, 130, 131, 132, 133, 138, 139, 142, 155, 158, 159, 160, 161 }
8	4	816	49	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 12, 13, 15, 16, 35, 38, 41, 43, 129, 130, 131, 132, 133, 139, 142, 155, 157, 158, 159, 160, 161 }
9	1	800	45	0	{ 1, 2, 3, 4, 5, 7, 9, 11, 12, 13, 14, 16, 34, 39, 41, 43, 129, 130, 131, 132, 133, 139, 142, 155, 157, 158, 159, 160, 161 }
10	4	800	45	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 14, 16, 34, 37, 38, 39, 41, 43, 129, 130, 131, 132, 133, 138, 142, 155, 156, 158, 159, 160, 161 }
11	2	792	43	0	{ 1, 2, 3, 4, 5, 7, 9, 12, 14, 16, 34, 36, 38, 39, 41, 43, 129, 130, 131, 132, 133, 139, 142, 155, 157, 158, 159, 160, 161 }
12	2	808	47	0	{ 1, 2, 3, 4, 5, 7, 9, 12, 13, 14, 15, 16, 38, 39, 41, 43, 129, 130, 131, 132, 133, 139, 142, 155, 157, 158, 159, 160, 161 }
13	12	832	53	0	{ 1, 2, 3, 5, 7, 9, 11, 12, 13, 14, 16, 34, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 137, 142, 155, 156, 157, 159, 161 }
14	4	808	47	0	{ 1, 2, 3, 5, 7, 9, 11, 12, 14, 16, 34, 36, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 137, 138, 142, 155, 156, 159, 161 }
15	4	1440	25	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 17, 20, 34, 38, 42, 65, 66, 129, 130, 131, 132, 133, 143, 146, 150, 155, 156, 160, 165, 166, 169 }
16	4	1400	28	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 25, 34, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 148, 156, 158, 165, 166, 167, 173 }
17	2	1392	26	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 35, 37, 41, 129, 130, 131, 132, 133, 134, 135, 143, 147, 148, 155, 156, 158, 165, 166, 167, 173 }
18	16	976	42	0	{ 1, 2, 3, 5, 9, 13, 17, 19, 34, 35, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 148, 155, 156, 158, 166, 167, 173, 174, 179 }
19	16	1240	32	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 17, 19, 21, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 147, 148, 155, 158, 165, 166, 167 }
20	2	632	52	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 21, 34, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 148, 155, 158, 159, 165, 166, 174, 179 }
21	2	624	50	0	{ 1, 2, 3, 5, 7, 9, 10, 13, 19, 21, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 155, 158, 159, 166, 174, 179 }
22	4	624	50	0	{ 1, 2, 3, 5, 7, 9, 11, 13, 18, 19, 21, 34, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 148, 155, 159, 167, 174, 179 }
23	4	640	54	0	{ 1, 2, 3, 5, 6, 7, 9, 10, 19, 25, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 156, 158, 159, 165, 166, 179 }
24	16	616	48	0	{ 1, 2, 3, 5, 7, 11, 18, 19, 25, 34, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 146, 148, 156, 159, 165, 167, 168, 174, 179 }
25	4	632	52	0	{ 1, 2, 3, 5, 7, 10, 18, 19, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 159, 165, 166, 168, 174, 179 }
26	32	648	56	0	{ 1, 2, 3, 5, 6, 11, 18, 19, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 159, 165, 167, 168, 173, 179 }
27	48	680	64	0	{ 1, 2, 3, 4, 5, 13, 21, 25, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 146, 147, 148, 157, 158, 159, 166, 167, 168, 173, 174 }
28	2	656	50	0	{ 1, 2, 3, 5, 7, 9, 12, 14, 16, 34, 36, 38, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 142, 155, 156, 158, 159, 161 }
29	2	648	48	0	{ 1, 2, 3, 5, 7, 9, 11, 14, 15, 16, 36, 37, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 137, 139, 142, 155, 157, 159, 161 }
30	1	664	52	0	{ 1, 2, 3, 5, 7, 9, 10, 12, 15, 16, 35, 36, 38, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 139, 142, 155, 158, 159, 161 }
31	4	664	52	0	{ 1, 2, 3, 5, 7, 9, 10, 12, 15, 16, 35, 36, 38, 41, 43, 45, 129, 130, 131, 132, 133, 137, 138, 139, 142, 155, 159, 160, 161 }

Nr.	$\|W_K\|$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
32	4	648	48	0	{ 1, 2, 3, 5, 7, 9, 10, 14, 15, 16, 36, 37, 38, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 140, 142, 156, 158, 159, 161 }
33	8	680	56	0	{ 1, 2, 3, 5, 7, 9, 10, 15, 16, 35, 36, 37, 38, 41, 43, 45, 129, 130, 131, 132, 133, 138, 139, 140, 143, 158, 159, 160, 161 }
34	4	656	50	0	{ 1, 2, 3, 5, 7, 9, 10, 12, 15, 16, 35, 36, 38, 41, 43, 45, 129, 130, 131, 132, 133, 137, 139, 140, 142, 157, 159, 160, 161 }
35	16	664	52	0	{ 1, 2, 3, 5, 7, 18, 20, 22, 23, 24, 36, 38, 40, 49, 51, 53, 129, 130, 131, 132, 133, 135, 141, 144, 165, 166, 167, 168, 170 }
36	2	656	50	0	{ 1, 2, 3, 5, 7, 18, 19, 20, 24, 34, 35, 36, 40, 49, 51, 53, 129, 130, 131, 132, 133, 138, 139, 141, 144, 167, 168, 169, 170 }
37	12	688	58	0	{ 1, 2, 3, 4, 5, 21, 22, 23, 24, 37, 38, 39, 40, 49, 50, 51, 129, 130, 131, 132, 133, 135, 139, 141, 144, 166, 167, 168, 170 }
38	48	1416	34	0	{ 1, 2, 3, 4, 5, 9, 13, 17, 34, 35, 37, 41, 49, 65, 129, 130, 131, 132, 133, 134, 148, 155, 156, 157, 158, 166, 167, 173, 174 }
39	48	1184	36	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 25, 35, 37, 41, 49, 65, 129, 130, 131, 132, 133, 134, 147, 148, 156, 157, 158, 165, 166, 167, 173 }
40	2	952	38	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 17, 18, 22, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 157, 165, 169, 173 }
41	2	968	42	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 14, 17, 18, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 156, 157, 169, 173 }
42	2	960	40	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 14, 17, 20, 34, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 146, 150, 155, 156, 160, 169, 173 }
43	1	952	38	0	{ 1, 2, 3, 4, 5, 6, 9, 14, 17, 20, 36, 42, 50, 65, 66, 129, 130, 131, 132, 133, 145, 149, 150, 155, 156, 160, 166, 169, 173 }
44	2	960	40	0	{ 1, 2, 3, 4, 5, 6, 9, 14, 17, 22, 36, 42, 50, 65, 66, 129, 130, 131, 132, 133, 145, 149, 150, 155, 157, 160, 166, 169, 173 }
45	4	640	56	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 21, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 155, 158, 159, 165, 166, 174, 179 }
46	2	616	50	0	{ 1, 2, 3, 5, 7, 9, 11, 19, 21, 34, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 148, 155, 158, 159, 165, 167, 174, 179 }
47	8	624	52	0	{ 1, 2, 3, 5, 7, 9, 19, 21, 34, 41, 49, 66, 129, 130, 131, 132, 133, 135, 145, 146, 148, 155, 158, 159, 165, 166, 167, 174, 179 }
48	2	624	52	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 21, 25, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 158, 159, 165, 166, 174, 179 }
49	24	656	60	0	{ 1, 2, 3, 5, 6, 9, 10, 19, 21, 25, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 158, 159, 165, 166, 173, 179 }
50	2	624	52	0	{ 1, 2, 3, 5, 7, 11, 13, 18, 25, 35, 49, 66, 129, 130, 131, 132, 133, 135, 144, 145, 147, 148, 156, 157, 159, 167, 168, 174, 179 }
51	2	616	50	0	{ 1, 2, 3, 5, 7, 10, 11, 18, 19, 25, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 146, 147, 148, 156, 159, 165, 168, 174, 179 }
52	4	624	52	0	{ 1, 2, 3, 5, 7, 11, 18, 25, 35, 49, 65, 66, 129, 130, 131, 132, 133, 144, 145, 147, 148, 156, 157, 159, 165, 167, 168, 174, 179 }
53	4	640	56	0	{ 1, 2, 3, 4, 5, 10, 11, 13, 21, 25, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 146, 147, 148, 157, 158, 159, 168, 173, 174 }
54	4	480	56	0	{ 1, 2, 3, 5, 10, 13, 21, 25, 35, 49, 65, 66, 129, 130, 131, 132, 133, 144, 145, 147, 148, 157, 158, 159, 166, 168, 173, 174, 179 }
55	4	488	58	0	{ 1, 2, 3, 5, 11, 13, 19, 25, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 146, 147, 148, 156, 158, 159, 167, 168, 173, 174, 179 }
56	48	504	62	0	{ 1, 2, 3, 5, 11, 13, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 157, 158, 159, 167, 168, 173, 174, 179 }
57	1152	1032	44	0	{ 1, 2, 3, 4, 9, 10, 11, 17, 18, 21, 25, 33, 34, 35, 41, 49, 67, 69, 129, 130, 131, 134, 135, 145, 157, 165, 173, 174, 175 }
58	4	376	62	0	{ 1, 2, 3, 5, 11, 18, 21, 25, 37, 41, 66, 67, 129, 130, 131, 132, 135, 143, 146, 147, 148, 157, 159, 165, 167, 168, 173, 174, 179 }
59	36	384	64	0	{ 1, 2, 3, 5, 11, 18, 21, 25, 34, 37, 41, 65, 66, 67, 129, 130, 131, 132, 143, 146, 148, 157, 159, 165, 167, 168, 173, 174, 179 }
60	16	392	66	0	{ 1, 2, 3, 5, 19, 21, 25, 34, 37, 41, 66, 67, 129, 130, 131, 132, 135, 143, 146, 148, 158, 159, 165, 166, 167, 168, 173, 174, 179 }
61	20	368	60	0	{ 1, 2, 3, 5, 10, 19, 25, 37, 41, 49, 66, 67, 129, 130, 131, 132, 135, 146, 147, 148, 156, 158, 159, 165, 166, 168, 173, 174, 179 }
62	96	424	74	0	{ 1, 2, 3, 4, 5, 10, 13, 19, 21, 25, 37, 41, 49, 65, 66, 67, 129, 130, 131, 132, 146, 147, 148, 158, 159, 166, 168, 173, 174 }
63	1440	488	90	0	{ 1, 2, 3, 4, 5, 13, 21, 25, 37, 41, 49, 65, 66, 67, 129, 130, 131, 132, 146, 147, 148, 157, 158, 159, 166, 167, 168, 173, 174 }
64	24	360	58	0	{ 1, 2, 3, 9, 11, 13, 18, 19, 37, 49, 66, 69, 129, 130, 131, 133, 135, 144, 146, 147, 148, 155, 156, 159, 167, 173, 174, 175, 179 }
65	240	1888	20	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 34, 35, 37, 41, 65, 129, 130, 131, 132, 133, 134, 143, 148, 155, 156, 157, 158, 165, 166, 173 }
66	4	1744	18	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 15, 17, 19, 20, 23, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 143, 144, 148, 149 }

Nr.	$\|W_K\|$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
67	4	1776	16	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 12, 13, 17, 18, 19, 22, 23, 26, 129, 130, 131, 132, 133, 134, 135, 138, 143, 144, 145, 147, 148 }
68	4	1616	20	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 14, 17, 18, 20, 21, 22, 23, 26, 129, 130, 131, 132, 133, 134, 139, 140, 143, 146, 147, 148, 149 }
69	24	1632	24	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 12, 17, 20, 33, 34, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 146, 155, 156, 160, 165, 169 }
70	192	2264	7	84	{ 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 17, 18, 21, 26, 129, 130, 131, 132, 133, 134, 135, 143, 144, 145, 146, 147, 148, 157, 165 }
71	8	872	42	0	{ 1, 2, 3, 4, 5, 9, 11, 13, 17, 19, 21, 34, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 148, 155, 158, 167, 173, 174 }
72	4	880	44	0	{ 1, 2, 3, 4, 5, 9, 11, 13, 17, 21, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 157, 158, 167, 173, 174 }
73	12	1344	33	0	{ 1, 2, 3, 4, 5, 9, 13, 17, 34, 35, 41, 49, 65, 129, 130, 131, 132, 133, 134, 145, 148, 155, 156, 157, 158, 166, 167, 173, 174 }
74	8	736	48	0	{ 1, 2, 3, 5, 7, 9, 18, 19, 21, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 155, 159, 165, 166, 167, 174, 179 }
75	16	1952	12	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 17, 18, 20, 23, 26, 129, 130, 131, 132, 133, 134, 135, 137, 138, 143, 144, 145, 148 }
76	4	1584	20	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 11, 17, 18, 21, 25, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 147, 148, 157, 165, 173 }
77	8	1592	22	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 11, 17, 19, 21, 25, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 147, 148, 158, 165, 173 }
78	240	1208	40	0	{ 1, 2, 3, 5, 7, 8, 9, 11, 12, 15, 33, 35, 36, 39, 43, 45, 129, 130, 131, 132, 133, 138, 139, 140, 142, 158, 159, 160, 161 }
79	6	1232	34	0	{ 1, 2, 3, 4, 5, 9, 13, 17, 18, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 155, 156, 157, 166, 167, 173, 174 }
80	8	520	53	0	{ 1, 2, 3, 5, 7, 18, 22, 23, 24, 36, 37, 38, 40, 49, 51, 53, 129, 130, 131, 132, 133, 135, 138, 141, 144, 165, 167, 168, 170 }
81	2	520	53	0	{ 1, 2, 3, 5, 7, 18, 19, 22, 24, 34, 36, 37, 40, 49, 51, 53, 129, 130, 131, 132, 133, 138, 139, 141, 144, 167, 168, 169, 170 }
82	4	536	57	0	{ 1, 2, 3, 5, 7, 18, 20, 22, 24, 34, 36, 38, 40, 49, 51, 53, 129, 130, 131, 132, 133, 138, 139, 141, 144, 167, 168, 169, 170 }
83	4	528	55	0	{ 1, 2, 3, 5, 7, 18, 24, 34, 35, 36, 37, 38, 40, 49, 51, 53, 129, 130, 131, 132, 133, 138, 139, 141, 144, 167, 168, 169, 170 }
84	2	528	55	0	{ 1, 2, 3, 5, 7, 18, 23, 24, 35, 36, 37, 38, 40, 49, 51, 53, 129, 130, 131, 132, 133, 138, 141, 144, 165, 167, 168, 169, 170 }
85	4	528	55	0	{ 1, 2, 3, 5, 7, 20, 22, 24, 34, 36, 38, 39, 40, 49, 51, 53, 129, 130, 131, 132, 133, 138, 141, 144, 165, 167, 168, 169, 170 }
86	16	544	59	0	{ 1, 2, 3, 5, 6, 18, 19, 23, 24, 35, 36, 37, 40, 49, 50, 53, 129, 130, 131, 132, 133, 138, 139, 141, 144, 167, 168, 169, 170 }
87	2	536	57	0	{ 1, 2, 3, 4, 5, 21, 22, 23, 24, 37, 38, 39, 40, 49, 50, 51, 129, 130, 131, 132, 133, 135, 138, 141, 144, 165, 167, 168, 170 }
88	20	528	55	0	{ 1, 2, 3, 4, 5, 19, 22, 23, 24, 36, 37, 39, 40, 49, 50, 51, 129, 130, 131, 132, 133, 135, 139, 141, 144, 166, 167, 168, 170 }
89	24	552	61	0	{ 1, 2, 3, 4, 5, 23, 24, 35, 36, 37, 38, 39, 40, 49, 50, 51, 129, 130, 131, 132, 133, 138, 139, 141, 144, 167, 168, 169, 170 }
90	8	560	63	0	{ 1, 2, 3, 4, 5, 21, 22, 23, 24, 37, 38, 39, 40, 49, 50, 51, 129, 130, 131, 132, 133, 138, 139, 141, 144, 167, 168, 169, 170 }
91	240	608	75	0	{ 1, 2, 3, 5, 9, 19, 20, 23, 27, 35, 36, 39, 43, 67, 68, 129, 130, 131, 132, 138, 143, 151, 162, 165, 167, 169, 174, 176, 184 }
92	12	784	49	0	{ 1, 2, 3, 5, 7, 19, 20, 21, 22, 24, 34, 39, 40, 49, 51, 53, 129, 130, 131, 132, 133, 135, 137, 144, 155, 165, 166, 168, 170 }
93	8	600	52	0	{ 1, 2, 3, 5, 11, 18, 19, 25, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 146, 147, 148, 156, 159, 165, 167, 168, 173, 174, 179 }
94	4	608	54	0	{ 1, 2, 3, 5, 10, 11, 13, 25, 35, 49, 65, 66, 129, 130, 131, 132, 133, 144, 145, 147, 148, 156, 157, 158, 159, 168, 173, 174, 179 }
95	240	1568	35	0	{ 1, 2, 3, 5, 8, 9, 12, 14, 15, 17, 20, 22, 23, 26, 27, 29, 129, 130, 131, 132, 133, 134, 137, 139, 140, 141, 146, 148, 149 }
96	12	1344	32	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 25, 34, 35, 37, 41, 65, 129, 130, 131, 132, 133, 134, 143, 148, 156, 157, 158, 165, 166, 167, 173 }
97	8	1520	26	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 34, 35, 37, 41, 65, 129, 130, 131, 132, 133, 134, 143, 148, 155, 156, 158, 165, 166, 167, 173 }
98	12	992	40	0	{ 1, 2, 3, 4, 5, 9, 13, 17, 19, 34, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 148, 155, 156, 158, 166, 167, 173, 174 }
99	2	1632	20	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 17, 25, 34, 35, 65, 129, 130, 131, 132, 133, 134, 143, 144, 145, 148, 156, 157, 158, 165, 167, 173 }
100	2	1632	20	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 19, 25, 34, 35, 41, 65, 129, 130, 131, 132, 133, 134, 143, 145, 148, 156, 158, 165, 166, 173 }
101	4	1808	15	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 13, 14, 17, 18, 21, 22, 23, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 143, 144, 148, 149 }

Nr.	$\|W_K\|$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
102	8	720	48	0	{ 1, 2, 3, 5, 7, 9, 13, 18, 19, 21, 25, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 159, 166, 167, 174, 179 }
103	2	1072	34	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 13, 17, 18, 19, 21, 35, 37, 41, 65, 66, 129, 130, 131, 132, 133, 143, 147, 148, 155, 167, 173 }
104	1	1080	36	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 11, 17, 19, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 147, 148, 155, 156, 158, 165, 173 }
105	2	1072	34	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 11, 13, 17, 19, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 158, 173 }
106	4	1088	38	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 17, 19, 21, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 158, 165, 166 }
107	2	496	56	0	{ 1, 2, 3, 5, 7, 11, 13, 18, 25, 34, 35, 49, 65, 66, 129, 130, 131, 132, 133, 144, 145, 148, 156, 157, 159, 167, 168, 174, 179 }
108	4	488	54	0	{ 1, 2, 3, 5, 7, 13, 18, 25, 35, 49, 65, 66, 129, 130, 131, 132, 133, 144, 145, 147, 148, 156, 157, 159, 166, 167, 168, 174, 179 }
109	8	504	58	0	{ 1, 2, 3, 5, 7, 11, 13, 18, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 157, 159, 167, 168, 174, 179 }
110	4	512	60	0	{ 1, 2, 3, 4, 5, 10, 13, 21, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 157, 158, 159, 166, 168, 173, 174 }
111	4	1376	28	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 18, 21, 35, 41, 65, 66, 129, 130, 131, 132, 133, 143, 145, 147, 148, 155, 157, 165, 166, 167, 173 }
112	12	1376	28	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 17, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 157, 158 }
113	4	1168	36	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 157, 158, 165, 166, 167, 173 }
114	4	736	46	0	{ 1, 2, 3, 5, 7, 9, 11, 21, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 155, 157, 158, 159, 165, 167, 174, 179 }
115	2	744	48	0	{ 1, 2, 3, 5, 7, 9, 21, 34, 35, 41, 49, 66, 129, 130, 131, 132, 133, 135, 145, 148, 155, 157, 158, 159, 165, 166, 167, 174, 179 }
116	8	760	52	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 21, 25, 35, 37, 41, 66, 129, 130, 131, 132, 133, 135, 143, 147, 148, 158, 159, 165, 166, 174, 179 }
117	2	744	48	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 21, 25, 35, 41, 49, 66, 129, 130, 131, 132, 133, 135, 145, 147, 148, 158, 159, 165, 166, 174, 179 }
118	8	1568	22	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 14, 15, 17, 20, 21, 23, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 143, 144, 148, 149 }
119	4	1992	12	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 12, 14, 17, 18, 20, 22, 23, 26, 129, 130, 131, 132, 133, 134, 135, 137, 139, 143, 144, 146, 148 }
120	2	784	45	0	{ 1, 2, 3, 4, 5, 7, 9, 12, 14, 15, 16, 36, 38, 39, 41, 43, 129, 130, 131, 132, 133, 137, 139, 142, 155, 157, 159, 160, 161 }
121	2	784	45	0	{ 1, 2, 3, 5, 7, 9, 13, 14, 16, 34, 37, 38, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 142, 155, 156, 158, 159, 161 }
122	16	792	47	0	{ 1, 2, 3, 5, 7, 9, 12, 16, 34, 35, 36, 38, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 139, 142, 155, 158, 159, 161 }
123	4	792	47	0	{ 1, 2, 3, 5, 7, 9, 10, 15, 16, 35, 36, 37, 38, 41, 43, 45, 129, 130, 131, 132, 133, 135, 139, 142, 155, 157, 158, 159, 161 }
124	4	792	47	0	{ 1, 2, 3, 5, 7, 9, 10, 15, 16, 35, 36, 37, 38, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 140, 143, 156, 158, 159, 161 }
125	24	416	64	0	{ 1, 2, 3, 5, 11, 13, 18, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 157, 159, 167, 168, 173, 174, 179 }
126	4	400	60	0	{ 1, 2, 3, 5, 10, 13, 19, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 158, 159, 166, 168, 173, 174, 179 }
127	16	432	68	0	{ 1, 2, 3, 5, 6, 11, 19, 25, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 147, 148, 156, 158, 159, 165, 167, 168, 173, 179 }
128	4	392	58	0	{ 1, 2, 3, 5, 11, 18, 25, 37, 41, 66, 67, 129, 130, 131, 132, 135, 143, 146, 147, 148, 156, 157, 159, 165, 167, 168, 173, 174, 179 }
129	4	408	62	0	{ 1, 2, 3, 5, 11, 18, 21, 25, 34, 37, 41, 66, 67, 129, 130, 131, 132, 135, 143, 146, 148, 157, 159, 165, 167, 168, 173, 174, 179 }
130	4	408	62	0	{ 1, 2, 3, 5, 11, 18, 21, 25, 35, 37, 41, 66, 67, 129, 130, 131, 132, 135, 143, 147, 148, 157, 159, 165, 167, 168, 173, 174, 179 }
131	16	416	64	0	{ 1, 2, 3, 5, 21, 25, 34, 37, 41, 66, 67, 129, 130, 131, 132, 135, 143, 146, 148, 157, 158, 159, 165, 166, 167, 168, 173, 174, 179 }
132	192	480	80	0	{ 1, 2, 3, 4, 5, 13, 19, 21, 25, 37, 41, 49, 65, 66, 67, 129, 130, 131, 132, 146, 147, 148, 158, 159, 166, 167, 168, 173, 174 }
133	128	416	64	0	{ 1, 2, 3, 4, 9, 10, 21, 25, 37, 41, 49, 67, 69, 129, 130, 131, 134, 135, 146, 147, 148, 157, 158, 159, 165, 166, 173, 174, 175 }
134	64	384	56	0	{ 1, 2, 3, 9, 11, 13, 18, 19, 35, 37, 49, 66, 69, 129, 130, 131, 133, 135, 144, 147, 148, 155, 156, 159, 167, 173, 174, 175, 179 }
135	720	728	60	0	{ 1, 2, 3, 5, 9, 19, 21, 25, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 158, 159, 165, 166, 167, 173, 174, 179 }
136	12	624	60	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 21, 25, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 158, 159, 165, 166, 174, 179 }

Nr.	$\lvert W_K\rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
137	4	592	52	0	{ 1, 2, 3, 5, 7, 9, 18, 21, 25, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 157, 159, 165, 166, 167, 174, 179 }
138	16	608	56	0	{ 1, 2, 3, 5, 11, 13, 25, 49, 65, 66, 129, 130, 131, 132, 133, 144, 145, 146, 147, 148, 156, 157, 158, 159, 167, 168, 173, 174, 179 }
139	24	592	52	0	{ 1, 2, 3, 5, 11, 13, 18, 25, 34, 41, 66, 67, 129, 130, 131, 132, 135, 143, 145, 146, 148, 156, 157, 159, 167, 168, 173, 174, 179 }
140	192	1344	33	0	{ 1, 2, 3, 5, 9, 10, 17, 34, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 148, 155, 156, 157, 158, 165, 166, 173, 174, 179 }
141	20	1808	15	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 20, 33, 42, 66, 129, 130, 131, 132, 133, 137, 143, 145, 146, 149, 155, 156, 160, 165, 166, 173 }
142	8	1960	12	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 17, 18, 19, 23, 26, 129, 130, 131, 132, 133, 134, 137, 143, 144, 145, 146, 148, 149 }
143	16	664	52	0	{ 1, 2, 3, 5, 7, 11, 13, 18, 25, 49, 66, 129, 130, 131, 132, 133, 135, 144, 145, 146, 147, 148, 156, 157, 159, 167, 168, 174, 179 }
144	18	648	48	0	{ 1, 2, 3, 5, 7, 10, 11, 13, 18, 19, 25, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 146, 147, 148, 156, 159, 168, 174, 179 }
145	24	664	52	0	{ 1, 2, 3, 4, 5, 10, 11, 13, 19, 21, 25, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 146, 147, 148, 158, 159, 168, 173, 174 }
146	432	696	60	0	{ 1, 2, 3, 4, 5, 6, 11, 13, 25, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 156, 157, 158, 159, 167, 168, 173 }
147	4	680	52	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 14, 15, 16, 36, 37, 38, 41, 43, 129, 130, 131, 132, 133, 138, 139, 142, 155, 158, 159, 160, 161 }
148	2	664	48	0	{ 1, 2, 3, 4, 5, 7, 9, 12, 13, 14, 16, 34, 38, 39, 41, 43, 129, 130, 131, 132, 133, 138, 139, 142, 155, 158, 159, 160, 161 }
149	6	672	50	0	{ 1, 2, 3, 5, 7, 9, 11, 12, 14, 16, 34, 36, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 137, 139, 142, 155, 157, 159, 161 }
150	4	672	50	0	{ 1, 2, 3, 5, 7, 9, 10, 12, 16, 34, 35, 36, 38, 41, 43, 45, 129, 130, 131, 132, 133, 138, 139, 140, 143, 158, 159, 160, 161 }
151	72	712	60	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 15, 16, 35, 36, 37, 38, 41, 42, 129, 130, 131, 132, 133, 138, 139, 140, 142, 158, 159, 160, 161 }
152	8	760	48	0	{ 1, 2, 3, 5, 7, 9, 11, 13, 19, 21, 34, 41, 49, 66, 129, 130, 131, 132, 133, 135, 145, 146, 148, 155, 158, 159, 167, 174, 179 }
153	48	1704	25	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 12, 14, 17, 18, 20, 22, 23, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 140, 141, 148, 149 }
154	4	1640	18	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 10, 17, 36, 42, 66, 129, 130, 131, 132, 133, 137, 143, 145, 149, 150, 155, 156, 157, 160, 165, 166 }
155	8	1648	20	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 14, 17, 20, 33, 36, 42, 66, 129, 130, 131, 132, 133, 137, 143, 145, 149, 155, 156, 160, 166, 173 }
156	4	1408	24	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 17, 21, 25, 35, 37, 41, 129, 130, 131, 132, 133, 134, 135, 143, 147, 148, 157, 158, 165, 167, 173 }
157	12	1200	34	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 13, 17, 19, 21, 35, 37, 41, 66, 129, 130, 131, 132, 133, 135, 143, 147, 148, 155, 158, 167, 173 }
158	24	1032	41	0	{ 1, 2, 3, 5, 7, 8, 9, 11, 15, 33, 35, 36, 37, 39, 43, 45, 129, 130, 131, 132, 133, 137, 138, 139, 140, 142, 159, 160, 161 }
159	12	1592	22	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 18, 34, 35, 41, 65, 66, 129, 130, 131, 132, 133, 143, 145, 148, 155, 156, 157, 166, 167, 173 }
160	24	1600	24	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 22, 34, 36, 42, 65, 66, 129, 130, 131, 132, 133, 143, 145, 150, 155, 157, 160, 165, 166, 173 }
161	2	1080	37	0	{ 1, 2, 3, 4, 5, 9, 14, 17, 34, 36, 42, 50, 66, 129, 130, 131, 132, 133, 137, 145, 150, 155, 156, 157, 160, 166, 169, 173, 176 }
162	4	1080	37	0	{ 1, 2, 3, 4, 5, 8, 9, 14, 17, 20, 22, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 146, 149, 150, 155, 160, 166, 169, 176 }
163	8	1088	39	0	{ 1, 2, 3, 5, 9, 10, 11, 13, 17, 34, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 148, 155, 156, 157, 158, 173, 174, 179 }
164	12	1072	35	0	{ 1, 2, 3, 5, 9, 12, 14, 17, 18, 20, 22, 36, 38, 50, 66, 129, 130, 131, 132, 133, 137, 144, 149, 150, 155, 169, 173, 176, 181 }
165	72	1088	39	0	{ 1, 2, 3, 5, 7, 8, 9, 11, 12, 16, 34, 35, 36, 39, 43, 45, 129, 130, 131, 132, 133, 135, 137, 138, 142, 155, 156, 159, 161 }
166	4	768	44	0	{ 1, 2, 3, 4, 5, 7, 9, 11, 18, 21, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 147, 148, 155, 157, 159, 165, 167, 174 }
167	2	776	46	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 18, 19, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 147, 148, 155, 156, 159, 166, 167, 174 }
168	8	800	52	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 13, 18, 21, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 155, 157, 159, 166, 174 }
169	2	768	44	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 13, 18, 19, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 147, 148, 155, 156, 159, 166, 174 }
170	12	1168	38	0	{ 1, 2, 3, 5, 8, 9, 12, 14, 17, 18, 20, 22, 36, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 149, 150, 155, 169, 176, 181 }
171	12	1776	19	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 15, 17, 20, 23, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 143, 144, 148, 149 }

Nr.	$\|W_K\|$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
172	14	1288	28	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 17, 18, 19, 35, 37, 41, 66, 129, 130, 131, 132, 133, 135, 143, 147, 148, 155, 156, 165, 167, 173 }
173	48	1304	32	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 17, 18, 19, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 165 }
174	240	1568	30	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 17, 25, 34, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 148, 156, 157, 158, 165, 166, 167 }
175	12	1936	14	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 17, 18, 23, 26, 129, 130, 131, 132, 133, 134, 135, 137, 143, 144, 145, 146, 148 }
176	48	896	50	0	{ 1, 2, 3, 5, 7, 9, 10, 17, 19, 21, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 158, 165, 166, 174, 179 }
177	4	1216	30	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 18, 19, 21, 35, 41, 65, 66, 129, 130, 131, 132, 133, 143, 145, 147, 148, 155, 166, 167, 173 }
178	2	1224	32	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 18, 19, 21, 35, 37, 41, 65, 66, 129, 130, 131, 132, 133, 143, 147, 148, 155, 165, 166, 167, 173 }
179	16	1760	19	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 18, 34, 35, 41, 65, 129, 130, 131, 132, 133, 134, 143, 145, 148, 155, 156, 157, 165, 166, 167, 173 }
180	2	1240	30	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 18, 21, 35, 41, 66, 129, 130, 131, 132, 133, 135, 143, 145, 147, 148, 155, 157, 166, 167, 173 }
181	2	1240	30	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 18, 19, 35, 37, 41, 66, 129, 130, 131, 132, 133, 135, 143, 147, 148, 155, 156, 165, 166, 167, 173 }
182	24	1264	36	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 17, 18, 21, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 155, 157, 166 }
183	4	1248	32	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 17, 18, 19, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 165, 166 }
184	16	1440	25	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 20, 36, 42, 65, 66, 129, 130, 131, 132, 133, 143, 145, 149, 150, 155, 156, 160, 165, 166, 169, 173 }
185	24	1440	25	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 12, 14, 17, 20, 22, 36, 38, 42, 66, 129, 130, 131, 132, 133, 137, 143, 149, 150, 155, 160, 173 }
186	32	1512	26	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 34, 41, 49, 65, 129, 130, 131, 132, 133, 134, 145, 146, 148, 155, 156, 157, 158, 165, 166, 167, 173 }
187	12	1480	27	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 20, 22, 33, 36, 38, 42, 65, 66, 129, 130, 131, 132, 133, 143, 149, 155, 160, 165, 166, 173 }
188	2	1464	23	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 20, 33, 38, 42, 66, 129, 130, 131, 132, 133, 137, 143, 146, 149, 155, 156, 160, 165, 166, 169, 173 }
189	192	808	46	0	{ 1, 2, 3, 4, 9, 10, 13, 17, 18, 21, 25, 33, 34, 35, 41, 49, 67, 69, 129, 130, 131, 134, 135, 145, 157, 166, 173, 174, 175 }
190	12	920	42	0	{ 1, 2, 3, 5, 7, 9, 10, 14, 16, 34, 36, 37, 38, 41, 43, 45, 129, 130, 131, 132, 133, 134, 138, 142, 155, 156, 158, 159, 160 }
191	8	928	44	0	{ 1, 2, 3, 5, 7, 9, 11, 14, 16, 34, 36, 37, 39, 41, 43, 45, 129, 130, 131, 132, 133, 134, 137, 142, 155, 156, 157, 159, 160 }
192	4	920	42	0	{ 1, 2, 3, 5, 7, 9, 12, 16, 34, 35, 36, 38, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 142, 155, 156, 157, 158, 159, 161 }
193	8	928	44	0	{ 1, 2, 3, 5, 7, 9, 10, 11, 15, 16, 35, 36, 37, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 139, 142, 155, 158, 159, 161 }
194	144	1216	36	0	{ 1, 2, 3, 4, 5, 7, 9, 11, 13, 34, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 148, 155, 156, 157, 158, 159, 167, 174 }
195	72	1200	32	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 11, 13, 21, 34, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 148, 155, 157, 158, 159, 174 }
196	8	1408	28	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 19, 25, 34, 35, 37, 41, 65, 129, 130, 131, 132, 133, 134, 143, 148, 156, 158, 165, 166, 173 }
197	2	1400	26	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 25, 34, 35, 37, 41, 129, 130, 131, 132, 133, 134, 135, 143, 148, 156, 158, 165, 166, 167, 173 }
198	24	1392	28	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 21, 35, 37, 41, 65, 129, 130, 131, 132, 133, 134, 143, 147, 148, 155, 158, 165, 166, 167, 173 }
199	4	1032	38	0	{ 1, 2, 3, 4, 5, 7, 9, 11, 13, 17, 21, 34, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 148, 155, 157, 158, 167, 174 }
200	24	1048	42	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 17, 21, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 157, 158, 166, 167, 174 }
201	24	1152	40	0	{ 1, 2, 3, 5, 7, 9, 17, 19, 34, 35, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 148, 155, 156, 158, 165, 166, 167, 174, 179 }
202	4	1424	24	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 11, 17, 21, 25, 34, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 148, 157, 158, 165, 173 }
203	48	1920	16	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 11, 13, 17, 18, 19, 21, 25, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 147, 148, 173 }
204	2	1424	26	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 19, 21, 25, 34, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 148, 158, 165, 166, 173 }
205	2	1416	24	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 17, 19, 21, 25, 34, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 148, 158, 165, 167, 173 }
206	48	1232	40	0	{ 1, 2, 3, 5, 9, 17, 34, 35, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 148, 155, 156, 157, 158, 165, 166, 167, 173, 174, 179 }

Nr.	$\lvert W_K\rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
207	2	920	40	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 19, 21, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 155, 158, 166, 167, 173 }
208	12	944	46	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 17, 18, 21, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 155, 157, 166, 167, 174 }
209	16	928	42	0	{ 1, 2, 3, 4, 5, 7, 9, 11, 13, 17, 18, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 147, 148, 155, 156, 157, 167, 174 }
210	12	1696	22	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 34, 35, 37, 41, 129, 130, 131, 132, 133, 134, 135, 143, 148, 155, 156, 157, 158, 165, 166, 167, 173 }
211	24	520	61	0	{ 1, 2, 3, 5, 11, 21, 25, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 147, 148, 157, 158, 159, 165, 167, 168, 173, 174, 179 }
212	16	496	55	0	{ 1, 2, 3, 4, 9, 10, 18, 21, 25, 33, 37, 41, 49, 67, 69, 129, 130, 131, 134, 135, 146, 147, 157, 159, 165, 166, 173, 174, 175 }
213	8	1072	38	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 21, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 155, 157, 158, 166, 167, 173 }
214	48	1000	45	0	{ 1, 2, 3, 5, 7, 9, 11, 15, 16, 35, 36, 37, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 139, 140, 142, 158, 159, 161 }
215	24	1416	31	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 14, 15, 17, 20, 22, 23, 25, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 143, 144, 148, 149 }
216	48	1368	32	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 18, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 157, 165, 166, 173 }
217	4	1072	36	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 17, 19, 21, 35, 37, 41, 65, 66, 129, 130, 131, 132, 133, 143, 147, 148, 155, 158, 165, 167, 173 }
218	4	1064	34	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 21, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 147, 148, 155, 158, 165, 166, 167, 173 }
219	8	1080	38	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 13, 17, 21, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 155, 157, 158, 166, 167 }
220	20	1568	20	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 34, 35, 41, 129, 130, 131, 132, 133, 134, 135, 143, 145, 148, 155, 156, 158, 165, 166, 167, 173 }
221	384	1376	40	0	{ 1, 2, 3, 5, 8, 9, 12, 14, 17, 20, 22, 26, 36, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 149, 150, 160, 169, 176, 181 }
222	1	1112	33	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 20, 36, 38, 42, 66, 129, 130, 131, 132, 133, 137, 143, 149, 150, 155, 156, 160, 165, 166, 169, 173 }
223	2	1120	35	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 17, 18, 20, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 156, 165, 169, 173 }
224	4	1128	37	0	{ 1, 2, 3, 4, 5, 6, 9, 14, 17, 22, 34, 36, 42, 50, 66, 129, 130, 131, 132, 133, 137, 145, 150, 155, 157, 160, 166, 169, 173 }
225	4	1120	35	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 14, 17, 20, 22, 34, 36, 42, 50, 66, 129, 130, 131, 132, 133, 137, 145, 150, 155, 160, 169, 173 }
226	24	2112	12	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 17, 18, 19, 21, 23, 26, 129, 130, 131, 132, 133, 134, 135, 138, 143, 144, 145, 147, 148 }
227	12	1136	35	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 12, 14, 15, 16, 36, 38, 39, 41, 129, 130, 131, 132, 133, 134, 135, 138, 142, 155, 156, 158, 159 }
228	2	1128	33	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 11, 14, 15, 16, 36, 37, 39, 41, 129, 130, 131, 132, 133, 134, 137, 138, 142, 155, 156, 159, 160 }
229	12	1120	31	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 15, 16, 35, 36, 39, 41, 129, 130, 131, 132, 133, 134, 137, 139, 142, 155, 157, 159, 160 }
230	8	1128	33	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 12, 14, 16, 34, 36, 38, 39, 41, 129, 130, 131, 132, 133, 135, 138, 142, 155, 156, 158, 159, 161 }
231	4	1592	22	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 34, 35, 41, 129, 130, 131, 132, 133, 134, 135, 143, 145, 148, 155, 156, 157, 158, 165, 166, 167, 173 }
232	24	1600	24	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 17, 19, 20, 22, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 140, 143, 148, 149 }
233	2	1080	34	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 18, 19, 35, 37, 41, 65, 66, 129, 130, 131, 132, 133, 143, 147, 148, 155, 156, 166, 167, 173 }
234	12	1104	40	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 19, 21, 35, 37, 41, 65, 66, 129, 130, 131, 132, 133, 143, 147, 148, 155, 158, 165, 166, 173 }
235	16	1080	34	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 13, 17, 18, 19, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 166, 173 }
236	2	1416	27	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 25, 34, 35, 41, 65, 129, 130, 131, 132, 133, 134, 143, 145, 148, 156, 157, 158, 165, 166, 167, 173 }
237	4	1416	27	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 12, 15, 17, 19, 20, 22, 23, 26, 129, 130, 131, 132, 133, 134, 138, 139, 140, 143, 147, 148, 149 }
238	2	808	43	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 11, 14, 16, 34, 36, 37, 41, 43, 129, 130, 131, 132, 133, 138, 139, 142, 155, 158, 159, 160, 161 }
239	4	808	43	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 12, 13, 16, 34, 35, 38, 41, 43, 129, 130, 131, 132, 133, 138, 139, 142, 155, 158, 159, 160, 161 }
240	4	824	47	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 12, 15, 16, 35, 36, 38, 41, 43, 129, 130, 131, 132, 133, 138, 139, 142, 155, 158, 159, 160, 161 }
241	8	808	43	0	{ 1, 2, 3, 4, 5, 7, 9, 11, 13, 14, 16, 34, 37, 39, 41, 43, 129, 130, 131, 132, 133, 137, 138, 142, 155, 156, 159, 160, 161 }

Nr.	$\lvert W_K \rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
242	48	528	62	0	{ 1, 2, 3, 5, 6, 11, 13, 18, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 157, 159, 167, 168, 173, 179 }
243	14	504	56	0	{ 1, 2, 3, 4, 5, 10, 11, 21, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 157, 158, 159, 165, 168, 173, 174 }
244	2	1272	28	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 19, 34, 35, 41, 65, 66, 129, 130, 131, 132, 133, 143, 145, 148, 155, 156, 158, 166, 167, 173 }
245	2	1280	30	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 22, 34, 36, 42, 66, 129, 130, 131, 132, 133, 137, 143, 145, 150, 155, 157, 160, 165, 166, 169, 173 }
246	4	1288	32	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 20, 36, 38, 42, 65, 66, 129, 130, 131, 132, 133, 143, 149, 150, 155, 156, 160, 165, 166, 173 }
247	24	1288	32	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 17, 34, 42, 50, 66, 129, 130, 131, 132, 133, 137, 145, 146, 150, 155, 156, 157, 160, 165, 166, 169 }
248	48	760	55	0	{ 1, 2, 3, 5, 19, 20, 21, 22, 23, 33, 39, 40, 49, 50, 51, 53, 129, 130, 131, 132, 133, 138, 139, 141, 144, 167, 168, 169, 170 }
249	8	888	42	0	{ 1, 2, 3, 5, 7, 9, 10, 11, 13, 19, 21, 34, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 148, 155, 158, 159, 174, 179 }
250	12	896	44	0	{ 1, 2, 3, 5, 7, 9, 11, 13, 21, 34, 35, 41, 49, 66, 129, 130, 131, 132, 133, 135, 145, 148, 155, 157, 158, 159, 167, 174, 179 }
251	2	1288	28	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 14, 17, 18, 20, 33, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 155, 156, 169, 173 }
252	12	1304	32	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 14, 17, 20, 22, 33, 34, 36, 42, 50, 65, 66, 129, 130, 131, 132, 133, 145, 155, 160, 169, 173 }
253	8	1288	28	0	{ 1, 2, 3, 4, 5, 6, 9, 14, 17, 22, 33, 36, 42, 50, 66, 129, 130, 131, 132, 133, 137, 145, 149, 155, 157, 160, 166, 169, 173 }
254	4	1256	32	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 15, 17, 19, 20, 22, 23, 25, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 140, 143, 148, 149 }
255	10	1248	30	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 13, 14, 15, 17, 22, 23, 25, 26, 129, 130, 131, 132, 133, 134, 137, 138, 140, 141, 145, 148, 149 }
256	4	896	42	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 13, 17, 19, 21, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 158, 167, 173 }
257	8	896	42	0	{ 1, 2, 3, 5, 7, 9, 11, 19, 21, 34, 35, 41, 49, 66, 129, 130, 131, 132, 133, 135, 145, 148, 155, 158, 159, 165, 167, 174, 179 }
258	12	2128	10	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 17, 18, 19, 21, 23, 26, 129, 130, 131, 132, 133, 134, 135, 137, 143, 144, 145, 146, 148 }
259	12	744	47	0	{ 1, 2, 3, 5, 7, 9, 12, 14, 15, 16, 36, 38, 39, 41, 43, 45, 129, 130, 131, 132, 133, 137, 138, 139, 142, 155, 159, 160, 161 }
260	4	752	49	0	{ 1, 2, 3, 5, 7, 9, 16, 34, 35, 36, 37, 38, 39, 41, 43, 45, 129, 130, 131, 132, 133, 138, 140, 142, 156, 158, 159, 160, 161 }
261	4	1344	30	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 25, 34, 35, 41, 49, 65, 129, 130, 131, 132, 133, 134, 145, 148, 156, 157, 158, 165, 166, 167, 173 }
262	24	1200	36	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 17, 21, 34, 35, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 148, 155, 157, 158, 166, 167, 174 }
263	8	1584	22	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 18, 25, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 147, 148, 156, 157, 165, 166, 173 }
264	12	1584	22	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 21, 25, 34, 35, 37, 129, 130, 131, 132, 133, 134, 135, 143, 144, 148, 158, 165, 166, 167, 173 }
265	4	760	47	0	{ 1, 2, 3, 5, 7, 9, 11, 14, 15, 16, 36, 37, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 139, 142, 155, 158, 159, 161 }
266	8	776	51	0	{ 1, 2, 3, 5, 7, 9, 10, 15, 16, 35, 36, 37, 38, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 139, 140, 142, 158, 159, 161 }
267	240	1208	40	0	{ 1, 2, 3, 5, 7, 9, 11, 17, 19, 34, 35, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 148, 155, 156, 158, 165, 167, 174, 179 }
268	1920	1608	40	0	{ 1, 2, 3, 5, 9, 17, 34, 35, 37, 41, 49, 65, 129, 130, 131, 132, 133, 134, 148, 155, 156, 157, 158, 165, 166, 167, 173, 174, 179 }
269	2	896	40	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 19, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 158, 166, 167, 173 }
270	2	912	44	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 17, 18, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 157, 166, 167, 174 }
271	2	904	42	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 17, 19, 34, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 148, 155, 156, 158, 166, 167, 174 }
272	2	904	42	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 18, 21, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 147, 148, 155, 157, 159, 166, 167, 174 }
273	12	920	46	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 34, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 148, 155, 156, 158, 159, 165, 166, 174, 179 }
274	4	968	38	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 16, 34, 35, 38, 39, 41, 129, 130, 131, 132, 133, 138, 139, 142, 155, 158, 159, 160, 161 }
275	16	960	36	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 11, 15, 16, 35, 36, 37, 39, 41, 129, 130, 131, 132, 133, 137, 139, 142, 155, 157, 159, 160, 161 }
276	32	992	44	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 15, 16, 35, 36, 37, 38, 41, 129, 130, 131, 132, 133, 138, 139, 142, 155, 158, 159, 160, 161 }

Nr.	$\lvert W_K\rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
277	48	1008	48	0	{ 1, 2, 3, 4, 5, 7, 8, 9, 10, 15, 16, 35, 36, 37, 38, 43, 129, 130, 131, 132, 133, 138, 139, 142, 155, 158, 159, 160, 161 }
278	4	1240	32	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 155, 156, 157, 158, 166, 167, 173 }
279	12	784	46	0	{ 1, 2, 3, 5, 6, 7, 9, 10, 21, 25, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 157, 158, 159, 165, 166, 179 }
280	4	912	42	0	{ 1, 2, 3, 4, 5, 7, 9, 11, 13, 18, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 147, 148, 155, 156, 157, 159, 167, 174 }
281	2	904	40	0	{ 1, 2, 3, 4, 5, 7, 9, 18, 19, 21, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 147, 148, 155, 159, 165, 166, 167, 174 }
282	4	912	42	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 13, 18, 19, 21, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 147, 148, 155, 159, 166, 174 }
283	4	912	42	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 13, 18, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 157, 159, 166, 174 }
284	8	912	42	0	{ 1, 2, 3, 4, 5, 7, 9, 11, 13, 21, 34, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 148, 155, 157, 158, 159, 167, 174 }
285	48	928	46	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 13, 21, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 157, 158, 159, 166, 174 }
286	16	928	46	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 21, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 147, 148, 155, 157, 158, 159, 166, 167, 174 }
287	4	1800	17	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 25, 34, 35, 65, 129, 130, 131, 132, 133, 134, 143, 144, 145, 148, 156, 157, 158, 165, 166, 173 }
288	48	1896	16	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 13, 17, 34, 35, 37, 41, 129, 130, 131, 132, 133, 134, 135, 143, 148, 155, 156, 157, 158, 167, 173 }
289	51840	1728	45	0	{ 1, 2, 3, 4, 10, 11, 12, 18, 19, 20, 28, 34, 35, 36, 44, 52, 72, 129, 130, 131, 135, 137, 138, 145, 156, 165, 175, 177, 178 }
290	16	384	66	0	{ 1, 2, 3, 5, 10, 19, 21, 25, 37, 41, 49, 65, 66, 67, 129, 130, 131, 132, 146, 147, 148, 158, 159, 165, 166, 168, 173, 174, 179 }
291	432	384	66	0	{ 1, 2, 3, 5, 10, 19, 21, 25, 35, 37, 41, 49, 65, 66, 67, 69, 129, 130, 131, 147, 148, 158, 159, 165, 166, 168, 173, 174, 179 }
292	24	368	62	0	{ 1, 2, 3, 4, 9, 10, 18, 21, 25, 35, 37, 41, 49, 65, 67, 69, 129, 130, 131, 134, 147, 148, 157, 159, 165, 166, 173, 174, 175 }
293	18	360	60	0	{ 1, 2, 3, 9, 11, 18, 19, 37, 41, 49, 66, 69, 129, 130, 131, 133, 135, 146, 147, 148, 155, 156, 159, 165, 167, 173, 174, 175, 179 }
294	103680	3528	0	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 17, 18, 19, 21, 25, 129, 130, 131, 132, 133, 134, 135, 143, 144, 145, 146, 147, 148 }
295	2	1064	38	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 19, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 158, 165, 166, 173 }
296	2	1056	36	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 13, 17, 19, 21, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 147, 148, 155, 158, 167, 173 }
297	2	1056	36	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 21, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 155, 157, 158, 165, 166, 167, 173 }
298	2	1768	16	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 17, 18, 23, 26, 129, 130, 131, 132, 133, 134, 135, 138, 143, 144, 145, 147, 148 }
299	8	1776	18	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 14, 17, 18, 20, 21, 23, 26, 129, 130, 131, 132, 133, 134, 135, 138, 143, 144, 145, 147, 148 }
300	2	928	40	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 11, 17, 19, 21, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 158, 165, 173 }
301	4	928	40	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 17, 19, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 147, 148, 155, 156, 158, 165, 167, 173 }
302	4	920	38	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 19, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 147, 148, 155, 156, 158, 166, 167, 173 }
303	8	936	42	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 13, 17, 19, 21, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 155, 158, 167, 173 }
304	1152	2112	13	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 18, 34, 41, 65, 66, 129, 130, 131, 132, 133, 143, 145, 146, 148, 155, 156, 157, 165, 166, 167, 173 }
305	4	1088	35	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 14, 17, 36, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 149, 150, 155, 156, 157, 160, 169, 173 }
306	12	1104	39	0	{ 1, 2, 3, 4, 5, 8, 9, 12, 14, 17, 34, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 146, 150, 155, 156, 157, 160, 169, 176 }
307	24	1096	37	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 13, 16, 34, 35, 37, 38, 41, 43, 129, 130, 131, 132, 133, 139, 142, 155, 157, 158, 159, 160, 161 }
308	16	1096	37	0	{ 1, 2, 3, 4, 5, 7, 9, 11, 13, 14, 16, 34, 37, 39, 41, 43, 129, 130, 131, 132, 133, 142, 155, 156, 157, 158, 159, 160, 161 }
309	4	1104	35	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 17, 20, 22, 36, 38, 42, 66, 129, 130, 131, 132, 133, 137, 143, 149, 150, 155, 160, 165, 169, 173 }
310	12	1112	37	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 20, 22, 36, 38, 42, 65, 66, 129, 130, 131, 132, 133, 143, 149, 150, 155, 160, 165, 166, 169, 173 }
311	2	1104	35	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 14, 17, 36, 42, 50, 65, 66, 129, 130, 131, 132, 133, 145, 149, 150, 155, 156, 157, 160, 169, 173 }

Nr.	$\lvert W_K \rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
312	8	1096	33	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 12, 17, 22, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 157, 160, 165, 169 }
313	4	512	55	0	{ 1, 2, 3, 5, 18, 19, 23, 24, 35, 36, 37, 40, 49, 50, 51, 53, 129, 130, 131, 132, 133, 139, 141, 144, 166, 167, 168, 169, 170 }
314	2	512	55	0	{ 1, 2, 3, 5, 19, 22, 23, 24, 36, 37, 39, 40, 49, 50, 51, 53, 129, 130, 131, 132, 133, 135, 141, 144, 165, 166, 167, 168, 170 }
315	4	528	59	0	{ 1, 2, 3, 5, 20, 22, 23, 24, 36, 38, 39, 40, 49, 50, 51, 53, 129, 130, 131, 132, 133, 135, 141, 144, 165, 166, 167, 168, 170 }
316	8	520	57	0	{ 1, 2, 3, 5, 20, 22, 23, 24, 36, 38, 39, 40, 49, 50, 51, 53, 129, 130, 131, 132, 133, 135, 138, 139, 141, 144, 167, 168, 170 }
317	16	552	65	0	{ 1, 2, 3, 5, 6, 19, 20, 21, 23, 24, 35, 39, 40, 49, 50, 53, 129, 130, 131, 132, 133, 141, 142, 165, 166, 167, 168, 169, 170 }
318	4	520	57	0	{ 1, 2, 3, 5, 11, 15, 20, 27, 35, 36, 39, 67, 68, 129, 130, 131, 132, 138, 143, 144, 151, 156, 160, 162, 167, 171, 174, 176, 184 }
319	48	520	57	0	{ 1, 2, 3, 4, 7, 12, 16, 18, 28, 34, 44, 52, 66, 68, 72, 129, 130, 131, 137, 145, 148, 150, 156, 159, 161, 170, 171, 175, 178 }
320	8	944	40	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 14, 17, 20, 22, 34, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 150, 155, 160, 169, 173 }
321	1	944	40	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 17, 20, 22, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 160, 165, 169, 173 }
322	4	936	38	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 17, 22, 36, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 149, 150, 155, 157, 160, 165, 169, 173 }
323	8	960	44	0	{ 1, 2, 3, 4, 5, 8, 9, 10, 17, 20, 22, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 160, 165, 166, 176 }
324	8	960	44	0	{ 1, 2, 3, 4, 5, 8, 9, 10, 14, 17, 22, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 157, 160, 166, 176 }
325	4	952	42	0	{ 1, 2, 3, 4, 5, 9, 12, 14, 17, 20, 34, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 146, 150, 155, 156, 160, 169, 173, 176 }
326	4	944	40	0	{ 1, 2, 3, 4, 5, 9, 12, 14, 17, 22, 34, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 146, 150, 155, 157, 160, 169, 173, 176 }
327	4	944	40	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 16, 34, 35, 37, 38, 39, 41, 43, 129, 130, 131, 132, 133, 137, 139, 142, 155, 157, 159, 160, 161 }
328	4	944	40	0	{ 1, 2, 3, 4, 5, 7, 9, 14, 16, 34, 36, 37, 38, 39, 41, 43, 129, 130, 131, 132, 133, 135, 139, 142, 155, 157, 158, 159, 161 }
329	4	952	42	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 14, 16, 34, 37, 38, 39, 41, 43, 129, 130, 131, 132, 133, 139, 142, 155, 157, 158, 159, 160, 161 }
330	6	768	46	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 18, 41, 49, 66, 129, 130, 131, 132, 133, 135, 145, 146, 147, 148, 155, 156, 157, 159, 166, 167, 174 }
331	2	760	44	0	{ 1, 2, 3, 4, 5, 7, 9, 18, 21, 41, 49, 66, 129, 130, 131, 132, 133, 135, 145, 146, 147, 148, 155, 157, 159, 165, 166, 167, 174 }
332	4	768	46	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 18, 19, 21, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 155, 159, 166, 167, 174 }
333	1	768	46	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 18, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 147, 148, 155, 156, 157, 159, 166, 167, 174 }
334	2	776	48	0	{ 1, 2, 3, 4, 5, 7, 9, 10, 18, 21, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 157, 159, 165, 166, 174 }
335	4	776	48	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 18, 21, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 157, 159, 166, 167, 174 }
336	4	784	50	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 21, 34, 35, 49, 65, 66, 129, 130, 131, 132, 133, 144, 145, 148, 155, 158, 159, 165, 166, 174, 179 }
337	12	760	44	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 25, 37, 41, 66, 129, 130, 131, 132, 133, 135, 143, 146, 147, 148, 156, 158, 159, 165, 166, 174, 179 }
338	24	1960	16	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 12, 14, 17, 18, 20, 22, 23, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 143, 144, 148, 149 }
339	48	976	43	0	{ 1, 2, 3, 5, 9, 12, 14, 17, 20, 22, 36, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 149, 150, 155, 160, 169, 173, 176, 181 }
340	144	632	60	0	{ 1, 2, 3, 5, 11, 13, 25, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 147, 148, 156, 157, 158, 159, 167, 168, 173, 174, 179 }
341	72	600	52	0	{ 1, 2, 3, 4, 9, 10, 18, 21, 25, 33, 35, 41, 49, 67, 69, 129, 130, 131, 134, 135, 145, 147, 157, 159, 165, 166, 173, 174, 175 }
342	360	288	65	0	{ 1, 2, 3, 5, 10, 11, 18, 21, 25, 35, 37, 41, 49, 65, 66, 67, 69, 129, 130, 131, 147, 148, 157, 159, 165, 168, 173, 174, 179 }
343	40320	448	105	0	{ 1, 2, 3, 4, 13, 21, 25, 37, 41, 49, 65, 66, 67, 129, 130, 131, 132, 146, 147, 148, 157, 158, 159, 166, 167, 168, 173, 174, 175 }
344	384	320	73	0	{ 1, 2, 3, 4, 9, 13, 18, 21, 25, 35, 37, 41, 49, 65, 66, 67, 69, 129, 130, 131, 147, 148, 157, 159, 166, 167, 173, 174, 175 }
345	96	288	65	0	{ 1, 2, 3, 11, 13, 18, 21, 35, 41, 49, 65, 66, 69, 129, 130, 131, 133, 145, 147, 148, 155, 157, 159, 167, 168, 173, 174, 175, 179 }
346	64	864	44	0	{ 1, 2, 3, 5, 7, 9, 11, 14, 15, 33, 36, 37, 39, 41, 43, 45, 129, 130, 131, 132, 133, 137, 138, 139, 140, 142, 159, 160, 161 }

Nr.	$\lvert W_K \rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
347	72	1792	18	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 17, 18, 25, 35, 37, 129, 130, 131, 132, 133, 134, 135, 143, 144, 147, 148, 156, 157, 165, 166 }
348	8	1256	32	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 17, 18, 21, 35, 41, 66, 129, 130, 131, 132, 133, 135, 143, 145, 147, 148, 155, 157, 165, 167, 173 }
349	4	1240	28	0	{ 1, 2, 3, 4, 5, 6, 9, 13, 17, 18, 19, 35, 41, 66, 129, 130, 131, 132, 133, 135, 143, 145, 147, 148, 155, 156, 166, 167, 173 }
350	4	1248	30	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 11, 17, 18, 19, 21, 35, 37, 41, 65, 66, 129, 130, 131, 132, 133, 143, 147, 148, 155, 165, 173 }
351	72	1600	30	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 34, 35, 37, 41, 49, 65, 129, 130, 131, 132, 133, 134, 148, 155, 156, 157, 158, 165, 166, 167, 173 }
352	12	728	52	0	{ 1, 2, 3, 5, 7, 9, 18, 19, 21, 25, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 159, 165, 166, 167, 174, 179 }
353	2	640	50	0	{ 1, 2, 3, 5, 7, 9, 12, 14, 15, 16, 36, 38, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 138, 139, 142, 155, 158, 159, 161 }
354	4	640	50	0	{ 1, 2, 3, 5, 7, 9, 12, 15, 16, 35, 36, 38, 39, 41, 43, 45, 129, 130, 131, 132, 133, 135, 139, 142, 155, 157, 158, 159, 161 }
355	24	680	60	0	{ 1, 2, 3, 5, 7, 9, 10, 15, 16, 35, 36, 37, 38, 41, 43, 45, 129, 130, 131, 132, 133, 138, 139, 142, 155, 158, 159, 160, 161 }
356	2	656	54	0	{ 1, 2, 3, 5, 7, 9, 10, 15, 16, 35, 36, 37, 38, 41, 43, 45, 129, 130, 131, 132, 133, 137, 138, 139, 140, 142, 159, 160, 161 }
357	8	648	52	0	{ 1, 2, 3, 5, 7, 18, 20, 23, 24, 35, 36, 38, 40, 49, 51, 53, 129, 130, 131, 132, 133, 135, 138, 141, 144, 165, 167, 168, 170 }
358	4	640	50	0	{ 1, 2, 3, 5, 7, 20, 24, 34, 35, 36, 38, 39, 40, 49, 51, 53, 129, 130, 131, 132, 133, 141, 144, 165, 166, 167, 168, 169, 170 }
359	720	848	60	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 19, 21, 25, 35, 37, 41, 65, 66, 129, 130, 131, 132, 133, 143, 147, 148, 158, 159, 165, 166, 173 }
360	2	1048	36	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 35, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 147, 148, 155, 156, 158, 165, 166, 167, 173 }
361	2	1056	38	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 21, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 158, 165, 166, 167, 173 }
362	12	1072	42	0	{ 1, 2, 3, 4, 5, 7, 9, 17, 19, 34, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 148, 155, 156, 158, 165, 166, 167, 174 }
363	2	1056	38	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 17, 34, 37, 41, 49, 66, 129, 130, 131, 132, 133, 135, 146, 148, 155, 156, 157, 158, 166, 167, 174 }
364	8	1056	38	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 34, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 148, 155, 156, 157, 158, 159, 166, 167, 174 }
365	4	1232	30	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 25, 35, 37, 41, 129, 130, 131, 132, 133, 134, 135, 143, 147, 148, 156, 157, 158, 165, 166, 167, 173 }
366	8	1240	32	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 19, 21, 25, 34, 35, 37, 41, 65, 129, 130, 131, 132, 133, 134, 143, 148, 158, 165, 166, 167, 173 }
367	4	480	54	0	{ 1, 2, 3, 5, 7, 10, 13, 25, 35, 49, 65, 66, 129, 130, 131, 132, 133, 144, 145, 147, 148, 156, 157, 158, 159, 166, 168, 174, 179 }
368	8	496	58	0	{ 1, 2, 3, 5, 7, 10, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 157, 158, 159, 165, 166, 168, 174, 179 }
369	4	496	58	0	{ 1, 2, 3, 5, 11, 13, 18, 19, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 159, 167, 168, 173, 174, 179 }
370	2	488	56	0	{ 1, 2, 3, 5, 10, 11, 21, 25, 35, 49, 65, 66, 129, 130, 131, 132, 133, 144, 145, 147, 148, 157, 158, 159, 165, 168, 173, 174, 179 }
371	2	496	58	0	{ 1, 2, 3, 5, 10, 19, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 158, 159, 165, 166, 168, 173, 174, 179 }
372	4	512	62	0	{ 1, 2, 3, 4, 5, 13, 25, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 156, 157, 158, 159, 166, 167, 168, 173, 174 }
373	24	544	70	0	{ 1, 2, 3, 5, 6, 11, 19, 25, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 156, 158, 159, 165, 167, 168, 173, 179 }
374	48	496	62	0	{ 1, 2, 3, 5, 11, 21, 25, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 157, 158, 159, 165, 167, 168, 173, 174, 179 }
375	4	1008	42	0	{ 1, 2, 3, 4, 5, 9, 13, 17, 21, 34, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 148, 155, 157, 158, 166, 167, 173, 174 }
376	2	1592	20	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 17, 18, 19, 23, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 143, 144, 148, 149 }
377	12	1600	22	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 17, 18, 23, 26, 129, 130, 131, 132, 133, 134, 138, 139, 143, 144, 147, 148, 149 }
378	16	1600	22	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 17, 19, 25, 34, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 148, 156, 158, 165, 166 }
379	4	1104	35	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 17, 18, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 146, 149, 150, 155, 156, 157, 165, 169, 173 }
380	8	1104	35	0	{ 1, 2, 3, 4, 5, 6, 9, 14, 17, 20, 34, 42, 50, 65, 66, 129, 130, 131, 132, 133, 145, 146, 150, 155, 156, 160, 166, 169, 173 }
381	8	1288	28	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 14, 16, 34, 36, 39, 41, 129, 130, 131, 132, 133, 134, 135, 139, 142, 155, 157, 158, 159 }

Nr.	$\|W_K\|$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
382	192	1568	24	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 34, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 148, 155, 156, 157, 158, 165, 166, 167, 173 }
383	4	1064	37	0	{ 1, 2, 3, 4, 5, 9, 12, 14, 17, 20, 34, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 150, 155, 156, 160, 169, 173, 176 }
384	4	1072	39	0	{ 1, 2, 3, 4, 5, 9, 14, 17, 22, 34, 36, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 150, 155, 157, 160, 166, 169, 173, 176 }
385	24	1160	40	0	{ 1, 2, 3, 5, 8, 9, 12, 17, 20, 22, 36, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 149, 150, 155, 160, 165, 169, 176, 181 }
386	12	1784	17	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 11, 17, 34, 49, 65, 129, 130, 131, 132, 133, 134, 144, 145, 146, 148, 155, 156, 157, 158, 165, 173 }
387	4	768	48	0	{ 1, 2, 3, 4, 5, 7, 9, 13, 18, 21, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 147, 148, 155, 157, 159, 166, 167, 174 }
388	2	760	46	0	{ 1, 2, 3, 4, 5, 7, 9, 18, 21, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 146, 147, 148, 155, 157, 159, 165, 166, 167, 174 }
389	144	816	60	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 21, 35, 37, 49, 65, 66, 129, 130, 131, 132, 133, 144, 147, 148, 155, 158, 159, 165, 166, 174, 179 }
390	12	752	44	0	{ 1, 2, 3, 5, 7, 9, 11, 18, 19, 21, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 155, 159, 165, 167, 174, 179 }
391	60	768	50	0	{ 1, 2, 3, 4, 5, 10, 13, 21, 25, 49, 65, 66, 129, 130, 131, 132, 133, 144, 145, 146, 147, 148, 157, 158, 159, 166, 168, 173, 174 }
392	4	1824	15	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 10, 17, 20, 34, 36, 42, 66, 129, 130, 131, 132, 133, 137, 143, 145, 150, 155, 156, 160, 165, 166 }
393	20	1008	40	0	{ 1, 2, 3, 5, 7, 9, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 155, 156, 157, 158, 159, 165, 166, 167, 174, 179 }
394	6	1568	24	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 25, 34, 35, 37, 65, 129, 130, 131, 132, 133, 134, 143, 144, 148, 156, 157, 158, 165, 166, 173 }
395	2	1560	22	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 17, 21, 34, 35, 37, 41, 129, 130, 131, 132, 133, 134, 135, 143, 148, 155, 157, 158, 165, 167, 173 }
396	144	1128	45	0	{ 1, 2, 3, 4, 5, 8, 9, 12, 17, 20, 34, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 146, 150, 155, 156, 160, 165, 169, 176 }
397	16	1096	37	0	{ 1, 2, 3, 4, 5, 8, 9, 14, 17, 22, 34, 38, 42, 50, 66, 129, 130, 131, 132, 133, 137, 146, 150, 155, 157, 160, 166, 169, 176 }
398	8	1448	23	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 14, 17, 20, 34, 36, 38, 42, 65, 66, 129, 130, 131, 132, 133, 143, 150, 155, 156, 160, 166, 169 }
399	2	1456	25	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 20, 22, 33, 36, 38, 42, 66, 129, 130, 131, 132, 133, 137, 143, 149, 155, 160, 165, 166, 169, 173 }
400	2	1456	25	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 17, 22, 33, 36, 38, 42, 66, 129, 130, 131, 132, 133, 137, 143, 149, 155, 157, 160, 165, 169, 173 }
401	2	1448	23	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 20, 34, 36, 42, 66, 129, 130, 131, 132, 133, 137, 143, 145, 150, 155, 156, 160, 165, 166, 169, 173 }
402	128	2136	8	112	{ 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 17, 18, 22, 26, 129, 130, 131, 132, 133, 134, 135, 143, 144, 145, 146, 147, 148, 157, 165 }
403	192	2344	6	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 17, 18, 20, 22, 26, 129, 130, 131, 132, 133, 134, 135, 143, 144, 145, 146, 147, 148 }
404	48	1800	17	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 17, 34, 35, 49, 65, 129, 130, 131, 132, 133, 134, 144, 145, 148, 155, 156, 157, 158, 165 }
405	12	832	46	0	{ 1, 2, 3, 5, 7, 9, 18, 19, 21, 25, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 147, 148, 159, 165, 166, 167, 174, 179 }
406	24	848	50	0	{ 1, 2, 3, 5, 7, 9, 11, 19, 21, 25, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 158, 159, 165, 167, 174, 179 }
407	384	2696	4	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 17, 18, 19, 21, 26, 129, 130, 131, 132, 133, 134, 135, 143, 144, 145, 146, 147, 148 }
408	2	752	48	0	{ 1, 2, 3, 5, 7, 9, 10, 21, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 155, 157, 158, 159, 165, 166, 174, 179 }
409	4	768	52	0	{ 1, 2, 3, 5, 7, 9, 10, 19, 21, 34, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 148, 155, 158, 159, 165, 166, 174, 179 }
410	24	752	48	0	{ 1, 2, 3, 5, 7, 9, 10, 18, 19, 21, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 147, 148, 155, 159, 165, 166, 174, 179 }
411	2	744	46	0	{ 1, 2, 3, 5, 7, 9, 11, 13, 19, 21, 34, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 148, 155, 158, 159, 167, 174, 179 }
412	2	1264	30	0	{ 1, 2, 3, 4, 5, 6, 9, 12, 17, 22, 34, 36, 42, 65, 66, 129, 130, 131, 132, 133, 143, 145, 150, 155, 157, 160, 165, 169, 173 }
413	2	1264	30	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 20, 22, 34, 36, 42, 65, 66, 129, 130, 131, 132, 133, 143, 145, 150, 155, 160, 165, 166, 169, 173 }
414	4	1264	30	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 12, 17, 18, 20, 22, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 165, 169 }
415	8	1272	32	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 18, 20, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 156, 165, 166, 173 }
416	4	1752	18	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 17, 18, 20, 23, 26, 129, 130, 131, 132, 133, 134, 137, 138, 139, 143, 144, 148, 149 }

Nr.	$\lvert W_K\rvert$	$\#K_5(1)$	$\#K_4^a(0)$	$\#K_4^b(0)$	K
417	8	800	45	0	{ 1, 2, 3, 4, 5, 9, 10, 14, 17, 20, 22, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 160, 166, 173, 176 }
418	16	800	45	0	{ 1, 2, 3, 4, 5, 9, 12, 14, 17, 20, 22, 34, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 146, 150, 155, 160, 169, 173, 176 }
419	32	800	45	0	{ 1, 2, 3, 5, 7, 18, 19, 20, 22, 24, 34, 36, 40, 49, 51, 53, 129, 130, 131, 132, 133, 138, 141, 144, 165, 167, 168, 169, 170 }
420	192	832	53	0	{ 1, 2, 3, 5, 6, 18, 19, 20, 23, 24, 35, 36, 40, 49, 50, 53, 129, 130, 131, 132, 133, 138, 141, 144, 165, 167, 168, 169, 170 }
421	12	632	52	0	{ 1, 2, 3, 5, 19, 20, 21, 23, 33, 35, 39, 40, 49, 50, 51, 53, 129, 130, 131, 132, 133, 137, 138, 139, 141, 144, 168, 169, 170 }
422	72	664	60	0	{ 1, 2, 3, 4, 5, 19, 20, 21, 22, 23, 33, 39, 40, 49, 50, 51, 129, 130, 131, 132, 133, 138, 139, 141, 142, 167, 168, 169, 170 }
423	4	1392	26	0	{ 1, 2, 3, 4, 5, 6, 9, 11, 13, 17, 18, 19, 21, 35, 41, 66, 129, 130, 131, 132, 133, 135, 143, 145, 147, 148, 155, 167, 173 }
424	8	1400	28	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 17, 18, 19, 35, 37, 41, 65, 66, 129, 130, 131, 132, 133, 143, 147, 148, 155, 156, 165, 166, 173 }
425	2	1216	32	0	{ 1, 2, 3, 4, 5, 6, 9, 10, 11, 17, 35, 37, 41, 49, 65, 66, 129, 130, 131, 132, 133, 147, 148, 155, 156, 157, 158, 165, 173 }
426	16	1216	32	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 10, 17, 19, 21, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 146, 147, 148, 155, 158, 165, 166 }
427	4	1216	32	0	{ 1, 2, 3, 4, 5, 6, 7, 9, 17, 19, 21, 35, 41, 49, 65, 66, 129, 130, 131, 132, 133, 145, 147, 148, 155, 158, 165, 166, 167 }
428	12	1760	16	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 17, 18, 23, 26, 129, 130, 131, 132, 133, 134, 135, 137, 138, 139, 143, 144, 148 }
430	16	1520	28	0	{ 1, 2, 3, 4, 5, 6, 9, 17, 34, 35, 41, 49, 65, 129, 130, 131, 132, 133, 134, 145, 148, 155, 156, 157, 158, 165, 166, 167, 173 }
431	2	904	42	0	{ 1, 2, 3, 4, 5, 9, 12, 14, 17, 22, 34, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 150, 155, 157, 160, 169, 173, 176 }
432	6	920	46	0	{ 1, 2, 3, 4, 5, 9, 14, 17, 22, 36, 38, 42, 50, 65, 66, 129, 130, 131, 132, 133, 149, 150, 155, 157, 160, 166, 169, 173, 176 }
433	120	1888	20	0	{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 15, 17, 19, 21, 23, 26, 129, 130, 131, 132, 133, 134, 135, 138, 139, 143, 144, 147, 148 }