Rational points on varieties, part II (surfaces)

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1. Picard group and canonical divisor

- Cartier divisors [4, Section II.6], [5, Section A2.2].
- Moving Lemma [5, Lemma A2.2.5].
- Morphism $f\colon X \to Y$ of varieties induces homomorphism $f^*\colon \operatorname{Pic} Y \to \operatorname{Pic} X$ [5, A2.2.6].
- Maps to projective space [4, Section II.7], [5, Section A3].
- Linear systems [4, Section II.7], [5, Section A3].
- Criterion for φ_L being a morphism in terms of linear system L [4, Lemma II.7.8 and Remark II.7.8.1], [5, Theorem A3.1.6 (read base points instead of fixed components)].
- Definitions of ample and very ample [5, Section A3.2].

Exercises

(1) Let $\varphi\colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ be an automorphism. Show that φ is linear, i.e., there is a linear map $\psi \in \mathrm{GL}_{n+1}(k)$ such that the induced automorphism on $(k^{n+1} - \{0\})/k^*$ coincides with φ.

(2) Let C be a smooth projective curve (irreducible) of genus 4. Let K be a canonical divisor on C. Assume that K is very ample, which is equivalent to C not being hyperelliptic (see [4, Proposition IV.5.2], [5, Exercise A4.2]). Show that the complete linear system $|K|$ embeds C as the complete intersection of a quadric and a cubic surface in $\mathbb{P}^3$. [Hint: use Riemann-Roch to compute the dimensions $\ell(K), \ell(2K), \ell(3K)$.]

(3) Let C be the image of the morphism

$$\mathbb{P}^1 \to \mathbb{P}^3,\ [s:t] \mapsto [s^3 : s^2t : st^2 : t^3].$$

Show that the ideal $I(C)$ associated to C can not be generated by two elements, i.e., show that C is not a complete intersection.

2. Next week

- Criterion for φ_L being a closed immersion in terms of linear system L [4, Remark II.7.8.2], [5, Theorem A3.2.1].
- Kodaira dimension [4, Section V.6], [5, Section F5.1].
- Classification of surfaces [4, Section V.6], [5, F5.1].
- General type or very canonical [4, Section V.6], [5, F5.2], [9, Section I.2].
- Bombieri–Lang conjecture [5, Section F5.2], [9, Section I.3].
- Extended moving lemma and intersection numbers constant within divisor classes [5, A2.3.1].
- Intersection pairing on $\operatorname{Pic} X$ when X is normal and projective surface [4, Theorem V.1.1], [5, Section A2.3], [6, Appendix B].
- Self intersection: $C \cdot D = \deg_C \mathcal{L}(D) \otimes \mathcal{O}_C$ restricted to $C = D$ [4, Lemma V.1.3].
- $X \subset \mathbb{P}^n$ a surface, $H \in \operatorname{Div} X$ a hyperplane section, $C \subset X$ a curve. Then $H^2 = H \cdot H = \deg X$ [5, A2.3], and $H \cdot C = \deg C$. [4, Exercise V.1.2].
- Adjunction formula $2g(C) - 2 = C \cdot (C + K_X)$ for smooth curve C on smooth projective surface X [4, Proposition V.1.5], [5, Theorem A4.6.2].
- Riemann-Roch for surfaces [4, Theorem V.1.6], [5, Theorem A4.6.3].
- Kodaira Vanishing [4, Remark II.7.15, Exercise V.4.12], [5, Remark A4.6.3.2].

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