DENSITY OF RATIONAL POINTS ON DEL PEZZO SURFACES OF DEGREE ONE

CECÍLIA SALGADO AND RONALD VAN LUIJK

Abstract. We state conditions under which the set $S(k)$ of k-rational points on a del Pezzo surface S of degree 1 over an infinite field k of characteristic not equal to 2 or 3 is Zariski dense. For example, it suffices to require that the elliptic fibration $S \dashrightarrow \mathbb{P}^1$ induced by the anticanonical map has a nodal fiber over a k-rational point of $\mathbb{P}^1$. It also suffices to require the existence of a point in $S(k)$ that does not lie on six exceptional curves of S and that has order 3 on its fiber of the elliptic fibration. This allows us to show that within a parameter space for del Pezzo surfaces of degree 1 over $\mathbb{R}$, the set of surfaces S defined over $\mathbb{Q}$ for which the set $S(\mathbb{Q})$ is Zariski dense, is dense with respect to the real analytic topology. We also include conditions that may be satisfied for every del Pezzo surface S and that can be verified with a finite computation for any del Pezzo surface S that does satisfy them.

1. Introduction

A del Pezzo surface over a field k is a smooth, projective, geometrically integral surface S over k with ample anticanonical divisor $-K_S$; the degree of S is defined to be the self-intersection number $d = K_S^2 \geq 1$. A del Pezzo surface is minimal if and only if there is no birational morphism over its ground field to a del Pezzo surface of higher degree. Every del Pezzo surface of degree d is geometrically isomorphic to $\mathbb{P}^2$ blown up at $9-d$ points in general position, or to $\mathbb{P}^1 \times \mathbb{P}^1$ if $d = 8$. Conversely, every smooth, projective surface that is geometrically birationally equivalent to $\mathbb{P}^2$ is birationally equivalent over the ground field to a del Pezzo surface or a conic bundle (see [12]).

A surface S over a field k is unirational if there is a dominant rational map $\mathbb{P}^2 \dashrightarrow S$ over k. Segre and Manin proved that every del Pezzo surface S of degree $d \geq 2$ over a field k with a k-rational point is unirational, at least if one assumes that the point is in general position in the case $d = 2$. For references, see [29, 30] for $d = 3$ and $k = \mathbb{Q}$, see [20, Theorem 29.4 and 30.1] for $d = 2$ and $d \geq 5$, as well as $d = 3, 4$ under the assumption that k is large enough, and see [16, Theorem 1.1] and [28, Proposition 5.19] for $d = 3$ and $d = 4$ in general. On the other hand, even though del Pezzo surfaces of degree 1 always have a rational point, we do not know whether any minimal del Pezzo surface of degree 1 that is not birationally equivalent to a conic bundle is unirational over its ground field. If k is infinite, then unirationality of S implies that the set $S(k)$ of k-rational points on S is Zariski dense. The following question asks whether this weaker property may hold for all del Pezzo surfaces of degree 1.

Question 1.1. *If S is a del Pezzo surface of degree 1 over an infinite field k, is the set $S(k)$ of k-rational points Zariski dense in S?*

Over number fields, a positive answer to this question is implied by the conjecture by Colliot-Thélène and Sansuc that the Brauer–Manin obstruction to weak approximation is the only one for geometrically rational varieties [3, Conjecture d), p. 319]. This conjecture may in fact hold more generally for geometrically rationally connected varieties over global fields (see [4, p. 3] for number fields).

The primary goal of this paper is to state conditions under which the answer to Question 1.1 is positive.

Let k be a field of characteristic not equal to 2 or 3, and S a del Pezzo surface of degree 1 over k with a canonical divisor K_S. Then the linear system $|-3K_S|$ induces an embedding of S in the weighted projective space $\mathbb{P}(2,3,1,1)$ with coordinates x, y, z, w. More precisely, there are homogeneous polynomials $f, g \in k[z,w]$ of degrees 4 and 6, respectively, such that S is isomorphic to the smooth sextic in $\mathbb{P}(2,3,1,1)$ given by

$$y^2 = x^3 + f(z,w)x + g(z,w). \tag{1}$$

For some special families of del Pezzo surfaces of degree 1 it is known that the set of rational points is Zariski dense. Examples that are minimal and have no conic bundle structure include those given by A. Várilly-Alvarado. He proves in [40, Theorem 2.1] that if we have $k = \mathbb{Q}$, while f is zero and g satisfies some technical conditions, then the set of $\mathbb{Q}$-rational points on the surface S given by (1) is Zariski dense if one also assumes that Tate–Shafarevich groups of elliptic curves over $\mathbb{Q}$ with j-invariant 0 are finite (cf. Example 7.3). These technical conditions are satisfied if $g = az^6 + bw^6$ for nonzero integers $a, b \in \mathbb{Z}$ with $3ab$ not a square, or with a and b relatively prime and $9 \nmid ab$ [40, Theorem 1.1].

M. Ulas [36, 37], as well as M. Ulas and A. Togbé [38, Theorem 2.1], also give various conditions on the homogeneous polynomials $f, g \in \mathbb{Q}[z,w]$ for the set of rational points on the surface $S \subset \mathbb{P}(2,3,1,1)$ over $\mathbb{Q}$ given by (1) to be Zariski dense. Besides hypotheses that imply that S is not smooth or not minimal, all their conditions imply that either (i) $f = 0$ and $g(t,1)$ is monic, or (ii) $g(t,1)$ has degree at most 4, or (iii) $f = 0$ and g vanishes on a rational point of $\mathbb{P}^1$. E. Jabara generalizes Ulas' work on case (iii) in [14, Theorems C and D] and treats the case over $\mathbb{Q}$ with $g(t,1)$ monic and the pair (f,g) sufficiently general.

The techniques in this paper are a generalization of a geometric interpretation of Ulas' work on case (iii); they are independent of the work of Jabara (see Remark 2.7). The projection $\varphi\colon \mathbb{P}(2,3,1,1) \dashrightarrow \mathbb{P}^1$ onto the last two coordinates is a morphism on the complement U of the line given by $z = w = 0$ in $\mathbb{P}(2,3,1,1)$. For any point $Q \in S(k)$ not equal to $\mathcal{O} = (1:1:0:0)$, we let $\mathcal{C}_Q(5)$ denote the family of sections of $U \to \mathbb{P}^1$ that meet S at Q with multiplicity at least 5; we will see that $\mathcal{C}_Q(5)$ has the structure of an affine curve, the components of which have genus at most 1 (see paragraph containing (8)).

The restriction $\varphi|_S\colon S \dashrightarrow \mathbb{P}^1$ corresponds to the linear system $|-K_S|$ and has a unique base point $\mathcal{O} \in S$. This map $\varphi|_S$ induces an elliptic fibration $\pi\colon \mathcal{E} \to \mathbb{P}^1$ of the blow-up $\mathcal{E}$ of S at $\mathcal{O}$. The exceptional curve on $\mathcal{E}$ above $\mathcal{O}$ is a section, also denoted by $\mathcal{O}$. For any $t = (z_0 : w_0) \in \mathbb{P}^1$, the fiber $\mathcal{E}_t$ is isomorphic to the intersection of S with the plane H_t given by $w_0 z = z_0 w$; the set $\mathcal{E}_t^{\mathrm{ns}}(k)$ of smooth k-points on $\mathcal{E}_t$ naturally carries a group structure characterized by the property that three points in $H_t \cap S$ sum to the identity $\mathcal{O}$ if and only if they are collinear. Our first main result is the following.

Theorem 1.2. *Let k be an infinite field of characteristic not equal to 2 or 3. Let $S \subset \mathbb{P}(2,3,1,1)$ be a del Pezzo surface given by (1) with $f, g \in k[z,w]$, and $\pi\colon \mathcal{E} \to \mathbb{P}^1$ the elliptic fibration induced by the anticanonical map $S \dashrightarrow \mathbb{P}^1$. Let $Q \in S(k)$ be a point that is not fixed by the automorphism of S that changes the sign of y. Let $\mathcal{C}_Q(5)$ be the curve of those sections of the projection $U \to \mathbb{P}^1$ that meet S at the point Q with multiplicity at least 5. Set $t = \pi(Q)$. Suppose that the following statements hold.*

- *The order of Q in $\mathcal{E}_t^{\mathrm{ns}}(k)$ is at least 3.*
- *If the order of Q in $\mathcal{E}_t^{\mathrm{ns}}(k)$ is at least 4, then $\mathcal{C}_Q(5)$ has infinitely many k-points.*
- *If the characteristic of k equals 5, then the order of Q in $\mathcal{E}_t^{\mathrm{ns}}(k)$ is not 5.*
- *If the order of Q in $\mathcal{E}_t^{\mathrm{ns}}(k)$ is 3 or 5, then Q does not lie on six (-1)-curves of S.*

Then the set $S(k)$ of k-points on S is Zariski dense in S.

Note that all four assumptions of Theorem 1.2 are hypotheses on the point Q. Given S, we provide an explicit zero-dimensional scheme of which the points correspond to the (-1)-curves of S going through Q (cf. Remark 2.6), so the first and the last two conditions of Theorem 1.2 are easy to check. If the set $S(k)$ is indeed Zariski dense in S, then the subset of those points $Q \in S(k)$ that satisfy these three conditions is also dense; Theorem 1.2 provides a proof of Zariski density of $S(k)$ as soon as $\mathcal{C}_Q(5)(k)$ is infinite for one of these points Q. If the answer to Question 1.1 is positive, then it may be true that for *every* del Pezzo surface S of degree 1, there exists such a point. Theorem 1.2 is the first result that states sufficient conditions for the set of rational points on an arbitrary del Pezzo surface of degree 1 to be Zariski dense.

Moreover, if k is an infinite field that is finitely generated over its ground field, then $\mathcal{C}_Q(5)(k)$ is infinite if and only if the curve $\mathcal{C}_Q(5)$ has a component that is birationally equivalent to $\mathbb{P}^1$ or a component of genus 1 whose Jacobian has a point of infinite order. The fact that the order of a point on an elliptic curve over such a field k is infinite is effectively verifiable by applying the Theorem of Nagell-Lutz to sufficiently many multiples of the point. This means that for such fields k, independent of Question 1.1, if $S(k)$ contains a point Q satisfying the conditions of Theorem 1.2, then we can find such a point, thus reducing the verification of Zariski density of $S(k)$ to a finite computation.

Note that if the order of Q in $\mathcal{E}_0^{\mathrm{ns}}(k)$ is 3 and Q does not lie on six (-1)-curves of S, then the assumptions in Theorem 1.2 are automatically satisfied without any further condition on $\mathcal{C}_Q(5)$. Besides verifying Zariski density of rational points on explicit surfaces, Theorem 1.2 also implies the following two results. Note that both show that our criterion is strong enough to prove Zariski density of the set of rational points on a set of del Pezzo surfaces of degree 1 over $\mathbb{Q}$ that is dense in the real analytic topology on the moduli space of such surfaces.

Theorem 1.3. *Let $f_0, \dots, f_4, g_0, \dots, g_6 \in \mathbb{Q}$ be such that the surface $S \in \mathbb{P}(2,3,1,1)$ given by*

$$y^2 = x^3 + \left(\sum_{i=0}^{4} f_i z^i w^{4-i} \right) x + \sum_{j=0}^{6} g_j z^j w^{6-j} = 0 \tag{2}$$

is smooth. Then for each $\ell \in \{0, \dots, 4\}$, $m \in \{0, \dots, 6\}$, and $\varepsilon > 0$, there exist $\lambda, \mu \in \mathbb{Q}$ with $|\lambda - f_\ell| < \varepsilon$ and $|\mu - g_m| < \varepsilon$ such that the surface $S' \in \mathbb{P}(2,3,1,1)$ given by (2) with the two values f_ℓ and g_m replaced by λ and μ, respectively, is smooth and the set $S'(\mathbb{Q})$ is Zariski dense in S'.

Theorem 1.4. *Suppose k is an infinite field of characteristic not equal to 2 or 3. If S is a del Pezzo surface of degree 1 and the associated elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ has a nodal fiber over a rational point in $\mathbb{P}^1$, then $S(k)$ is Zariski dense in S.*

Our strategy to prove Theorem 1.2 is to exhibit a rational map $\sigma: \mathcal{C}_Q(5) \dashrightarrow S$ such that its image has a component whose strict transform on $\mathcal{E}$ is a multisection of π of infinite order (cf. [1]). In the next section, we will construct σ. To show that the image $\sigma(\mathcal{C}_Q(5))$ always has a horizontal component under the conditions in Theorem 1.2, we first choose a completion $\overline{\mathcal{C}}_Q(5)$ of the affine curve $\mathcal{C}_Q(5)$ and show that the added points correspond naturally to limits of the sections in $\mathcal{C}_Q(5)$, which allows us to show that σ extends to the extra points in $\overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$, sending them to $-4Q$ or $-5Q$ on the fiber of π containing Q (Section 3). This allows us to characterize all cases where no component of $\overline{\mathcal{C}}_Q(5)$ has a horizontal image under σ in Sections 4 and 5. In Section 6, we show that the base change of $\pi: \mathcal{E} \to \mathbb{P}^1$ by a curve of genus at most 1 has no nonzero torsion sections. Finally, we apply this to a horizontal component of $\sigma(\overline{\mathcal{C}}_Q(5))$ to prove all our main results in Section 7.

Remark 1.5. For any explicit surface S with a point Q, it is easy to check whether $\sigma(\mathcal{C}_Q(5))$ has a horizontal component, and if so, whether that component is a multisection of infinite order. Since this is indeed the case for some specific examples, we may already conclude that it is true for S and Q sufficiently general.

While we consider only surfaces given by (1) that are smooth, i.e., del Pezzo surfaces of degree 1, one could also consider *generalized del Pezzo surfaces* of degree 1, which have a birational model given by (1) that may have isolated rational double points. As for del Pezzo surfaces, there is a natural elliptic fibration on the blow-up of a generalized del Pezzo surface at the point corresponding to $\mathcal{O}$. All our results up to and including Section 5 also hold for generalized del Pezzo surfaces of degree 1, as long as we assume that the point Q does not lie on a reducible fiber, with the exception of Proposition 5.3. The proof of Proposition 5.3 shows that there is one more singular surface that we should add to the list of examples where $\sigma(\overline{\mathcal{C}}_Q(5))$ is not horizontal. One can actually generalize many of our results to the case that Q lies on a reducible fiber, but given the significant amount of additional computations required, this is not included in this paper.

In the proof of Theorem 1.4, we will view the family of the curves $\overline{\mathcal{C}}_Q(5)$ as Q runs through the points on the nodal fiber as an elliptic surface. We may also consider the family of *all* curves

$\overline{\mathcal{C}}_Q(5)$ as an elliptic threefold over S, possibly adding some extra component in some fibers to achieve flatness. This threefold has real points for any surface S over $\mathbb{R}$; it would be interesting to study the Hasse principle and weak approximation for this elliptic threefold.

All computations were done using MAGMA [2]. We thank Peter Bruin, Jean-Louis Colliot-Thélène, Bas Edixhoven, Marc Hindry, Christian Liedtke, Bjorn Poonen, Alexei Skorobogatov, Damiano Testa, Anthony Várilly-Alvarado, and Bianca Viray for useful discussions. We thank the anonymous referee for their useful comments and suggestions. The first author thanks Universiteit Leiden and the Max-Planck-Institut in Bonn and the second author the Centre Interfacultaire Bernoulli in Lausanne for their hospitality and support.

2. A family of sections

By a variety over a field we mean a separated scheme of finite type over that field. In particular, we do not assume that varieties are irreducible or reduced. By a component we always mean an irreducible component. Curves are varieties whose components all have dimension 1 and surfaces are varieties whose components all have dimension 2.

Let k be a field of characteristic not equal to 2 or 3 and let $\mathbb{P}$ denote the weighted projective space $\mathbb{P}(2,3,1,1)$ over k with coordinates x, y, z, w. Let $\mathbb{P}^1$ be the projective line over k with coordinates z, w. The subset $Z \subset \mathbb{P}$ given by $z = w = 0$ contains the two singular points $(1:0:0:0)$ and $(0:1:0:0)$ of $\mathbb{P}$, so the complement $U = \mathbb{P} - Z$ is nonsingular. The projection $\varphi\colon \mathbb{P} \dashrightarrow \mathbb{P}^1$ onto the last two cooordinates is well defined on U. For each field extension ℓ of k, let $\mathcal{C}(\ell)$ denote the family of all curves C in U_ℓ, defined over ℓ, for which the restriction $\varphi|_C\colon C \to \mathbb{P}^1_\ell$ is an isomorphism, that is, $\mathcal{C}(\ell)$ is the family of sections of $\varphi\colon U_\ell \to \mathbb{P}^1_\ell$. Whenever convenient, we will freely switch between viewing the elements of $\mathcal{C}(\ell)$ as curves and viewing them as morphisms $\mathbb{P}^1_\ell \to U_\ell$. The following lemma shows that there is an algebraic variety whose ℓ-points are naturally in bijection with the curves in $\mathcal{C}(\ell)$.

Lemma 2.1. *For every field extension ℓ of k, there is a bijection $\mathbb{A}^7(\ell) \to \mathcal{C}(\ell)$ sending the point $(x_0, y_0, a, b, c, p, q)$ to the curve defined by*

$$x = qz^2 + pzw + x_0w^2 \qquad \text{and} \qquad y = cz^3 + bz^2w + azw^2 + y_0w^3. \tag{3}$$

Proof. Without loss of generality, we assume $\ell = k$. Clearly, the described map is well defined and injective. To show surjectivity, let $\sigma\colon \mathbb{P}^1 \to U$ be a section of $\varphi\colon U \to \mathbb{P}^1$. If we set $t = z/w$, then there are polynomials $r_1, r_2, s_1, s_2 \in k[t]$ such that σ is given on $\mathbb{A}^1 \subset \mathbb{P}^1 - \{(1:0)\}$ by

$$t \mapsto \left[\frac{r_1(t)}{s_1(t)} : \frac{r_2(t)}{s_2(t)} : t : 1\right].$$

The fact that the image C of σ is contained in U implies that s_1 and s_2 are constant and the degrees of r_1 and r_2 bounded by 2 and 3 respectively. This shows that indeed there are $x_0, y_0, a, b, c, p, q \in k$ such that C is given by (3). □

Let $f, g \in k[z, w]$ be homogeneous of degree 4 and 6, respectively, and let $S \subset \mathbb{P}$ be the surface given by (1). The number of (-1)-curves on S is finite. Over a separable closure k^{sep} of k there are 240 such curves; those that are defined over k are characterized by the following lemma.

Lemma 2.2. *The curves in $\mathcal{C}(k)$ that are contained in S are exactly the (-1)-curves of S that are defined over k.*

Proof. The (-1)-curves are defined over a separable extension of k by [5, Theorem 1]. This shows that the assumption that k be perfect is not necessary in [39, Thm. 1.2], which therefore implies that the (-1)-curves on $S_{k^{\mathrm{sep}}}$ are exactly the curves given by (3) for some $x_0, y_0, a, b, c, p, q \in k^{\mathrm{sep}}$, which also follows from [32, Lemma 10.9]. The lemma follows from taking Galois invariants. □

Proposition 2.3. *For each curve $C \in \mathcal{C}(k^{sep})$ that is not contained in S, we have $C \cdot S = 6$.*

Proof. The equations (3) show that C has degree 6. Also, C is contained in U, so the intersection $C \cap S$ with the hypersurface S of degree 6 is contained in U, which is smooth. Therefore, intersection multiplicities are defined as usual, and the weighted analogue of Bézout's Theorem gives $C \cdot S = \mu^{-1}(\deg C) \cdot (\deg S)$, where $\mu = 6$ is the product of the weights of $\mathbb{P}$. The statement follows. □

The intersection $S \cap Z$ consists of the single point $\mathcal{O} = (1:1:0:0)$. For any point $Q \in S(k) - \{\mathcal{O}\}$, and for $1 \le n \le 6$, we let $\mathcal{C}_Q(n) \subset \mathbb{A}^7$ denote the subvariety of all points whose associated curve, through the bijection of Lemma 2.1, intersects S at Q with multiplicity at least n. Note that for $n = 5$ this coincides with the definition of $\mathcal{C}_Q(5)$ in the introduction.

Let $Q \in S(k) - \{\mathcal{O}\}$. After applying an automorphism of $\mathbb{P}^1$ (and the corresponding automorphism of $\mathbb{P}$), we assume without loss of generality that $\varphi(Q) = 0 = (0:1)$, say $Q = (x_0 : y_0 : 0 : 1)$ for some $x_0, y_0 \in k$. The variety $\mathcal{C}_Q(1) \subset \mathbb{A}^7$ consists of the points of $\mathbb{A}^7$ whose first two coordinates equal x_0 and y_0, respectively, so the projection onto the last five coordinates gives an isomorphism $\mathcal{C}_Q(1) \to \mathbb{A}^5$. From now on we will freely use this isomorphism to identify $\mathcal{C}_Q(1)$ and $\mathbb{A}^5$ with coordinates a, b, c, p, q.

As in the introduction, we let $\mathcal{E}$ denote the blow-up of S at $\mathcal{O}$ and $\pi\colon \mathcal{E} \to \mathbb{P}^1$ the elliptic fibration induced by the anticanonical map $\varphi|_S\colon S \dashrightarrow \mathbb{P}^1$. We will sometimes identify the fiber $\mathcal{E}_t$ above $t = (z_0 : w_0)$ with its isomorphic image on S, equal to the intersection of S with the hyperplane H_t given by $w_0 z = z_0 w$ and denoted by S_t. The intersection $H_t \cap S$ is given by a Weierstrass equation; in particular, all fibers are irreducible, and therefore all singular fibers have type I_1 or II. Whenever we speak of vertical or horizontal curves or of fibers on S or $\mathcal{E}$, we refer to this fibration. We write

$$f = f_4 z^4 + f_3 z^3 w + \cdots + f_0 w^4,$$
$$g = g_6 z^6 + g_5 z^5 w + \cdots + g_0 w^6,$$

so the fiber $\mathcal{E}_0$ above $t = 0$, containing Q, is given by $y^2 = x^3 + f_0 x + g_0$.

We can give equations for $\mathcal{C}_Q(n)$ inside $\mathcal{C}_Q(1) = \mathbb{A}^5$ as follows. Note that $t = z/w$ is a local parameter at the point $(0:1)$ on $\mathbb{P}^1$. Hence, around Q, the curve associated to $(a, b, c, p, q) \in \mathcal{C}_Q(1)$ is parametrized by

$$(4) \qquad \begin{cases} x &= qt^2 + pt + x_0, \\ y &= ct^3 + bt^2 + at + y_0, \\ z &= t, \\ w &= 1. \end{cases}$$

For $0 \le i \le 6$, let $F_i \in k[a, b, c, p, q]$ be the coefficient of t^i in

$$(5) \qquad -y^2 + x^3 + f(t,1)x + g(t,1),$$

with x and y as in (4). Then we have

$$(6) \qquad \begin{aligned} F_0 &= 0, \\ F_1 &= -2y_0 a + (3x_0^2 + f_0)p + f_1 x_0 + g_1, \\ F_2 &= -a^2 - 2y_0 b + 3x_0 p^2 + f_1 p + (3x_0^2 + f_0)q + f_2 x_0 + g_2, \\ F_3 &= -2ab - 2y_0 c + p^3 + 6x_0 pq + f_2 p + f_1 q + f_3 x_0 + g_3, \\ F_4 &= -2ac - b^2 + 3p^2 q + f_3 p + 3x_0 q^2 + f_2 q + f_4 x_0 + g_4, \\ F_5 &= -2bc + 3pq^2 + f_4 p + f_3 q + g_5, \\ F_6 &= -c^2 + q^3 + f_4 q + g_6, \end{aligned}$$

and the variety $\mathcal{C}_Q(n) \subset \mathcal{C}_Q(1) = \mathbb{A}^5$ is given by the equations $F_1 = F_2 = \ldots = F_{n-1} = 0$.

We define the polynomials

$$(7) \qquad \begin{aligned} \Phi_2 &= 4(x^3 + f_0 x + g_0), & \Phi_4 &= \Psi\Phi_3 - \Phi_2^2, \\ \Psi &= \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}x}\Phi_2, & \Phi_5 &= \Phi_2^2\Phi_4 - \Phi_3^3, \\ \Phi_3 &= 3x\Phi_2 - \tfrac{1}{4}\Psi^2, & \Phi_6 &= \Phi_5 - \Phi_4^2. \end{aligned}$$

in $k[x]$. For every integer j with $2 \le j \le 6$, the polynomial Φ_j is the factor of the j-th division polynomial of the fiber $\mathcal{E}_0$ that corresponds to the *primitive* j-torsion. In particular, the polynomials $\Phi_2, \Phi_3, \Phi_2\Phi_4$, Φ_5, and $\Phi_2\Phi_3\Phi_6$ are the j-th division polynomials for $j = 2,3,4,5,6$, respectively. For notational convenience, we set $\phi_j = \Phi_j(x_0)$ for all $j \ge 2$, as well as

$$\psi = \Psi(x_0), \qquad h_i = (f_i x_0 + g_i)\phi_2^{i-1}, \qquad l_i = f_i\phi_2^i - h_i\psi,$$

for $1 \le i \le 6$, where we set $f_5 = f_6 = 0$.

Lemma 2.4. *If $y_0 \ne 0$, then the projection of $\mathcal{C}_Q(1) = \mathbb{A}^5$ onto its last two coordinates restricts to an isomorphism $\mathcal{C}_Q(4) \to \mathbb{A}^2$. The inverse is given by $(p,q) \mapsto (a,b,c,p,q)$ with*

$$\begin{aligned}
a &= \frac{\psi p + 2h_1}{4y_0},\\
b &= \frac{\psi\phi_2 q + 2\phi_3 p^2 + 2l_1 p + 2h_2 - 2h_1^2}{4y_0\phi_2},\\
c &= \frac{\zeta q + \eta}{2y_0\phi_2^2}, \qquad \textit{with}\\
&\qquad \zeta = \phi_2(2\phi_3 p + l_1),\\
&\qquad \eta = -\phi_4 p^3 - (2h_1\phi_3 + l_1\psi)p^2 + (l_2 - 2h_1 l_1 + h_1^2\psi)p + h_3 - 2h_1h_2 + 2h_1^3.
\end{aligned}$$

Proof. Since F_1 is linear in a, the projection of $\mathcal{C}_Q(1) = \mathbb{A}^5$ along the a-axis induces an isomorphism ρ_1 from $\mathcal{C}_Q(2)$ to $\mathbb{A}^4$ with coordinates (b,c,p,q), of which the inverse is determined by the given expression for a. The image $\rho_1(\mathcal{C}_Q(3)) \subset \mathbb{A}^4$ has a defining equation that is linear in b, as F_2 is linear b and F_1 is independent of b. Therefore, the projection from $\rho_1(\mathcal{C}_Q(2)) = \mathbb{A}^4$ along the b-axis restricts to an isomorphism ρ_2 from $\rho_1(\mathcal{C}_Q(3))$ to $\mathbb{A}^3$ with coordinates (c,p,q), of which the inverse is determined by the given expression for b. Finally, the defining equation of the image $\rho_2(\rho_1(\mathcal{C}_Q(4))) \subset \mathbb{A}^3$ is linear in c, as F_3 is linear in c and F_1 and F_2 are independent of c. Therefore, the projection of $\rho_2(\rho_1(\mathcal{C}_Q(3))) = \mathbb{A}^3$ along the c-axis restricts to an isomorphism ρ_3 from $\rho_2(\rho_1(\mathcal{C}_Q(4)))$ to $\mathbb{A}^2$ with coordinates (p,q), of which the inverse determined by the given expression for c. The composition $\rho_3 \circ \rho_2 \circ \rho_1 : \mathcal{C}_Q(4) \to \mathbb{A}^2$ is the isomorphism of the lemma. □

From now on we will assume $y_0 \ne 0$, or equivalently $\phi_2 \ne 0$, and we identify $\mathcal{C}_Q(4)$ and $\mathbb{A}^2$ with coordinates (p,q) through the isomorphism of Lemma 2.4. We may eliminate the variables a,b,c from the equation $F_4 = 0$; after multiplying all coefficients by ϕ_2^3, we find that the variety $\mathcal{C}_Q(5) \subset \mathcal{C}_Q(4) = \mathbb{A}^2$ is defined by

$$c_1q^2 + (c_2p^2 + c_3p + c_4)q = c_5p^4 + c_6p^3 + c_7p^2 + c_8p + c_9 \tag{8}$$

with

$$\begin{aligned}
c_1 &= \phi_2^2\phi_3,\\
c_2 &= -3\phi_2\phi_4,\\
c_3 &= -2\phi_2(l_1\psi + 2h_1\phi_3),\\
c_4 &= \phi_2(h_1^2\psi - 2l_1h_1 + l_2),\\
c_5 &= \phi_3^2 - \phi_4\psi,\\
c_6 &= 2l_1\phi_3 - 2h_1\phi_2^2 - 4h_1\phi_4 - l_1\psi^2,\\
c_7 &= h_1^2\psi^2 - 2(3h_1^2 - h_2)\phi_3 - (4l_1h_1 - l_2)\psi + l_1^2,\\
c_8 &= (4h_1^3 - 2h_1h_2)\psi - 6l_1h_1^2 + 2l_1h_2 + 2l_2h_1 - l_3,\\
c_9 &= 5h_1^4 - 6h_1^2h_2 + 2h_1h_3 + h_2^2 - h_4.
\end{aligned}$$

As we assumed that y_0, ϕ_2 are nonzero and that the characteristic of k is not 2 or 3, the vanishing of ϕ_3 and ϕ_4 would imply that Q has both order 3 and 4 in $\mathcal{E}_0^{\rm ns}(k)$, which is a contradiction, so the coefficients c_1 and c_2 do not both vanish, and $\mathcal{C}_Q(5)$ is a curve, though not necessarily reduced or irreducible. We will identify $\mathcal{C}_Q(5)$ with its image in $\mathbb{A}^2$ and we view the coordinates a, b, and c as functions on $\mathcal{C}_Q(4)$ or $\mathcal{C}_Q(5)$, as given in Lemma 2.4.

Remark 2.5. The functions F_4, F_5, and F_6 are regular on $\mathcal{C}_Q(4) \cong \mathbb{A}^2$ and can therefore be identified with polynomials in $k[p,q]$.

Remark 2.6. The (-1)-curves on S going through Q correspond to the points of the subscheme in $\mathcal{C}_Q(4) \cong \mathbb{A}^2$ given by $F_4 = F_5 = F_6 = 0$.

Remark 2.7. A special case of Theorem 1.2 is Theorem 2.1(2) of [37]; indeed, when $f = 0$ and g vanishes at $(1:0) \in \mathbb{P}^1$, and $Q = (1:1:1:0)$, then the curve $\mathcal{C}_Q(5)$ is isomorphic to the curve given in Theorem 2.1(2) of [37]. The generalizations of this theorem given in [14, Theorems C and D] are also a special case of our Theorem 1.2, where one uses $Q = (0:1:1:0)$, which has order 3 in its fiber in the case of Theorem C. The proofs of Theorems C and D in [14] are incomplete, but they do work for surfaces S that are sufficiently general. More precisely, it is not shown that the rational function $T(\varrho)$ in the proof of Theorem C (and its implicit equivalent for Theorem D) is always nonconstant. In our geometric interpretation, this is equivalent to $\sigma(\mathcal{C}_Q(5))$ having a horizontal component. Also, there is no proof of the claim that $X(2 \cdot P_\varrho)$ is never contained in $Q[\varrho]$ in the proof of Theorem C (and its implicit equivalent for Theorem D), which is crucial for the argument that the point P_ϱ has infinite order on the elliptic curve $\mathcal{E}'_\varrho$.

Every curve $C \in \mathcal{C}(k)$ corresponding to a point $P \in \mathcal{C}_Q(5)(k)$ and not contained in S,intersects S with multiplicity at least 5 at Q, so by Proposition 2.3, there is a unique sixth point of intersection, which is also defined over k. We define a rational map

$$\sigma : \mathcal{C}_Q(5) \dashrightarrow S$$

by sending P to the sixth intersection point of C with S. The map σ is defined over k. By Proposition 2.3, it is well defined at each point $P \in \mathcal{C}_Q(5)$ whose corresponding curve is not contained in S, and thus there are at most 240 points $P \in \mathcal{C}_Q(5)$ where σ is not well defined (see Lemma 2.2 and the sentences before). Every horizontal component of the image of σ, or its strict transform on $\mathcal{E}$, yields a multisection of the elliptic fibration $\pi : \mathcal{E} \to \mathbb{P}^1$.

We can describe the map σ very explicitly. The curve C corresponding to $(a,b,c,p,q) \in \mathcal{C}_Q(5)$ is parametrized by (4). When we substitute the expressions of (4) into equation (5), we obtain $t^5(F_5 + F_6 t)$, so the sixth intersection point of $C \cap S$ is given by (4) with $t = -F_5/F_6$.

3. A completion of the family of sections

We keep the notation of the previous section. In particular, the field k, the weighted projective space $\mathbb{P} = \mathbb{P}(2,3,1,1)$ over k with coordinates x, y, z, w, the projective line $\mathbb{P}^1$ over k with coordinates z, w, the surface $S \subset \mathbb{P}$, and the points $\mathcal{O}, Q \in S(k)$ are as before, and so are the objects that depend on them, including the elliptic fibration $\pi : \mathcal{E} \to \mathbb{P}^1$, the elements $\psi, \phi_j, c_i \in k$, the curve $\mathcal{C}_Q(5) \subset \mathbb{A}^2$, the coordinates p, q of $\mathbb{A}^2$, the functions a, b, c, F_i on $\mathcal{C}_Q(5)$, and the map $\sigma : \mathcal{C}_Q(5) \dashrightarrow S$.

We will see in Theorem 6.4 that when the closure of the image $\sigma(\mathcal{C}_Q(5)) \subset S$ contains a horizontal component with respect to the natural elliptic fibration $\pi : \mathcal{E} \to \mathbb{P}^1$, then we can use such a component to construct a base change of π with a section of infinite order. Unfortunately, in some cases the image $\sigma(\mathcal{C}_Q(5))$ does not contain such a component. In order to investigate when this happens, we extend the map $\sigma : \mathcal{C}_Q(5) \dashrightarrow S$ to a projective completion $\overline{\mathcal{C}}_Q(5)$ of the affine curve $\mathcal{C}_Q(5)$ and first determine the image of the limit points in $\Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$ (see Proposition 3.9).

For every extension ℓ of k, the points in $\mathcal{C}_Q(5)(\ell)$ correspond to elements of $\mathcal{C}(\ell)$, which are curves in U_ℓ. So the curve $\mathcal{C}_Q(5)$ parametrizes a family of curves in $U \subset \mathbb{P}$. The elements of Ω correspond to the limit curves of this family. Viewing the elements of $\mathcal{C}(\ell)$ as sections $\mathbb{P}^1_\ell \to U_\ell$ of φ, we define the morphism

$$\gamma : \mathcal{C}_Q(5) \times \mathbb{P}^1 \to \mathbb{P}$$

by $\gamma(P,R) = \chi(R)$, where $\chi \in \mathcal{C}(\ell)$ is the section of φ corresponding to $P \in \mathcal{C}_Q(5)(\ell)$. The morphism γ is defined over k. In terms of the coordinates (p,q) on $\mathcal{C}_Q(5) \subset \mathbb{A}^2$, the map γ sends $\big((p,q),(z:w)\big)$ to $(x:y:z:w)$ with x and y as in (3) and a,b,c as in Lemma 2.4. For each point $P \in \mathcal{C}_Q(5)(\ell)$ with corresponding section $\chi \in \mathcal{C}(\ell)$, the image $\chi(\mathbb{P}^1_\ell) \subset U_\ell \subset \mathbb{P}_\ell$ is the image under γ of the fiber of the trivial $\mathbb{P}^1$-bundle $\mathcal{C}_Q(5) \times \mathbb{P}^1$ over P. Therefore, we may find an appropriate completion $\overline{\mathcal{C}}_Q(5)$, as well as the limit curves corresponding to the elements in $\overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$ as follows. Start with an arbitrary completion $\mathcal{C}^0_Q(5)$ of $\mathcal{C}_Q(5)$ and the trivial $\mathbb{P}^1$-bundle $\Gamma^0 = \mathcal{C}^0_Q(5) \times \mathbb{P}^1$ over it. Now γ is defined on an open subset of Γ^0. After an appropriate sequence of blow-ups and blow-downs, we obtain a surface Γ that is birational to Γ^0 to which γ extends as a morphism, as well as a new completion $\overline{\mathcal{C}}_Q(5)$ such that the $\mathbb{P}^1$-bundle structure $\mathcal{C}_Q(5) \times \mathbb{P}^1 \to \mathcal{C}_Q(5)$ extends to a conic bundle structure $\Gamma \to \overline{\mathcal{C}}_Q(5)$. Note that it is not necessary to require that $\overline{\mathcal{C}}_Q(5)$ be smooth. The limit curves are then the images under γ of the fibers of $\Gamma \to \overline{\mathcal{C}}_Q(5)$ over the points in $\Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$.

The problem with the process above, in which we construct $\overline{\mathcal{C}}_Q(5)$ and Γ, is that we are not working with a single del Pezzo surface of degree 1, but with all of them, and we have to distinguish several cases of monoidal transformations, based on the types of singularities at the points in $\mathcal{C}^0_Q(5) - \mathcal{C}_Q(5)$. Therefore, instead of presenting this process here, we will immediately introduce the result: a completion $\overline{\mathcal{C}}_Q(5)$ together with a conic bundle $\Gamma \to \overline{\mathcal{C}}_Q(5)$ that works in all cases, in the sense that γ extends to it.

3.1. **Compactifying $\overline{\mathcal{C}}_Q(5)$.** Let $\overline{p}, \overline{q}, \overline{r}$ be the coordinates of the weighted projective space $\mathbb{P}(1,2,1)$, and let $\mathbb{H} \to \mathbb{P}(1,2,1)$ be the blow-up at the singular point $(0:1:0)$. Since $\mathbb{P}(1,2,1)$ is isomorphic to a cone in $\mathbb{P}^3$, the surface $\mathbb{H}$ is smooth; it is in fact a Hirzebruch surface. By sending (p,q) to $(p:q:1)$, we identify $\mathbb{A}^2$ with an open subset of $\mathbb{P}(1,2,1)$ and hence with an open subset of $\mathbb{H}$. In doing so, we also identify the function field $k(p,q)$ of $\mathbb{A}^2$ with that of $\mathbb{H}$.

Let $\overline{\mathcal{C}}_Q(5)$ denote the completion of $\mathcal{C}_Q(5)$ inside $\mathbb{H}$. Note that the completion of $\mathcal{C}_Q(5)$ inside $\mathbb{P}(1,2,1)$ contains the singular point $(0:1:0)$ if and only if the coefficient c_1 of q^2 in (8) vanishes, i.e., if and only if Q has order 3 in $\mathcal{E}^{\mathrm{ns}}_0(k)$. Hence, if Q does not have order 3, we may identify $\overline{\mathcal{C}}_Q(5)$ with the completion of $\mathcal{C}_Q(5)$ inside $\mathbb{P}(1,2,1)$; as c_1, c_2, and c_5 do not all vanish, this completion is given by

$$c_1\overline{q}^2 + (c_2\overline{p}^2 + c_3\overline{pr} + c_4\overline{r}^2)\overline{q} = c_5\overline{p}^4 + c_6\overline{p}^3\overline{r} + c_7\overline{p}^2\overline{r}^2 + c_8\overline{pr}^3 + c_9\overline{r}^4. \tag{9}$$

We identify $\mathbb{H}$ with the variety in $\mathbb{P}(1,2,1) \times \mathbb{P}^1(s,t)$ given by $\overline{p}t = \overline{r}s$. Denoting the zeroset in $\mathbb{H}$ of a doubly homogeneous polynomial h in $k[\overline{p},\overline{q},\overline{r}][s,t]$ by $Z(h)$, we define the open subsets

$$\mathbb{H}_1 = \mathbb{H} - Z(\overline{r}), \qquad \mathbb{H}_2 = \mathbb{H} - Z(\overline{p}), \qquad \mathbb{H}_3 = \mathbb{H} - Z(\overline{q}t), \qquad \mathbb{H}_4 = \mathbb{H} - Z(\overline{q}s)$$

of $\mathbb{H}$. In the function field of $\mathbb{H}$, we have $p = \frac{s}{t} = \frac{\overline{p}}{\overline{r}}$ and $q = \frac{\overline{q}}{\overline{r}^2}$. We define the functions

$$\begin{array}{llll} \lambda_1 = p, & \lambda_2 = p^{-1}, & \lambda_3 = p, & \lambda_4 = p^{-1}, \\ \mu_1 = q, & \mu_2 = qp^{-2}, & \mu_3 = q^{-1}, & \mu_4 = p^2q^{-1} \\ \mathfrak{a}_1 = a, & \mathfrak{a}_2 = a\lambda_2, & \mathfrak{a}_3 = a, & \mathfrak{a}_4 = a\lambda_4, \\ \mathfrak{b}_1 = b, & \mathfrak{b}_2 = b\lambda_2^2, & \mathfrak{b}_3 = b\mu_3, & \mathfrak{b}_4 = b\lambda_4^2\mu_4, \\ \mathfrak{c}_1 = c, & \mathfrak{c}_2 = c\lambda_2^3, & \mathfrak{c}_3 = c\mu_3, & \mathfrak{c}_4 = c\lambda_4^3\mu_4 \end{array} \tag{10}$$

in the function field $k(p,q)$ of $\mathbb{H}$.

Lemma 3.1. *For each $i \in \{1,2,3,4\}$, the functions $\lambda_i, \mu_i, \mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i$ are regular on $\mathbb{H}_i$ and the map $\mathbb{H}_i \to \mathbb{A}^2$ sending R to $(\lambda_i(R), \mu_i(R))$ is an isomorphism. The sets $\mathbb{H}_1$, $\mathbb{H}_2$, $\mathbb{H}_3$, and $\mathbb{H}_4$ cover $\mathbb{H}$.*

Proof. Suppose $i \in \{1,2,3,4\}$. The fact that λ_i and μ_i are regular on $\mathbb{H}_i$ and define an isomorphism to $\mathbb{A}^2$ is a standard computation. So is the last statement. Using Lemma 2.4, one can express $\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i$ as polynomials in λ_i and μ_i, which shows that $\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}_i$ are regular as well. □

For each $i \in \{1,2,3,4\}$, set $\mathcal{C}^i_Q(5) = \overline{\mathcal{C}}_Q(5) \cap \mathbb{H}_i$. The union of the four affine curves $\mathcal{C}^i_Q(5)$, with $1 \le i \le 4$, is $\overline{\mathcal{C}}_Q(5)$. Note that $\mathbb{H}_1 = \mathbb{A}^2(p,q)$ and $\mathcal{C}^1_Q(5) = \mathcal{C}_Q(5)$. The affine curve $\mathcal{C}^2_Q(5)$ coincides with the affine part with $\overline{p} \neq 0$ of the curve in $\mathbb{P}(1,2,1)$ given by (9); the affine coordinates (λ_2, μ_2) correspond with $(\overline{r}/\overline{p}, \overline{q}/\overline{p}^2)$. By abuse of notation, we will denote the restrictions of λ_i, μ_i, $\mathfrak{a}_i$, $\mathfrak{b}_i$, and $\mathfrak{c}_i$ to $\mathcal{C}^i_Q(5)$ by the same symbol.

3.2. **Extending γ.** We define the conic bundles

$$\begin{aligned}
&\Delta_1 = \mathbb{H}_1 \times \mathbb{P}^1(z,w),\\
&\Delta_2 = \mathbb{H}_2 \times \mathbb{P}^1(z',w'),\\
&\Delta_3 \subset \mathbb{H}_3 \times \mathbb{P}^2(u_0,u_1,u_2) \quad \text{given by } \bar{r}^2 u_0u_2 = \bar{q}u_1^2, \quad \text{and}\\
&\Delta_4 \subset \mathbb{H}_4 \times \mathbb{P}^2(u_0',u_1',u_2') \quad \text{given by } \bar{p}^2 u_0'u_2' = \bar{q}{u_1'}^2
\end{aligned}$$

over $\mathbb{H}_1$, $\mathbb{H}_2$, $\mathbb{H}_3$, and $\mathbb{H}_4$, respectively. We glue these conic bundles to a conic bundle Δ over $\mathbb{H}$ as follows. We glue Δ_1 and Δ_2 above the intersection $\mathbb{H}_1 \cap \mathbb{H}_2$ by identifying $(z:w) \in \mathbb{P}^1(z,w)$ with $(\bar{p}z : \bar{r}w) \in \mathbb{P}^1(z',w')$. We also glue Δ_1 and Δ_3 above the intersection $\mathbb{H}_1 \cap \mathbb{H}_3$ by identifying $(z:w) \in \mathbb{P}^1(z,w)$ with $(\bar{q}z^2 : \bar{r}^2 zw : \bar{r}^2 w^2) \in \mathbb{P}^2(u_0,u_1,u_2)$. We glue Δ_3 and Δ_4 above $\mathbb{H}_3 \cap \mathbb{H}_4$ by identifying $(u_0 : u_1 : u_2) \in \mathbb{P}^2(u_0,u_1,u_2)$ with $(tu_0 : su_1 : tu_2) \in \mathbb{P}^2(u_0',u_1',u_2')$. One easily checks that these identifications also induce an isomorphism between the parts of Δ_i and Δ_j above the intersection $\mathbb{H}_i \cap \mathbb{H}_j$ for the remaining pairs $(i,j) \in \{(1,4),(2,3),(2,4)\}$.

The map $\gamma : \mathcal{C}_Q(5) \times \mathbb{P}^1 \to \mathbb{P}$ extends to $\mathcal{C}_Q(4) \times \mathbb{P}^1 = \mathbb{A}^2 \times \mathbb{P}^1 = \mathbb{H}_1 \times \mathbb{P}^1 = \Delta_1$ by setting $\gamma(P,R) = \chi(R)$ where, for any field extension ℓ of k, we have $R \in \mathbb{P}^1(\ell)$, and the section $\chi \in \mathcal{C}(\ell)$ of φ corresponds to $P \in \mathcal{C}_Q(4)(\ell)$. The extended map, also denoted by γ, sends $\big(P,(z:w)\big) \in \mathcal{C}_Q(4) \times \mathbb{P}^1$ to $(x:y:z:w)$ with x and y as in (3), with $(p,q) = (p(P),q(P)) = (\lambda_1(P),\mu_1(P))$, and with a,b,c as in Lemma 2.4. The following proposition shows that γ extends to a morphism $\Delta \to \mathbb{P}$.

Proposition 3.2. *The map γ extends to a morphism $\Delta \to \mathbb{P}$ that is given on Δ_2 by sending $(P,(z':w'))$ to*

$$(\mu_2(P)z'^2 + z'w' + x_0w'^2 : \mathfrak{c}_2(P)z'^3 + \mathfrak{b}_2(P)z'^2w' + \mathfrak{a}_2(P)z'w'^2 + y_0w'^3 : \lambda_2(P)z' : w'),$$

on Δ_3 by sending $(P,(u_0:u_1:u_2))$ to

$$(u_2(u_0 + \lambda_3(P)u_1 + x_0u_2) : u_2(\mathfrak{c}_3(P)u_0u_1 + \mathfrak{b}_3(P)u_0u_2 + \mathfrak{a}_3(P)u_1u_2 + y_0u_2^2) : u_1 : u_2),$$

and on Δ_4 by sending $(P,(u_0':u_1':u_2'))$ to

$$(u_2'(u_0' + u_1' + x_0u_2') : u_2'(\mathfrak{c}_4(P)u_0'u_1' + \mathfrak{b}_4(P)u_0'u_2' + \mathfrak{a}_4(P)u_1'u_2' + y_0{u_2'}^2) : \lambda_4(P)u_1' : u_2').$$

Proof. It is easy to check that the given maps coincide with γ wherever they are well defined. Hence, it suffices to show that they are well defined on the claimed subsets.

Suppose the first map is not well defined at a point $(P,(z':w')) \in \Delta_2$. By Lemma 3.1, the functions $\lambda_2, \mu_2, \mathfrak{a}_2, \mathfrak{b}_2, \mathfrak{c}_2$ are all regular at P, so the fact that the map is not well defined at $(P,(z':w'))$ implies that the four given polynomials that are claimed to define the map on Δ_2 vanish. This yields $w' = 0$, so $z' \neq 0$, and thus λ_2, μ_2, and $\mathfrak{c}_2$ all vanish at P. From Lemma 2.4 and $\lambda_2(P) = \mu_2(P) = 0$, we obtain $\mathfrak{c}_2(P) = -\phi_4/(2y_0\phi_2^2)$, so the vanishing of $\mathfrak{c}_2$ at P gives $\phi_4 = 0$. From $\lambda_2(P) = \mu_2(P) = 0$ and equation (9), we find $c_5 = 0$, so we also have $\phi_3^2 = \phi_4\psi = 0$. This is a contradiction as Q can not have both order 3 and 4 on $\mathcal{E}_0^{\mathrm{ns}}(k)$. Hence, the first map is well defined on Δ_2.

It is clear that the second map is well defined at any point $(P,(u_0:u_1:u_2)) \in \Delta_3$ with $u_1 \neq 0$ or $u_2 \neq 0$. To see that it is also well defined at points with $(u_0:u_1:u_2) = (1:0:0)$, we identify $\mathbb{P}$ with its image under the closed immersion to $\mathbb{P}^{22}$ corresponding to $\mathcal{O}(6)$ on $\mathbb{P}$. Substituting the expressions for the second map into the 23 monomials of weighted degree 6 in the variables x, y, z, and w gives 23 polynomials of total degree 6 in u_0,u_1,u_2, which after replacing u_1^2 by $\mu_3u_0u_2$ (the conic bundle Δ_3 is given by $\mu_3u_0u_2 = u_1^2$) are all divisible by u_2^3. The composition $\Delta_3 \to \mathbb{P} \to \mathbb{P}^{22}$ is given by these 23 polynomials, each divided by u_2^3. The coordinate corresponding to the monomial x^3 is given by $(u_0 + \lambda_3u_1 + x_0u_2)^3$, which does not vanish at $(P,(1:0:0))$, so this composition, and thus the map $\Delta_3 \to \mathbb{P}$, is well defined.

The third map is well defined whenever $u_2' \neq 0$. On the other hand, if $u_2' = 0$, then also $u_1' = 0$, and one uses the composition $\Delta_4 \to \mathbb{P} \to \mathbb{P}^{22}$ to check that the map $\Delta_4 \to \mathbb{P}$ is well defined at points with $(u_0':u_1':u_2') = (1:0:0)$, as in the previous case. □

Let Γ be the inverse image of $\overline{\mathcal{C}}_Q(5)$ under the map $\Delta \to \mathbb{H}$. We denote the restriction $\Gamma \to \overline{\mathcal{C}}_Q(5)$ of the conic bundle $\Delta \to \mathbb{H}$ by τ. By abuse of notation, we denote the restriction $\Gamma \to \mathbb{P}$ of the map $\gamma\colon \Delta \to \mathbb{P}$ by γ as well.

3.3. **The limit curves and their images.** Set $\Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$. Then the limit curves described in the beginning of this section are the images under γ of the fibers of τ above the points in Ω. These images are described in Lemmas 3.3, 3.4, 3.5, and 3.6. Recall that $\mathcal{C}_Q^2(5)$ can be identified with the open subset of $\mathbb{P}(1,2,1)$ given by $\overline{p} \neq 0$.

Lemma 3.3. *Each point $P \in \mathcal{C}_Q^2(5) - \mathcal{C}_Q(5)$ corresponds to $(\overline{p} : \overline{q} : \overline{r}) = (1 : \alpha : 0)$ for some $\alpha \in \overline{k}$ satisfying $c_1\alpha^2 + c_2\alpha - c_5$; the map γ sends $(P, (z' : w')) \in \Gamma$ to $(x : y : z : w)$ with*

$$\begin{cases} x &= \alpha z'^2 + z'w' + x_0 w'^2, \\ y &= y_0\left(\frac{4\alpha\phi_2\phi_3 - 2\phi_4}{\phi_2^3} z'^3 + \frac{\alpha\psi\phi_2 + 2\phi_3}{\phi_2^2} z'^2 w' + \frac{\psi}{\phi_2} z' w'^2 + w'^3 \right), \\ z &= 0, \\ w &= w', \end{cases}$$

and the image of the fiber $\tau^{-1}(P) \subset \Gamma$ under γ is a curve in $\mathbb{P}$ of degree 6 that intersects S at Q with multiplicity at least 5.

Proof. Let $P \in \mathcal{C}_Q^2(5) - \mathcal{C}_Q(5)$. The first part of the statement is obvious. We have $\lambda_2(P) = 0$ and $\mu_2(P) = \alpha$. From Lemma 2.4 we deduce

$$\mathfrak{a}_2(P) = \frac{\psi}{4y_0}, \qquad \mathfrak{b}_2(C) = \frac{\psi\phi_2\alpha + 2\phi_3}{4y_0\phi_2}, \qquad \mathfrak{c}_2(P) = \frac{2\phi_2\phi_3\alpha - \phi_4}{2y_0\phi_2^2}.$$

Hence, according to Proposition 3.2, the map γ sends $(P, (z' : w')) \in \Delta_2$ to $(x : y : 0 : w')$ with $x = \alpha z'^2 + z'w' + x_0 w'^2$ and $y = \mathfrak{c}_2(P) z'^3 + \mathfrak{b}_2(P) z'^2 w' + \mathfrak{a}_2(P) z' w'^2 + y_0 w'^3$. From $4y_0^2 = \phi_2$, it follows that the latter equals the expression given for y in the lemma.

The curve $D = \gamma(\tau^{-1}(P))$ in $\mathbb{P}$ lies inside the hyperplane given by $z = 0$, which is isomorphic to the weighted projective space $\mathbb{P}(2,3,1)$. The intersection of D with the curve D' in this hyperplane given by $y = \lambda xw + \mu w^3$ yields three intersection points for general λ and μ. Bézout's Theorem tells us that the product of the weights $(2,3,1)$ times this intersection number 3 equals $(\deg D)(\deg D')$, so we find $\deg D = 18/\deg D' = 6$.

Since the degree of D is 6, it is a full limit of images under γ of fibers of $\tau\colon \mathcal{C}_Q(5) \times \mathbb{P}^1 \to \mathcal{C}_Q(5)$, all of which intersect S at Q with multiplicity at least 5, so D does this as well. This can also be checked computationally by substituting the parametrization given in the lemma into the polynomial

$$-y^2 + x^3 + f(z,w)x + g(z,w),$$

and checking that the coefficients of $z'^i w'^{6-i}$ vanish for $0 \le i \le 4$. $\square$

Recall that S_0 is the image of $\mathcal{E}_0$ on S, which is the intersection of S with the plane given by $z = 0$. The following two lemmas give more information about the image under γ of the fibers of τ above points in $\mathcal{C}_Q^2(5) - \mathcal{C}_Q(5)$ in the case that S_0 is singular. In particular, they show that S_0 is one of the limit curves in this case.

Lemma 3.4. *Suppose $\mathcal{E}_0$ has a node. Then $\mathcal{C}_Q^2(5) - \mathcal{C}_Q(5)$ contains the point $P_1 = (1 : \alpha_1 : 0) \in \mathbb{P}(1,2,1)$ with*

$$\alpha_1 = \frac{f_0}{4(f_0x_0 - 3g_0)}.$$

The map γ restricts to a birational morphism from the fiber $\tau^{-1}(P_1)$ to S_0. If $\phi_3 = 0$, then P_1 is the only point in $\mathcal{C}_Q^2(5) - \mathcal{C}_Q(5)$. If $\phi_3 \neq 0$, then $\mathcal{C}_Q^2(5) - \mathcal{C}_Q(5)$ contains a unique second point $P_2 = (1 : \alpha_2 : 0) \in \mathbb{P}(1,2,1)$ with

$$\alpha_2 = \frac{f_0(2f_0x_0 - 21g_0)}{4(f_0x_0 - 3g_0)(2f_0x_0 - 9g_0)};$$

the image under γ of the fiber $\tau^{-1}(P_2)$ is not contained in S.

Proof. Since $\mathcal{E}_0$ has a node, we have $4f_0^3+27g_0^2=0$ with $f_0,g_0\neq 0$, so for $d=-\frac{3g_0}{2f_0}$ we have $f_0=-3d^2$ and $g_0=2d^3$. The curve $\mathcal{E}_0\cong S_0$ is given by $y^2=(x-d)^2(x+2d)$, and we have

$$\begin{aligned}\Phi_2&=4(x-d)^2(x+2d),\\ \Phi_3&=3(x-d)^3(x+3d),\\ \Phi_4&=2(x-d)^5(x+5d),\\ \Phi_5&=(x-d)^{10}(5x^2+50dx+89d^2).\end{aligned}$$

If $\phi_3\neq 0$, then $c_1\neq 0$ and the polynomial $c_1T^2+c_2T-c_5$ factors as $c_1(T-\alpha_1)(T-\alpha_2)$ with $\alpha_1=\frac{1}{4}(x_0+2d)^{-1}$ and $\alpha_2=\frac{1}{4}(x_0+7d)(x_0+2d)^{-1}(x_0+3d)^{-1}$, which equal the expressions given in the proposition. If $\phi_3=0$, then $c_1=0$ and $x_0=-3d$, so the only root of $c_1T^2+c_2T-c_5$ is $\alpha_1=c_5/c_2=-\frac{1}{4}d^{-1}$, which equals the expression for α_1 given in the proposition. It follows from Lemma 3.3 that the points in $\mathcal{C}_Q^2(5)-\mathcal{C}_Q(5)$ are as claimed.

It follows from Lemma 3.3 and the identities above that the restriction of γ to $\tau^{-1}(P_1)=\{P_1\}\times\mathbb{P}^1(z',w')$ factors as the composition of the isomorphism

$$\{P_1\}\times\mathbb{P}^1(z',w')\to\mathbb{P}^1,\quad (P_1,(z':w'))\mapsto((x_0-d)(z'+2(x_0+2d)w'):2y_0w')$$

and the birational morphism

$$\mathbb{P}^1\to S_0,\quad (s:1)\mapsto(s^2-2d:s^3-3ds:0:1).$$

This proves the second statement.

For the last statement, we assume $\phi_3\neq 0$, take $\alpha=\alpha_2$ and substitute the corresponding parametrization of Lemma 3.3 in the equation

$$-y^2+x^3+f(z,w)x+g(z,w)=0,$$

which defines S. The obtained equation in z' and w', multiplied by

$$-16d^{-3}(x_0-d)^{10}(x_0+2d)^5(x_0+3d)^3,$$

is

$$z'^5\big(\phi_5\cdot z'+(x_0-d)^6\phi_2\phi_3\cdot w'\big)=0.$$

As the left-hand side does not vanish identically, the curve $\gamma(\tau^{-1}(P_2))$ is not contained in S. □

Lemma 3.5. *Suppose that $\mathcal{E}_0$ has a cusp. Then $\overline{\mathcal{C}}_Q(5)$ equals $\mathcal{C}_Q(5)\cup\mathcal{C}_Q^2(5)=\mathcal{C}_Q^1(5)\cup\mathcal{C}_Q^2(5)$ and $\mathcal{C}_Q^2(5)-\mathcal{C}_Q(5)$ contains exactly one point, namely $P=(2:x_0^{-1}:0)\in\mathbb{P}(1,2,1)$. The map γ restricts to a birational morphism from the fiber $\tau^{-1}(P)$ to S_0.*

Proof. Since $\mathcal{E}_0$, or equivalently S_0, has a cusp, we have $f_0=g_0=0$. The cusp $(0:0:0:1)$ is the only point on S_0 with x-coordinate 0, so from $y_0\neq 0$ we find $x_0\neq 0$. The Φ_i are as in the proof of Lemma 3.4 with $d=0$. From $c_1=48x_0^{10}\neq 0$ we get $\overline{\mathcal{C}}_Q(5)=\mathcal{C}_Q(5)\cup\mathcal{C}_Q^2(5)$. The polynomial $c_1T^2+c_2T-c_5$ factors as $3x_0^8(4x_0T-1)^2$ with the unique root $\alpha=(4x_0)^{-1}$, which implies by Lemma 3.3 that $P=(2:x_0^{-1}:0)$ is the only point in $\mathcal{C}_Q^2(5)-\mathcal{C}_Q(5)$. One checks by a computation that it also follows from Lemma 3.3 that the restriction of γ to $\tau^{-1}(P)=\{P\}\times\mathbb{P}^1(z',w')$ factors as the composition of the isomorphism

$$\{P\}\times\mathbb{P}^1(z',w')\to\mathbb{P}^1,\quad (P,(z':w'))\mapsto(z'+2x_0w':2x_0w')$$

and the birational morphism $\mathbb{P}^1\to S_0$ that sends $(s:1)$ to $(x_0s^2:y_0s^3:0:1)$. This proves the second statement. □

The points of $\overline{\mathcal{C}}_Q(5)-\mathcal{C}_Q(5)$ that are not handled by the previous lemmas are the points in $\overline{\mathcal{C}}_Q(5)-\big(\mathcal{C}_Q^1(5)\cup\mathcal{C}_Q^2(5)\big)$, that is, the points above the singular point $(0:1:0)$ in $\mathbb{P}(1,2,1)$. The next lemma takes care of these points.

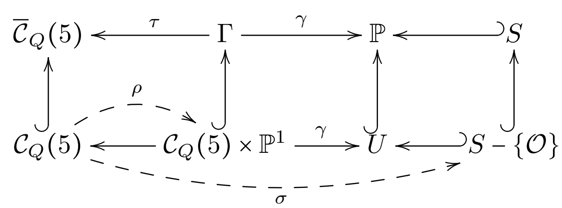
Lemma 3.6. *For each point $P\in\overline{\mathcal{C}}_Q(5)-\big(\mathcal{C}_Q^1(5)\cup\mathcal{C}_Q^2(5)\big)$, the map γ sends the fiber $\tau^{-1}(P)$ to the curve in $\mathbb{P}$ given by $z=0$ and $4y_0y=\psi xw+(\phi_2-\psi x_0)w^3$; this curve intersects S with multiplicity 3 at Q and nowhere else.*

Proof. By Lemma 3.1, the open subset $\mathbb{H}_3$ has affine coordinates (λ_3, μ_3). If P lies in $\mathcal{C}_Q^3(5) - \big(\mathcal{C}_Q^1(5) \cup \mathcal{C}_Q^2(5)\big)$, then it corresponds with a point with $(\lambda_3, \mu_3) = (p, 0)$ for some $p \in \overline{k}$ and the fiber $\tau^{-1}(P)$ is given by $u_1^2 = 0$ in $\mathbb{P}^2(u_0, u_1, u_2)$. The map γ sends $(P, (u : 0 : 1)) \in \Delta_3$ to $(u + x_0 : \mathfrak{b}_3(P)u + y_0 : 0 : 1)$ by Lemma 3.2. From Lemma 2.4, we find $4y_0\mathfrak{b}_3(P) = \psi$. It follows that the image of the fiber is the claimed curve. In the affine plane given by $z = 0$ and $w = 1$, this curve is a line going through Q with slope $\mathfrak{b}_3(P) = (3x_0^2 + f_0)/(2y_0)$, so it is exactly the tangent line to the curve S_0 at Q. Note that the existence of P implies that Q has order 3 on $S_0^{\mathrm{ns}}(k)$, so this tangent line intersects S_0 with multiplicity 3 at Q and nowhere else. As the curve intersects the surface S only in the curve S_0, the lemma follows. If P lies in $\mathcal{C}_Q^4(5) - \big(\mathcal{C}_Q^1(5) \cup \mathcal{C}_Q^2(5)\big)$, then the argument is analogous. □

Remark 3.7. Scheme theoretically, the image under γ of the fiber of τ above P in Lemma 3.6 is not reduced, but given by $z^2 = 0$ and $4y_0 y = \psi x w + (\phi_2 - \psi x_0) w^3$. This nonreduced curve is also a limit curve as mentioned in the beginning of the section, and it intersects S with multiplicity 6 at Q.

Remark 3.8. Let T be the image of $\gamma\colon \Gamma \to \mathbb{P}$. Then T is the union of all curves $C \subset U$ corresponding to points $P \in \mathcal{C}_Q(5)$ and the limit curves corresponding to points $P \in \Omega$. The closure of the image $\sigma(\mathcal{C}_Q(5))$ in S is contained in the intersection $S \cap T$. This intersection also contains all (-1)-curves on S that go through Q. See also Remarks 5.7 and 5.8.

The rational map $\sigma\colon \mathcal{C}_Q(5) \dashrightarrow S$ from the end of Section 2 factors as $\sigma = \gamma \circ \rho$, where $\rho\colon \mathcal{C}_Q(5) \dashrightarrow \mathcal{C}_Q(5) \times \mathbb{P}^1(z, w)$ is a rational section of $\tau\colon \Gamma \to \overline{\mathcal{C}}_Q(5)$ that sends $P \in \mathcal{C}_Q(5)$ to $\big(P, (-F_5(P) : F_6(P))\big)$. Here, for $0 \le i \le 6$, we view F_i as in (6) as a function on $\overline{\mathcal{C}}_Q(5)$. We use this in Proposition 3.9 to show that σ extends to a map that is well defined at every point in $\Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$.



Proposition 3.9. *The rational map σ extends to a rational map $\overline{\mathcal{C}}_Q(5) \dashrightarrow S$ that is well defined at the points in Ω. For every $P \in \Omega$, we have $\sigma(P) = -4Q \in S_0^{\mathrm{ns}}(k) \subset S$ if S_0 has a cusp or S_0 has a node and $P = P_1$ as in Lemma 3.4, and we have $\sigma(P) = -5Q \in S_0^{\mathrm{ns}}(k) \subset S$ otherwise.*

Proof. Let $P \in \Omega$. Then by Lemmas 3.3, 3.4, 3.5, 3.6, and Remark 3.7, the scheme-theoretic image of $\tau^{-1}(P)$ under γ is a curve of degree 6 in the plane given by $z = 0$ in $\mathbb{P}$. We denote this curve by C. A parametrization of C is given in Lemma 3.3 if $P \in \mathcal{C}_Q^2(5) - \mathcal{C}_Q(5)$; the curve C is nonreduced if $P \in \overline{\mathcal{C}}_Q(5) - \big(\mathcal{C}_Q^1(5) \cup \mathcal{C}_Q^2(5)\big)$. The intersection $C \cap S$ is the same as the intersection of C with with $S_0 = S \cap \{z = 0\}$, and C intersects S_0 with multiplicity at least 5 at Q.

If S_0 is smooth, then S_0 has genus 1, so C has no components in common with S_0. The curves S_0 and C also have no components in common if $P \in \overline{\mathcal{C}}_Q(5) - \big(\mathcal{C}_Q^1(5) \cup \mathcal{C}_Q^2(5)\big)$ (Lemma 3.6 and Remark 3.7) or $P = P_2$ as in Lemma 3.4. Hence, in all these cases there is a unique sixth intersection point in $C \cap S = C \cap S_0$, and we can extend σ to P by sending P to this sixth intersection point, say R; the divisor $5(Q) + (R)$ on S_0 is a hypersurface section inside the plane given by $z = 0$, so it is linearly equivalent to a multiple of $3(\mathcal{O} \cap S_0)$ on S_0, and we find $R = -5Q$ in $S_0^{\mathrm{ns}}(k)$.

We are left with the case that S_0 has a cusp (Lemma 3.5), or S_0 has a node and $P = P_1$ as in Lemma 3.4. In both cases, there is a $d \in k$ such that $f_0 = -3d^2$ and $g_0 = 2d^3$ and, in terms of the coordinates $(\overline{p} : \overline{q} : \overline{r})$ on $\mathbb{P}(1,2,1)$, we have $P = (1 : \alpha : 0) \in \mathcal{C}_Q^2(5)$ with $\alpha = \frac{1}{4}(x_0 + 2d)^{-1}$. By Lemma 3.1, the functions $\lambda_2 = \overline{r}/\overline{p}$ and $\mu_2 = \overline{q}/\overline{p}^2$ are affine coordinates for $\mathbb{H}_2$, with P corresponding to $(\lambda_2, \mu_2) = (0, \alpha)$, and the functions $\mathfrak{a}_2, \mathfrak{b}_2$, and $\mathfrak{c}_2$ are regular on $\mathbb{H}_2$. As before, we denote the restrictions of λ_2, μ_2, $\mathfrak{a}_2, \mathfrak{b}_2$, and $\mathfrak{c}_2$ to $\mathcal{C}_Q^2(5)$ by the same symbols.

Using (6) and (10), we can express, for each i, the function $F_i' = \lambda_2^i F_i$ on $\overline{\mathcal{C}}_Q(5)$ as a polynomial in terms of λ_2, μ_2, $\mathfrak{a}_2, \mathfrak{b}_2$, and $\mathfrak{c}_2$, which shows that F_i' is regular on $\mathcal{C}_Q^2(5)$. In particular, we have

$$F_5' = -2\mathfrak{b}_2\mathfrak{c}_2 + 3\mu_2^2 + f_4\lambda_2^4 + f_3\lambda_2^3\mu_2 + g_5\lambda_2^5,$$
$$F_6' = -\mathfrak{c}_2^2 + \mu_2^3 + f_4\lambda_2^4\mu_2 + g_6\lambda_2^6.$$

Recall from Subsection 3.2 that Δ_1 and Δ_2 are glued by setting $(z' : w') = (\overline{p}z : \overline{r}w) = (z : \lambda_2 w)$. Hence, on $\mathcal{C}_Q^2(5)$, the rational map $\rho: \mathcal{C}_Q^2(5) \dashrightarrow \mathcal{C}_Q^2(5) \times \mathbb{P}^1(z', w') \subset \Gamma$ is given by

$$\rho(P) = \big(P, (-F_5(P) : \lambda_2(P)F_6(P))\big) = \big(P, (-F_5'(P) : F_6'(P))\big).$$

The functions λ_2 and $\mu_2 - \alpha$ are local parameters for $\mathbb{H}_2$ at P, so their restrictions generate the maximal ideal $\mathfrak{m}$ of the local ring A_P of $\mathcal{C}_Q^2(5)$ at P. From (9), we find that in A_P we have

$$c_1{\mu_2}^2 + c_2\mu_2 - c_5 \equiv (c_6 - c_3\mu_2)\lambda_2 \tag{11}$$

modulo λ_2^2.

Now suppose $\phi_3 \neq 0$. Then $c_1 \neq 0$ and the left-hand side of (11) factors as $c_1(\mu_2-\alpha)(\mu_2-\alpha')$ with $\alpha' = \frac{1}{4}(x_0+7d)(x_0+2d)^{-1}(x_0+3d)^{-1}$. In fact, α and α' correspond to α_1 and α_2 of Proposition 3.4. Modulo $\mathfrak{m}^2$, the left- and right-hand side of (11) are congruent to $c_1(\mu_2-\alpha)(\alpha-\alpha')$ and $(c_6-c_3\alpha)\lambda_2$, respectively. Assume $d \neq 0$ as well. Then $\alpha' \neq \alpha$, so we find that modulo $\mathfrak{m}^2$ we have $\mu_2 - \alpha \equiv \delta\lambda_2$ with

$$\delta = \frac{c_6 - c_3\alpha}{c_1(\alpha - \alpha')}.$$

Hence, $\mathfrak{m}$ is generated by λ_2 and one checks, preferably with the help of a computer, that we have

$$F_5' \equiv \frac{(f_1 d + g_1)\phi_5}{(x_0 - d)^{10}\phi_2^2} \cdot \lambda_2 \qquad \text{and} \qquad F_6' \equiv \frac{(f_1 d + g_1)\phi_4}{(x_0 - d)^{5}\phi_2^2} \cdot \lambda_2 \qquad \pmod{\lambda_2\mathfrak{m}}. \tag{12}$$

We claim that (12) also holds when $d = 0$ or $\phi_3 = 0$. Indeed, if $\phi_3 = 0$, then one uses $x_0 = -3d$, while $c_1 = 0$ and $c_2 \neq 0$, so (11) yields $\mu_2 - \alpha \equiv c_2^{-1}(c_6 - \alpha c_3)\lambda_2 \pmod{\mathfrak{m}^2}$; it follows that λ_2 generates $\mathfrak{m}$, and one checks (12) again by computer. If $d = 0$, then $\mathfrak{m}$ may not be principal, so being congruent modulo $\lambda_2\mathfrak{m}$ is potentially stronger than being congruent modulo $\mathfrak{m}^2$; but using that modulo $\lambda_2\mathfrak{m}$ we have (11) and $\mu_2\lambda_2 \equiv \alpha\lambda_2$, one can again check that (12) holds. Hence, (12) holds in all cases.

Now $f_1 d + g_1$ is nonzero because the surface S is smooth at the singular point of S_0. Also, since Q is not the singular point of S_0, we have $x_0 \neq d$ and ϕ_4 and ϕ_5 do not both vanish. We conclude that $\rho: \mathcal{C}_Q^2(5) \dashrightarrow \mathcal{C}_Q^2(5) \times \mathbb{P}^1(z', w')$ is well defined at P, sending P to $\big(P, (-F_5'(P) : F_6'(P))\big) = \big(P, (-\phi_5 : (x_0 - d)^5\phi_4)\big)$. Substituting this into the parametrization of Lemma 3.3, we find $\sigma(P) = \gamma(\rho(P)) = (x_1 : y_1 : 0 : 1)$, with

$$x_1 = d + \frac{(x_0 - d)^4}{16(x_0 + 2d)(x_0 + 5d)^2} \qquad \text{and} \qquad y_1 = -\frac{(x_0 - d)^3(x_0^2 + 22dx_0 + 49d^2)y_0}{64(x_0 + 2d)^2(x_0 + 5d)^3}.$$

It is easy to check that this point equals $-4Q$ in the group $S_0^{\mathrm{ns}}(k)$, using the fact that the tangent line to S_0 at Q intersects S_0 also in $-2Q$, the tangent line to S_0 at $-2Q$ intersects S_0 also in $4Q$, and the inverse of a point is obtained by negating the y-coordinate. □

Corollary 3.10. *The following statements hold. The multiples of Q are taken in the group $S_0^{\mathrm{ns}}(k)$.*

1. *We have $\sigma(\Omega) = \{-5Q\}$ if and only if S_0 is smooth.*
2. *If $\sigma(\Omega) = \{-4Q, -5Q\}$, then S_0 is nodal. The converse holds if $3Q \neq \mathcal{O}$.*
3. *If S_0 is cuspidal, then $\sigma(\Omega) = \{-4Q\}$. The converse holds if $3Q \neq \mathcal{O}$.*
4. *If $4Q \neq \mathcal{O}$ and $5Q \neq \mathcal{O}$, then $\sigma(\Omega) \subset S_0^{\mathrm{ns}}(k) - \{\mathcal{O}\}$ and $\varphi(\sigma(\Omega)) = \{(0 : 1)\}$.*

Proof. The 'if'-part of (1) follows immediately from Proposition 3.9. For the 'only if'-part, note that if S_0 is singular, then by Proposition 3.9 there exists a $P \in \Omega$ with $\sigma(P) = -4Q$: when S_0 is cuspidal, this holds for any $P \in \Omega$ and when S_0 is nodal, we can take $P = P_1$ as in Lemma 3.4.

The first part of (3) follows directly from Proposition 3.9. Together with (1), this also implies the first part of (2). If $3Q \neq \mathcal{O}$ and S_0 is nodal, then for the points P_1 and P_2 as in Lemma 3.4, we have $\sigma(P_1) = -4Q$ and $\sigma(P_2) = -5Q$ by Proposition 3.9, which proves the second part of (2). The second part of (3) now follows from (1) and (2).

Statement (4) follows immediately from Proposition 3.9. □

The next two sections investigate the conditions under which $\sigma: \overline{\mathcal{C}}_Q(5) \dashrightarrow S$ sends an irreducible component of $\overline{\mathcal{C}}_Q(5)$ to a fiber of $\varphi|_S: S \dashrightarrow \mathbb{P}^1$. The following lemma shows that if this is the fiber that contains Q, then σ is the constant map to Q on the component.

Lemma 3.11. *Let $\mathcal{C}_0$ be a component of $\overline{\mathcal{C}}_Q(5)$ for which $\varphi(\sigma(\mathcal{C}_0)) = (0:1)$. Then $\sigma(\mathcal{C}_0) = Q$.*

Proof. Without loss of generality, we assume k is algebraically closed. Let $P \in \mathcal{C}_0 \cap \mathcal{C}_Q(5)$ be such that the associated section $C \in \mathcal{C}(k)$ is not entirely contained in S. Then σ is well defined at P and $\sigma(P)$ is the unique sixth intersection point of C with S. Since C is a section of $\varphi: U \dashrightarrow \mathbb{P}^1$, it intersects the fiber S_0 only once, namely in Q, and as this sixth intersection point lies in S_0 as well, we conclude $\sigma(P) = Q$. Thus all but finitely many points of $\mathcal{C}_0$ map to Q under σ, so $\sigma(\mathcal{C}_0) = Q$. □

4. Examples

In this section, k still denotes a field of characteristic not equal to 2 or 3. We will give examples of surfaces $S \subset \mathbb{P}$ over k given by (1), together with a point $Q \in S(k)$ for which the map $\sigma: \overline{\mathcal{C}}_Q(5) \dashrightarrow S$ sends at least one irreducible component of $\overline{\mathcal{C}}_Q(5)$ to a fiber of $\varphi|_S: S \dashrightarrow \mathbb{P}^1$. In the next section we will see that, at least outside characteristic 5, these examples include all cases where every component of $\overline{\mathcal{C}}_Q(5)$ is sent to a fiber on S.

In view of Theorem 1.2, it is important to note that in all examples, there are at least six (-1)-curves on S going through Q. Recall from Remark 2.6 that these correspond to the points of $\mathbb{A}^2 \cong \mathcal{C}_Q(4)$ with $F_4 = F_5 = F_6 = 0$. Recall also, from the last paragraph of Section 2, that the map $\varphi \circ \sigma: \mathcal{C}_Q(5) \to \mathbb{P}^1$ is given by $[-F_5 : F_6]$.

Example 4.1. Let $\beta, \delta \in k^*$ and assume the characteristic of k is not 5. Set

$$x_0 = 3(\beta^2 + 6\beta + 1), \qquad f_0 = -27(\beta^4 + 12\beta^3 + 14\beta^2 - 12\beta + 1),$$
$$y_0 = 108\beta, \qquad g_0 = 54(\beta^2 + 1)(\beta^4 + 18\beta^3 + 74\beta^2 - 18\beta + 1),$$

and let $S \subset \mathbb{P}$ be the surface given by (1) with $f = f_0 w^4$ and $g = \delta z^5 w + g_0 w^6$, and with point $Q = (x_0 : y_0 : 0 : 1)$. Assume that S is smooth, so that it is a del Pezzo surface. The curve S_0 is nonsingular if and only if $\beta(\beta^2 + 11\beta - 1) \neq 0$. The point Q has order 5 in $S_0^{\mathrm{ns}}(k)$. Generically, in particular over a field in which β and δ are independent transcendentals, the surface S is smooth and the fibration $\pi: \mathcal{E} \to \mathbb{P}^1$ has 10 nodal fibers (type I_1) and one cuspidal fiber (type II) above $(z:w) = (1:0)$.

Let α be an element in a field extension of k satisfying $\alpha^2 = \alpha + 1$. Then $\overline{\mathcal{C}}_Q(5)$ splits over $k(\alpha)$ into two components. The function F_6 vanishes on $\overline{\mathcal{C}}_Q(5)$ and the map $\sigma: \mathcal{C}_Q(5) \dashrightarrow S$ sends each component birationally to the cuspidal fiber. The conic bundle Γ splits into two components as well. Both components of the image T of $\gamma: \Gamma \to \mathbb{P}$ (cf. Remark 3.8) intersect S in the cuspidal fiber and, over an extension of $k(\alpha)$ of degree at most 5, five (-1)-curves; the surface T intersects S doubly in the cuspidal curve, as well as in ten (-1)-curves going through Q, corresponding to the points on the affine part $\mathcal{C}_Q(5)$ where F_5 vanishes. Indeed, if α, ϵ in an extension of k satisfy

$$\alpha^2 = \alpha + 1 \qquad \text{and} \qquad \delta = -6(\beta + \alpha^5)\epsilon^5,$$

then we have a section over $k(\alpha, \epsilon)$ going through Q with

$$x = \epsilon^2 z^2 + 6\alpha\epsilon zw + x_0 w^2,$$
$$y = -\epsilon^3 z^3 + 3(\beta + 2\alpha + 3)\epsilon^2 z^2 w + 18\alpha(\beta + 1)\epsilon z w^2 + y_0 w^3.$$

Example 4.2. Let k be a field of characteristic 5 containing elements $\alpha, \beta \in k$. Let $S \subset \mathbb{P}$ be the surface given by (1) with $f = \alpha z^4$ and $g = \beta z^6 + (3\alpha + 1)z^5 w + z w^5$, and with point $Q = (1:1:0:1)$. Assume that S is smooth, so that it is a del Pezzo surface. Generically, and in particular when α and β are independent transcendentals, this is the case, and the fibration $\pi: \mathcal{E} \to \mathbb{P}^1$ has 10 nodal fibers (type I_1) and one cuspidal fiber (type II), namely S_0. The curve $\mathcal{C}_Q(5)$ is given by

$$q^2 + (2p^2 - 1)q + p^4 - p^2 + 3\alpha = 0,$$

and F_5 vanishes on $\mathcal{C}_Q(5)$. By Lemma 3.11, the map σ is constant and sends $\mathcal{C}_Q(5)$ to Q. Generically, the curve $\mathcal{C}_Q(5)$ is geometrically irreducible. There are at least ten (-1)-curves going through Q.

Example 4.3. For any $\beta \neq 0$, the point $(x_0, y_0) = (3, \beta)$ has order 3 on the Weierstrass curve given by $y^2 = x^3 + f_0 x + g_0$ with $f_0 = 6\beta - 27$ and $g_0 = \beta^2 - 18\beta + 54$; this curve is nonsingular if and only if $\beta \neq 4$.
Subexample (i). For any $\alpha_1, \alpha_2, \alpha_3 \in k$ we consider the surface $S \subset \mathbb{P}$ given by (1) with

$$f = -3\alpha_1^2 z^4 + 3\alpha_2 z^3 w + (18 - 3\beta)\alpha_1 z^2 w^2 + f_0 w^4,$$
$$g = \alpha_3 z^6 + 3\alpha_1\alpha_2 z^5 w + (18 - 6\beta)\alpha_1^2 z^4 w^2 + (\beta - 9)\alpha_2 z^3 w^3 + (15\beta - 54)\alpha_1 z^2 w^4 + g_0 w^6,$$

and with $Q = (3 : \beta : 0 : 1)$, so that Q has order 3 on $S_0^{\mathrm{ns}}(k)$. Assume S is smooth, so that it is a del Pezzo surface. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by

$$(p^2 - \beta\alpha_1)(\beta q - p^2 + 2\beta\alpha_1) = 0.$$

The function $F_5 = 3\beta^{-1} p(q + \alpha_1)(\beta q - p^2 + 2\beta\alpha_1)$ vanishes on the component given by the vanishing of the second factor; by Lemma 3.11, this component is contracted by the map $\sigma\colon \overline{\mathcal{C}}_Q(5) \dashrightarrow S$, which sends it to Q. There are at least six (-1)-curves on S going through Q.
Subexample (ii). For any $\alpha_4, \alpha_5, \alpha_6 \in k$ we consider the surface $S \subset \mathbb{P}$ given by (1) with

$$f = 3\alpha_4 z^3 w + f_0 w^4,$$
$$g = \alpha_6 z^6 + \alpha_5 z^3 w^3 + g_0 w^6,$$

and with $Q = (3 : \beta : 0 : 1)$. Assume S is smooth, so that it is a del Pezzo surface. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by

$$p(\beta p q - p^3 + (\beta - 9)\alpha_4 - \alpha_5) = 0.$$

Again, the function $F_5 = 3\beta^{-1} q(\beta p q - p^3 + (\beta - 9)\alpha_4 - \alpha_5)$ vanishes on the component given by the vanishing of the second factor; again by Lemma 3.11, this component is contracted by the map $\sigma\colon \overline{\mathcal{C}}_Q(5) \dashrightarrow S$, which sends it to Q. There are at least nine (-1)-curves on S going through Q.
Subexample (iii). Let S be any smooth surface that fits in both families of these examples, i.e., with $\alpha_1 = 0$, $\alpha_4 = \alpha_2$, $\alpha_5 = (\beta - 9)\alpha_2$, and $\alpha_6 = \alpha_3$. Writing $\epsilon = \alpha_2$ and $\delta = \alpha_3$, we have

$$f = 3\epsilon z^3 w + f_0 w^4,$$
$$g = \delta z^6 + (\beta - 9)\epsilon z^3 w^3 + g_0 w^6.$$

Generically, say over a field in which β, δ, and ϵ are independent transcendentals, the surface S is smooth and the fibration $\pi\colon \mathcal{E} \to \mathbb{P}^1$ has twelve nodal fibers. Suppose S is indeed smooth. Then $\beta \notin \{0, 4\}$. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by

$$p^2(\beta q - p^2) = 0,$$

so it consists of two components. The function F_5 vanishes on both components, so by Lemma 3.11, they are contracted to Q by $\sigma\colon \mathcal{C}_Q(5) \dashrightarrow S$. There are at least nine (-1)-curves on S going through Q.

Example 4.4. For any $\beta \in k^*$, the point $(0, \beta)$ has order 3 on the elliptic curve given by $y^2 = x^3 + \beta^2$. In the following three subexamples, we take $g = \epsilon z^6 + \delta z^3 w^3 + \beta^2 w^6$ for some $\delta, \epsilon \in k$ and the point $Q = (0 : \beta : 0 : 1) \in \mathbb{P}$, which in all cases has order 3 on S_0.
Subexample (i). Let S be the surface given by (1) with $f = \alpha z^2 w^2$ for some $\alpha \in k$ and assume that S is smooth. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by $(3p^2 + \alpha)q = 0$. The function $F_5 = 3pq^2$ vanishes on the component given by $q = 0$; by Lemma 3.11, this component is contracted by the map $\sigma\colon \overline{\mathcal{C}}_Q(5) \dashrightarrow S$, which sends it to Q. There are at least six (-1)-curves on S going through Q. Generically, there are twelve nodal fibers.
Subexample (ii). Let S be the surface given by (1) with $f = \alpha z^3 w$ for some $\alpha \in k$ and assume that S is smooth. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by $p(3pq + \alpha) = 0$. The function $F_5 = q(3pq + \alpha)$ vanishes on one of the components; by Lemma 3.11, this component is contracted

by the map $\sigma\colon\overline{\mathcal{C}}_Q(5) \dashrightarrow S$, which sends it to Q. There are at least nine (-1)-curves on S going through Q. Generically, there are twelve nodal fibers.

Subexample (iii). Let S be the surface given by (1) with $f = 0$ and assume that S is smooth. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by $p^2q = 0$. The function F_5 vanishes on both components, so we have $\sigma(\overline{\mathcal{C}}_Q(5)) = Q$ by Lemma 3.11. The surface is isotrivial; all fibers have j-invariant 0. There are at least nine (-1)-curves on S going through Q, and there are six cuspidal fibers.

5. A multisection

We continue the notation of Sections 2 and 3. In particular, the field k with characteristic not equal to 2 or 3, the surface S, and the point Q are fixed as before, as are all the objects that depend on them.

As we have seen in the previous section, not every component of $\overline{\mathcal{C}}_Q(5)$ necessarily has its image under $\sigma\colon\overline{\mathcal{C}}_Q(5) \dashrightarrow S$ map dominantly to $\mathbb{P}^1$ under the projection $\varphi|_S\colon S \dashrightarrow \mathbb{P}^1$. Proposition 5.1 states that this does hold for every component if the order of Q is larger than 6. Moreover, Proposition 5.1 is sharp in the sense that there are examples where the order of Q is 6 and $\mathcal{C}_Q(5)$ has a component that maps under σ to Q.

Proposition 5.1. *Suppose the order of Q in $S_0^{\mathrm{ns}}(k)$ is larger than 5 and $\overline{\mathcal{C}}_Q(5)$ has a component $\mathcal{C}_0$ that maps under $\sigma\colon\overline{\mathcal{C}}_Q(5) \dashrightarrow S$ to a fiber of φ. Then Q has order 6 and $\sigma(\mathcal{C}_0) = Q$. The curve $\overline{\mathcal{C}}_Q(5)$ has a unique second component, which is sent under σ to a horizontal curve on S.*

Proof. Since $\mathcal{C}_0$ is projective, it contains a point in $\Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$, say R. By Corollary 3.10, part (4), we have $\varphi(\sigma(R)) = (0:1) \in \mathbb{P}^1$. Suppose σ does not send $\mathcal{C}_0$ to a horizontal curve. Then the composition $\varphi\circ\sigma$ sends $\mathcal{C}_0$ to $(0:1)$. From Lemma 3.11 we find $\sigma(\mathcal{C}_0) = Q$ and we obtain $Q = \sigma(R) = -4Q$ or $Q = \sigma(R) = -5Q$ from Proposition 3.9. As the order of Q is larger than 5, we find that the order is 6 and $\sigma(R) = -5Q$.

We have $c_2^2 + 4c_1c_5 = \phi_2^2(9\phi_4^2 - 4\phi_3\phi_4\psi + 4\phi_3^3)$. From equations (7) we find

$$\phi_3\phi_4\psi - \phi_3^3 = \phi_4(\phi_4 + \phi_2^2) - \phi_3^3 = \phi_4^2 + \phi_5 = 2\phi_4^2 + \phi_6, \tag{13}$$

so the factor $9\phi_4^2 - 4\phi_3\phi_4\psi + 4\phi_3^3$ equals

$$9\phi_4^2 - 4\phi_3\phi_4\psi + 4\phi_3^3 = 9\phi_4^2 - 4(2\phi_4^2 + \phi_6) = \phi_4^2 - 4\phi_6. \tag{14}$$

As $\phi_6 = 0$ (together with $y_0 \neq 0$) implies $\phi_4 \neq 0$, we get $c_2^2 + 4c_1c_5 \neq 0$, which in turn, together with $c_1 \neq 0$, implies that $\mathcal{C}_Q(5)$ is reduced. Suppose that each component of $\overline{\mathcal{C}}_Q(5)$ maps under σ to a fiber of φ. Then as above, we find $(\varphi\circ\sigma)(\overline{\mathcal{C}}_Q(5)) = (0:1)$ and as the composition $\varphi\circ\sigma$ is given by $(-F_5 : F_6)$, we find that F_5 vanishes on $\mathcal{C}_Q(5)$; as $\mathcal{C}_Q(5)$ is reduced, this implies that if we view F_4 and F_5 as polynomials in $k[p,q]$ (cf. Remark 2.5), then F_5 is a multiple of F_4. Viewing F_4 and F_5 as quadratic polynomials in q over $k[p]$, and comparing the coefficients in $k[p]$ of q^2 in

$$\begin{aligned} \phi_2^3F_4 &= \phi_2^2\phi_3q^2 + (-3\phi_2\phi_4p^2 + \ldots)q + \ldots, \\ \phi_2^3F_5 &= \phi_2\big((\phi_2^2 - 2\phi_4)p - \psi l_1\big)q^2 + \big((\phi_4\psi - 4\phi_3^2)p^3 + \ldots\big)q + \ldots, \end{aligned}$$

we find

$$\phi_2\phi_3F_5 = \big((\phi_2^2 - 2\phi_4)p - \psi l_1\big)F_4.$$

Comparing the coefficient of p^3q in this equality gives

$$\phi_3(\phi_4\psi - 4\phi_3^2) = -3\phi_4(\phi_2^2 - 2\phi_4). \tag{15}$$

Since $\phi_4 - \psi\phi_3 + \phi_2^2 = 0$ by equations (7), we find from (14) that the difference of the two sides in (15) equals

$$-3\phi_4(\phi_2^2 - 2\phi_4) - \phi_3(\phi_4\psi - 4\phi_3^2) + 3\phi_4(\phi_4 - \psi\phi_3 + \phi_2^2) = 9\phi_4^2 - 4\phi_3\phi_4\psi + 4\phi_3^3 = \phi_4^2 - 4\phi_6.$$

Hence, the equality (15) is equivalent to $4\phi_6 = \phi_4^2$, so we obtain $\phi_4 = \phi_6 = 0$, a contradiction from which we conclude that not all components map to a vertical component. It follows that there is a second component, which is unique as $c_1 \neq 0$ implies that there are at most two components. This second component maps to a horizontal curve on S. □

We say that two pairs (X_1, Q_1) and (X_2, Q_2) of a variety with a point on it are isomorphic if there is an isomorphism from X_1 to X_2 that maps Q_1 to Q_2. For example, the involution $\iota\colon \mathbb{P} \to \mathbb{P}$ that sends $(x : y : z : w) \in \mathbb{P}$ to $(x : y : -z : w - z)$ fixes Q, so it induces an isomorphism, also denoted ι, from the pair (S, Q) to $(\iota(S), Q)$; the surface $\iota(S)$ is given by $y^2 = x^3 + \tilde{f}(z, w)x + \tilde{g}(z, w)$, with

$$\tilde{f}(z, w) = f(-z, w - z) = f_0 w^4 + (-4f_0 - f_1)w^3 z + \dots,$$
$$\tilde{g}(z, w) = g(-z, w - z) = g_0 w^6 + (-6g_0 - g_1)w^5 z + \dots.$$

Note that ι fixes the points in the fiber above $(0 : 1)$ and it switches the fibers above $(1 : 1)$ and $(1 : 0)$. It also fixes f_0 and g_0 and it replaces f_1 and g_1 by $-4f_0 - f_1$ and $-6g_0 - g_1$, respectively.

The following lemma is well known (see [6, Proposition 8.2.8.] for $n = 5$, [10, pp. 457] for $n = 3$, and [17, Table 3] for characteristic 0). The lemma is used in Propositions 5.3 and 5.5.

Lemma 5.2. *Let E be an elliptic curve over k and $n \in \{3, 5\}$ an integer. Let $P \in E(k)$ be a point of order n. Then there exist elements $\beta \in k$ and $e \in \{0, 1\}$ such that the pair (E, P) is isomorphic to the pair $(E', (0, 0))$, with E' given by*

$$\begin{cases} y^2 + exy + \beta y = x^3 & \text{if } n = 3, \\ y^2 + (\beta + 1)xy + \beta y = x^3 + \beta x^2 & \text{if } n = 5. \end{cases}$$

Proof. After choosing an initial Weierstrass model for E, we may apply a linear change of variables to obtain a model E' in which P corresponds to $(0, 0)$. Given that the order of P is not 2, we may also assume that the tangent line to the model E' at P is given by $y = 0$. Then there are $a_1, a_2, a_3 \in k$ with $a_3 \neq 0$ such that E' is given by $y^2 + a_1xy + a_3y = x^3 + a_2x^2$. We have $-2P = (-a_2, 0)$, so $3P = 0$, or, equivalently, $P = -2P$, holds if and only if $a_2 = 0$.

If $n = 3$, so $3P = 0$, then either $a_1 = 0$ or $a_1 \neq 0$, and in the latter case, we may scale x and y such that we have $a_1 = 1$. These two cases are exactly the claimed cases, with $\beta = a_3$ and $e = a_1$.

If $n = 5$, then we have $a_2 \neq 0$ and $a_3 \neq 0$, so we may scale x and y such that we have $a_2 = a_3$. Then we have $3P = (-a_1 + 1, a_1 - a_2 - 1)$, so the property $5P = 0$, or, equivalently, $3P = -2P$, yields $a_1 = a_2 + 1$, which yields the claimed case with $\beta = a_2$. □

Proposition 5.3. *Suppose that the characteristic of k is not 5, and $5Q = \mathcal{O}$ in $S_0^{\mathrm{ns}}(k)$. If no component of $\overline{\mathcal{C}}_Q(5)$ maps under σ to a horizontal curve on S, then there exist $\beta, \delta \in k$ such that the pair (S, Q) is isomorphic to the pair of Example 4.1.*

Proof. If E is an elliptic curve over k with a point P of order 5, then by Lemma 5.2, there exists a $\beta \in k$ such that E is isomorphic to the elliptic curve given by $y^2 + (\beta + 1)xy + \beta y = x^3 + \beta x^2$, with P corresponding to the point $(0, 0)$. A short Weierstrass model for this curve is given by $v^2 = u^3 + Au + B$, with the point $(0, 0)$ corresponding to (u_0, v_0), where

$$\begin{aligned} u_0 &= 3(\beta^2 + 6\beta + 1), \\ v_0 &= 108\beta, \\ A &= -27(\beta^4 + 12\beta^3 + 14\beta^2 - 12\beta + 1), \\ B &= 54(\beta^2 + 1)(\beta^4 + 18\beta^3 + 74\beta^2 - 18\beta + 1). \end{aligned}$$

If S_0 is smooth, then, as Q has order 5 on S_0 and isomorphisms between short Weierstrass models are all given by appropriate scaling of the coordinates, there are $\beta, \eta \in k$ such that

$$(x_0, y_0, f_0, g_0) = (u_0\eta^2, v_0\eta^3, A\eta^4, B\eta^6). \tag{16}$$

Another way to phrase this is that (16) gives a parametrization of the quadruples (x_0, y_0, f_0, g_0) with $y_0^2 = x_0^3 + f_0x_0 + g_0$ for which the associated fifth division polynomial $\Phi_5 \in k[f_0, g_0][x]$ vanishes at x_0. Hence, also in the case that S_0 is singular, there exist $\beta, \eta \in k$ for which (16) holds. From $y_0 \neq 0$, we get $\beta, \eta \neq 0$. Without loss of generality, we assume $\eta = 1$. The fiber S_0 is singular if and only if $D = \beta(\beta^2 + 11\beta - 1)$ is zero, and in this case S_0 is nodal. Note that because Q has order 5, we have $\phi_5 = 0$ and $\phi_3, c_1 \neq 0$.

We first state two claims, both with a computational proof.

Claim 1: *If $D = 0$ and F_4 divides F_5F_6, then S is singular.*

Proof. Since the main coefficient c_1 of F_4 as a polynomial in q over $k[p]$ is invertible, we can compute (by computer, with $x_0, f_0, \dots, f_4, g_0, \dots, g_6$ independent transcendentals) the remainder of F_5F_6 upon division by F_4, which is a polynomial $L = \mu q + \nu$, with $\mu, \nu \in k[p]$ of degree 9 and 11, respectively. Our special values of x_0, f_0, g_0 already imply that the coefficients of p^{11} and $p^9 q$ in L specialize to 0, and the fact that F_4 divides F_5F_6 implies that L specializes to 0. We consider two cases, based on the characteristic of k.
Case 1: the characteristic of k is not $11, 17, 23$, or 29.

In this case, the vanishing of the (specialization of the) coefficients of p^8q, p^7q, p^6q, and p^5q in L determine, in that order, the values of f_1, f_2, f_3, and f_4 in terms of $g_0, \dots, g_6$. The vanishing of the coefficient of p^8 then implies that we have one of two subcases: (a) $g_1 = 0$ or (b) $g_2 = \lambda g_1^2$ for some specific constant λ.

Assume we are in subcase (a), i.e., $g_1 = 0$. The vanishing of the coefficient of p^7 yields $g_2 = 0$; then the vanishing of the coefficients of p^5 and p^3q implies $g_3 = g_6 = 0$, and finally the vanishing of the coefficients of p^3 gives $g_4 = 0$, which shows that the pair (S, Q) is isomorphic to the pair in Example 4.1, with $\delta = g_5$, though F_6 vanishes on both components $\mathcal{C}_0$ and $\mathcal{C}_1$, and S is singular.

Assume we are in subcase (b), i.e., $g_2 = \lambda g_1^2$. We may assume $g_1 \neq 0$, and the vanishing of the coefficients of p^7, p^6, p^5, and finally p^3q, express g_3, g_4, g_6, g_5, in that order, in terms of the remaining unknown coefficients of g, which in the end yields a surface S that is singular.
Case 2: the characteristic of k is not $7, 13$, or 19.

As in case 1, we similarly solve for the parameters $f_1, \dots, f_4$ and $g_1, \dots, g_6$, except that we start by expressing $g_1, \dots, g_4$ in terms of $f_1, \dots, f_4$. We conclude also in these characteristics that S is singular, thus proving the claim.

Claim 2: *If $D \neq 0$ and $\mathcal{C}_Q(5)$ is reduced and F_4 divides $(F_5 + F_6)F_6$, then either S is singular, or there exists a $\delta \in k$, such that the pair (S, Q) is isomorphic to the pair of Example 4.1.*

Proof. Since $\mathcal{C}_Q(5)$ is reduced, i.e., F_4 has no multiple factors, the condition that F_4 divides $(F_5 + F_6)F_6$ is equivalent to all components of $\mathcal{C}_Q(5)$ being sent under the map $\varphi \circ \sigma$ to $(1:1)$ or $(1:0)$. As the isomorphism ι described before Lemma 5.2 switches the fibers above these two points, the hypotheses of this claim hold for the pair (S, Q) if and only if they hold for the pair $(\iota(S), Q)$. Hence, without loss of generality we may apply ι at some point.

Viewing F_4, F_5, F_6 as polynomials in q over $k[p]$, we find that generically, say over a field in which $x_0, f_0, \dots, f_4, g_0, \dots, g_6$ are independent transcendentals, there are $d_0, \dots, d_{10}$ and $e_0, \dots, e_{12}$, in terms of these transcendentals, such that

$$(F_5 + F_6)F_6 \equiv (d_{10}p^{10} + \cdots + d_1 p + d_0)q + e_{12}p^{12} + \cdots + e_1 p + e_0 \pmod{F_4}.$$

The fact that Q has order 5 implies that $d_{10}, d_9, e_{12}, e_{11}$ specialize to 0. In our case, the other coefficients $d_0, \dots, d_8, e_0, \dots, e_{10}$ specialize to 0 as well. We claim that from the fact that e_{10} and d_8 specialize to 0, it follows that

$$\left\{ \begin{array}{l} f_1 = 0 \\ g_1 = 0 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} f_1 = -4f_0 \\ g_1 = -6g_0 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} f_1 = -2f_0 - 54\gamma^{-1}\lambda \\ g_1 = -3g_0 + 54\gamma^{-1}\mu \end{array} \right. \tag{17}$$

for some element $\gamma \in k$ with $\gamma^2 = 5$ and with

$$\lambda = (\beta^2 + 1)(\beta^2 + 10\beta - 1),$$
$$\mu = 3(\beta^6 + 16\beta^5 + 49\beta^4 - 40\beta^3 - 49\beta^2 + 16\beta - 1).$$

Indeed, for any γ in an extension of k with $\gamma^2 = 5$ and $\omega = \frac{1}{2}(3-\gamma)$, the linear combinations

$$\begin{aligned}
&\frac{1}{2^5 3^3}\phi_2^4\left((3\beta-1)(7\beta+1)d_8 + 180\beta(11\beta-2)e_{10}\right)\\
&\quad = \big(3g_0f_1 - 2f_0g_1\big)\cdot\big((f_1+2f_0)\mu + (g_1+3g_0)\lambda\big) \quad \text{and}\\
&\frac{1}{4}\phi_2^4\left(\omega^4\beta^2 + (2\gamma-4)\beta + \omega^{-1}\right)(d_8 + 36\omega e_{10})\\
&\quad = \big(\gamma(3g_0f_1 - 2f_0g_1) - 54(g_1\lambda + f_1\mu)\big)\\
&\qquad \cdot\big(\gamma(3g_0f_1 - 2f_0g_1) - 54((g_1+6g_0)\lambda + (f_1+4f_0)\mu)\big)
\end{aligned}$$

of d_8 and e_{10} factor into two linear factors. Therefore, the vanishing of d_8 and e_{10} implies the vanishing of one of the first two factors and one of the second two. The four combinations give four systems of two linear equations in the two variables f_1 and g_1. For each combination, the determinant of the system is a nonzero multiple of D and therefore nonzero itself. The systems yield exactly the four claimed pairs for (f_1, g_1) in (17).

Note that as the isomorphism ι replaces f_1 and g_1 by $-4f_0 - f_1$ and $-6g_0 - g_1$, respectively, it switches the first two cases in (17), as well as the last two cases given by the third pair for $\pm\gamma$. Therefore, after applying the isomorphism ι if necessary, we may assume we have only two subcases.

Case 1: We have $(f_1, g_1) = (0, 0)$.

The equations $d_7 = e_9 = 0$ determine a system of two linear equations in f_2 and g_2, of which the determinant is a nonzero multiple of D and therefore nonzero itself. The unique solution is $f_2 = g_2 = 0$. Subsequently, the system $d_6 = e_8 = 0$ gives $f_3 = g_3 = 0$ and then the system $d_5 = e_7 = 0$ yields $f_4 = g_4 = 0$. At this point, the coefficients d_4 and e_6 specialize to 0 automatically, and the equation $d_3 = 0$ determines $g_6 = 0$. With $g_5 = \delta$, we obtain exactly the surface of Example 4.1.

Case 2: We have $(f_1, g_1) = (-2f_0 - 54\gamma^{-1}\lambda, -3g_0 + 54\gamma^{-1}\mu)$.

As in the previous subcase, the linear systems $d_{9-i} = e_{11-i} = 0$ determine f_i and g_i inductively for $i = 2, 3, 4$. Again, the coefficients d_4 and e_6 then specialize to 0 automatically. Finally, the system $d_3 = e_6 = 0$ is linear in g_5 and g_6 and determines these two parameters uniquely. However, this yields a surface S that is singular. More specifically, the associated minimal elliptic surface has two singular fibers of type I_5. This proves the claim.

We continue the proof of the proposition. Suppose no component of $\overline{\mathcal{C}}_Q(5)$ maps under σ to a horizontal curve on S, so $\varphi \circ \sigma$ has finite image. If we had $\varphi(\sigma(\overline{\mathcal{C}}_Q(5)) = (0:1)$, then we would have $\sigma(\overline{\mathcal{C}}_Q(5)) = Q = -4Q$ by Lemma 3.11, so by Corollary 3.10, part (3), the fiber S_0 would be cuspidal. From this contradiction we conclude that there is a component $\mathcal{C}_1$ with $\varphi(\sigma(\mathcal{C}_1)) \neq (0:1)$. Without loss of generality, we assume $\varphi(\sigma(\mathcal{C}_1)) = (1:0) =: \infty$ and we write $S_\infty = \varphi^{-1}(\infty)$.

We will distinguish the following three cases.

(A) There is a component $\mathcal{C}_0$ of $\overline{\mathcal{C}}_Q(5)$ with $\varphi(\sigma(\mathcal{C}_0)) = (0:1)$.
(B) There is a component $\mathcal{C}_0$ of $\overline{\mathcal{C}}_Q(5)$ with $\varphi(\sigma(\mathcal{C}_0)) \neq (0:1), (1:0)$.
(C) There is no component $\mathcal{C}_0$ of $\overline{\mathcal{C}}_Q(5)$ with $\varphi(\sigma(\mathcal{C}_0)) \neq (1:0)$.

Since $c_1 \neq 0$, the curve $\overline{\mathcal{C}}_Q(5)$ has at most two components and both are reduced if there are two, so in cases (A) and (B), the components are $\mathcal{C}_0$ and $\mathcal{C}_1$, and $\mathcal{C}_Q(5)$ is reduced.

We start with case (A). Assume that there is a component $\mathcal{C}_0$ of $\overline{\mathcal{C}}_Q(5)$ with $\varphi(\sigma(\mathcal{C}_0)) = (0:1)$. From Lemma 3.11 we find $\sigma(\mathcal{C}_0) = Q = -4Q$, and as $\mathcal{C}_0$ contains points of Ω, we conclude that S_0 is singular from Corollary 3.10, part (1), so $\beta^2 + 11\beta - 1 = 0$. If we consider F_4, F_5, and F_6 as polynomials in q over $k[p]$ (cf. Remark 2.5), then F_5 and F_6 vanish on $\mathcal{C}_0$ and $\mathcal{C}_1$, respectively, so F_4 divides F_5F_6. Claim 1 implies that S is singular, a contradiction.

We continue with case (B). Assume that there is a component $\mathcal{C}_0$ of $\overline{\mathcal{C}}_Q(5)$ with $\varphi(\sigma(\mathcal{C}_0)) \neq (0:1), (1:0)$. After applying an automorphism of the base curve $\mathbb{P}^1$ that fixes $(0:1)$ and $(1:0)$, we may assume $\varphi(\sigma(\mathcal{C}_0)) = (1:1)$, so that $F_5 + F_6$ and F_6 vanish on $\mathcal{C}_0$ and $\mathcal{C}_1$, respectively, and the product $(F_5+F_6)F_6$ is divisible by F_4. Since the points in $\Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$ map under σ to S_0, and $\varphi(S_0 - \{\mathcal{O}\}) = (0:1)$, the points in Ω map under σ to $\mathcal{O} = -5Q$; it follows from Corollary 3.10, part (1), that S_0 is smooth, so we find $D \neq 0$. Hence, we are done by Claim 2.

We finish with case (C). In that case, we have $\sigma(\overline{\mathcal{C}}_Q(5)) \subset S_\infty$, so F_6 vanishes on $\mathcal{C}_Q(5)$, and also $\sigma(\Omega) \subset S_\infty$. From Proposition 3.9 we conclude $\sigma(\Omega) \subset S_0 \cap S_\infty = \{\mathcal{O}\} = \{-5Q\}$; it follows from Corollary 3.10, part (1), that S_0 is smooth, so we find $D \neq 0$. From (13) we obtain

$$c_2^2 + 4c_1c_5 = \phi_2^2(9\phi_4^2 - 4\phi_3\phi_4\psi + 4\phi_3^3) = \phi_2^2(9\phi_4^2 - 4(\phi_4^2 + \phi_5)) = \phi_2^2(5\phi_4^2 - 4\phi_5).$$

As $\phi_5 = 0$ (together with $y_0 \neq 0$) implies $\phi_4 \neq 0$, we get $c_2^2 + 4c_1c_5 \neq 0$, which in turn, together with $c_1 \neq 0$, implies that $\mathcal{C}_Q(5)$ is reduced. Therefore, F_6 is a multiple of F_4, so we are done by claim 2. □

Indeed, in characteristic 5, there are other examples than those mentioned in Proposition 5.3 where Q has order 5 and no component of $\overline{\mathcal{C}}_Q(5)$ maps under σ to a horizontal curve on S (see Example 4.2). It takes less computational force to deal with the case that Q has order 4.

Proposition 5.4. *Suppose $4Q = \mathcal{O}$ in $S_0^{\text{ns}}(k)$. Then $\mathcal{C}_Q(5)$ has a component that maps under $\sigma\colon \mathcal{C}_Q(5) \dashrightarrow S$ to a horizontal curve on S.*

Proof. First note that the fiber S_0 does not have a cusp, as the additive reduction together with the identity $4Q = \mathcal{O}$ would imply that the characteristic of k is 2, which it is not by assumption. Therefore, by Corollary 3.10, parts (1) and (2), at least one of the points in Ω maps to $-5Q = -Q$. Let R be such a point and let $\mathcal{C}_0$ be a component of $\overline{\mathcal{C}}_Q(5)$ that contains R. Suppose that $\mathcal{C}_0$ is sent by σ to a fiber on S, so that $\varphi(\sigma(\mathcal{C}_0))$ is a point on $\mathbb{P}^1$. From $\sigma(R) = -Q \in S_0$ we conclude $\varphi(\sigma(\mathcal{C}_0)) = (0:1)$, and Lemma 3.11 implies $\sigma(\mathcal{C}_0) = Q$, which contradicts $\sigma(R) = -Q$, so $\mathcal{C}_0$ is sent to a horizontal curve on S. □

Finally, we deal with the case that Q has order 3.

Proposition 5.5. *Suppose $3Q = \mathcal{O}$ in $S_0^{\text{ns}}(k)$. Then $\mathcal{C}_Q(5) \subset \mathbb{A}^2(p,q)$ has a unique component that projects birationally to $\mathbb{A}^1(p)$. If this component maps under $\sigma\colon \mathcal{C}_Q(5) \dashrightarrow S$ to a vertical curve on S, then the pair (S,Q) is isomorphic to one of the pairs described in Examples 4.3 and 4.4. Moreover, if another component of $\mathcal{C}_Q(5)$ maps to a vertical curve on S as well, then the pair (S,Q) is isomorphic to one of the pairs described in Subexamples 4.3(iii) and 4.4(iii).*

Proof. If E is an elliptic curve over k with a point $P \in E(k)$ of order 3, then by Lemma 5.2, there exist elements $\beta' \in k$ and $e \in \{0,1\}$ such that E is isomorphic to the elliptic curve given by $y^2 + exy + \beta' y = x^3$, with the point P corresponding to $(0,0)$. Associated short Weierstrass models are given by $v^2 = u^3 + A_e u + B_e$, with the point $(0,0)$ corresponding to (u_e, v_e), where

$$\begin{aligned} u_e &= 3e, \\ v_e &= \beta, \\ A_e &= (6\beta - 27)e, \\ B_e &= \beta^2 - 18(\beta - 3)e, \end{aligned}$$

and where $\beta = 108\beta'$. We also simplified these expressions using $e^2 = e$. Since Q has order 3 in $S^{\text{ns}}(k)$, we find, as in the proof of Proposition 5.3, that there are $\beta, \eta \in k^*$ and $e \in \{0,1\}$, such that

$$(x_0, y_0, f_0, g_0) = (u_e\eta^2, v_e\eta^3, A_e\eta^4, B_e\eta^6),$$

also in the case that S_0 is singular; the property $\beta \neq 0$ follows from the assumption $y_0 \neq 0$. Without loss of generality we assume $\eta = 1$.

Any component $\mathcal{C}_0$ of $\overline{\mathcal{C}}_Q(5)$ contains a point $R \in \Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$, which satisfies $\varphi(\sigma(R)) = (0:1)$ by Corollary 3.10, part (4), so if $\sigma(\mathcal{C}_0)$ is contained in a fiber of $S \dashrightarrow \mathbb{P}^1$, then F_5 vanishes on $\mathcal{C}_0$.

As the order of Q is 3, we have $\phi_3 = 0$ and thus $\phi_2\phi_4 \neq 0$, so $c_1 = 0$ and $c_2 \neq 0$, and the curve $\mathcal{C}_Q(5)$ is given by $mq = n$ with

$$m = c_2p^2 + c_3p + c_4 \qquad \text{and} \qquad n = c_5p^4 + c_6p^3 + c_7p^2 + c_8p + c_9.$$

From $c_2 \neq 0$, we find that m is not identically 0, so indeed, there is a unique component of $\mathcal{C}_Q(5)$, say $\mathcal{C}_1$, that projects birationally to $\mathbb{A}^1(p)$. Assume that $\sigma(\mathcal{C}_1)$ is contained in a fiber on S. Then F_5 vanishes on $\mathcal{C}_1$ by the above. If we write F_5 as $F_5 = \delta_1q^2 + \delta_2q + \delta_3$, with $\delta_j \in k[p]$ of degree $2j - 1$, then we find that

$$L = \delta_1n^2 + \delta_2mn + \delta_3m^2$$

vanishes.

In the case $e = 1$, so $(x_0, y_0, f_0, g_0) = (u_1, v_1, A_1, B_1)$, we claim that the pair (S, Q) is isomorphic to one of the pairs of Example 4.3. We sketch a sequence of computations that proves the claim.

> A priori, say over a field in which the elements $x_0, f_0, \dots, f_4, g_0, \dots, g_6$ are independent transcendentals, the polynomial $L \in k[p]$ has degree 9, but from $\phi_3 = 0$, it already follows that the degree is at most 8. We will use the vanishing of all coefficients to identify the pair (S, Q).
>
> The vanishing of the coefficient of p^8 in L gives $3f_1g_0 = 2f_0g_1$. Since f_0 and g_0 do not both vanish, there is a $\delta \in k$ such that $f_1 = 2\delta f_0$ and $g_1 = 3\delta g_0$. After applying an automorphism of $\mathbb{P}^1$ given by $(z : w) \mapsto (2z : \delta z + 2w)$, we may assume without loss of generality that $\delta = 0$, so $f_1 = g_1 = 0$. Then the vanishing of the coefficient of p^7 in L shows that there is an $\alpha_1 \in k$ such that $f_2 = (18 - 3\beta)\alpha_1$ and $g_2 = (15\beta - 54)\alpha_1$. The coefficient of p^6 now vanishes automatically and the vanishing of the coefficient of p^5 yields $g_4 = (\beta - 3)(f_4 - 3\alpha_1^2)$. Subsequently, the vanishing of the coefficient of p^4 gives $g_5 = \big((2\beta - 9)f_3 - 3g_3\big)\alpha_1\beta^{-1}$. Then the coefficient of p^3 vanishes automatically. The vanishing of the coefficient of p^2 yields $f_4 = -3\alpha_1^2$ or $3g_3 = (\beta - 9)f_3$, but in the latter case, the vanishing of the coefficient of p yields the former, so we have $f_4 = -3\alpha_1^2$ in any case. Finally, the vanishing of the coefficient of p gives $3g_3 = (\beta - 9)f_3$ or $\alpha_1 = 0$; the former case yields Example 4.3(i) with $\alpha_2 = \frac{1}{3}f_3$ and $\alpha_3 = g_6$, while the latter case yields Example 4.3(ii) with $\alpha_4 = \frac{1}{3}f_3$, $\alpha_5 = g_3$ and $\alpha_6 = g_6$. This proves the claim.

We continue (still) with $e = 1$. Suppose we are in the case of Example 4.3(i). If σ sends one of the components of $\mathcal{C}_Q(5)$ given by $p^2 - \beta\alpha_1 = 0$ to a fiber of φ, then the first argument of this proof shows that $\frac{1}{3}\beta F_5 = p(q + \alpha_1)(\beta q - p^2 + 2\beta\alpha_1)$ vanishes on this component, which implies $\alpha_1 = 0$, so the pair (S, Q) belongs to the family described in Example 4.3(iii). Now suppose we are in the case of Example 4.3(ii). If σ sends the component of $\mathcal{C}_Q(5)$ given by $p = 0$ to a fiber of φ, then similarly $F_5 = 3\beta^{-1}q(\beta pq - p^3 + (\beta - 9)\alpha_4 - \alpha_5)$ vanishes on this component, which implies $\alpha_5 = (\beta - 9)\alpha_4$, so again the pair (S, Q) belongs to the family described in Example 4.3(iii). This finishes the case $e = 1$.

We now consider the case $e = 0$, so $(x_0, y_0, f_0, g_0) = (0, \beta, 0, \beta^2)$. We claim that the pair (S, Q) is isomorphic to one of the pairs of Example 4.4. We sketch a sequence of computations that proves the claim.

> Since $g_0 \neq 0$, we may apply an automorphism of $\mathbb{P}^1(z, w)$ given by $(z : w) \mapsto (6g_0z : g_1z + 6g_0w)$ to reduce to the case $g_1 = 0$. As in the case $e = 1$, we will use the vanishing of all coefficients in L to identify the pair (S, Q). The vanishing of the coefficients of p^8, p^7, p^5, and p^4 yields $f_1 = g_2 = f_4 = g_5 = 0$. Then the vanishing of the coefficient of p^2 yields $f_3g_4 = 0$; if $f_3 = 0$, then the vanishing of the coefficient of p in L gives $g_4 = 0$, so we have $g_4 = 0$ in any case. Then the vanishing of the coefficient of p gives $f_2 = 0$ or $f_3 = 0$ and these cases correspond to Subexamples 4.4(ii) and 4.4(i), respectively. This proves the claim.

Suppose we are in the case of Example 4.4(i), so $f_3 = 0$ and $f_2 = \alpha$. If σ sends one of the components of $\mathcal{C}_Q(5)$ given by $3p^2 + \alpha = 0$ to a fiber of φ, then the first argument of this proof shows that $F_5 = 3pq^2$ vanishes on this component, which implies $\alpha = 0$, so the pair (S, Q) belongs to the family described in Example 4.4(iii). Now suppose we are in the case of Example 4.4(ii), so $f_2 = 0$ and $f_3 = \alpha$. If σ sends the component of $\mathcal{C}_Q(5)$ given by $p = 0$ to a fiber of φ, then similarly

$F_5 = q(3pq + \alpha)$ vanishes on this component, which implies $\alpha = 0$, so again the pair (S, Q) belongs to the family described in Example 4.4(iii). This finishes the case $e = 0$ and thus the proof. □

Corollary 5.6. *Let k, S, and Q be as before. Assume that the pair (S, Q) is not isomorphic to the one of the pairs described in Examples 4.1, 4.3(iii), and 4.4(iii). If the order of Q in $S_0^{\mathrm{ns}}(k)$ is 5, then also assume that the characteristic of k is not equal to 5. Then the rational map $\sigma\colon \mathcal{C}_Q(5) \dashrightarrow S$ sends at least one component of $\mathcal{C}_Q(5)$ to a horizontal curve on S.*

Proof. From the assumption $y_0 \neq 0$, it follows that the order of Q is at least 3. The statement now follows immediately from Propositions 5.1, 5.3, 5.4, and 5.5. □

Remark 5.7. In Remark 3.8, we have seen that the closure of the image $\sigma(\overline{\mathcal{C}}_Q(5))$ in S is contained in the intersection $S \cap T$, where T is the image of $\gamma\colon \Gamma \to \mathbb{P}$. Generically, this containment is in fact an equality, but if there are any (-1)-curves of S going through Q, then these are components of the intersection $S \cap T$ as well. In degenerate cases, the intersection $S \cap T$ may contain even more components (see Remark 5.8). When studying our del Pezzo surfaces of degree one in families, it is more natural to look at the divisor $S \cap T$ on S than at the closure of the image $\sigma(\overline{\mathcal{C}}_Q(5))$.

We will now describe this intersection $S \cap T$ and its arithmetic genus in terms of the Picard group of S, at least in the generic case. Generically, the Picard group $\operatorname{Pic} S$ of S is generated by the divisor class of $-K_S$. Also generically, the surface T has degree 12 and the intersection $S \cap T$ is irreducible and reduced, so $\sigma(\overline{\mathcal{C}}_Q(5)) = S \cap T$ is linearly equivalent with $-12K_S$. The arithmetic genus of $\sigma(\overline{\mathcal{C}}_Q(5))$ is 67 in this case, and $\sigma(\overline{\mathcal{C}}_Q(5))$ has multiplicity 10 at the point Q, multiplicity 2 at $-5Q$, and is also singular at 20 more points, which agrees with the fact that the geometric genus of the normalization equals

$$67 - \tfrac{1}{2} \cdot 10 \cdot (10 - 1) - \tfrac{1}{2} \cdot 2 \cdot (2 - 1) - 20\left(\tfrac{1}{2} \cdot 2 \cdot (2 - 1)\right) = 1.$$

These 22 singular points of $\sigma(\overline{\mathcal{C}}_Q(5))$ are the intersection points of S with the singular locus of T, which is a curve with an embedded point at Q.

Conversely, the family of intersections of S with a hypersurface of degree 12 has dimension 78, as can be seen from the fact that the space of polynomials in x, y, z, w of weighted degree 12 modulo the multiples of the defining equation of S has dimension $102 - 23 = 79$ or from the fact that the linear system of curves in $\mathbb{P}^2$ of degree $3d$ having multiplicity at least d at each of 8 given points has dimension $\binom{3d+2}{2} - 1 - 8 \cdot \binom{d+1}{2} = \binom{d+1}{2}$, which equals 78 for $d = 12$. Hence, the subfamily of those intersections that have multiplicity 10 at Q, multiplicity 2 at $-5Q$, and 20 more singularities, has dimension

$$78 - \tfrac{1}{2} \cdot 10 \cdot (10 + 1) - \tfrac{1}{2} \cdot 2 \cdot (2 + 1) - 20 = 0,$$

so there are only finitely many curves satisfying these conditions.

We now give yet another description of $\sigma(\overline{\mathcal{C}}_Q(5))$ that narrows it down to one of only finitely many curves. Note that the projection $\nu\colon S \to \mathbb{P}(2,1,1)$ from S to the weighted projective space with coordinates x, z, w, gives S the structure of a double cover of a cone that is ramified at the singular point $\mathcal{O}$ (corresponding to the vertex of the cone), as well as over the curve given by $x^3 + fx + g = 0$, i.e., the locus of nontrivial 2-torsion points. The involution induced by this double cover is multiplication by -1 on the elliptic fibration. If we let $-\sigma(\overline{\mathcal{C}}_Q(5)) \subset S$ denote the image of $\sigma(\overline{\mathcal{C}}_Q(5))$ under this involution, then $\sigma(\overline{\mathcal{C}}_Q(5))$ and $-\sigma(\overline{\mathcal{C}}_Q(5))$ intersect each other in 36 points on the ramification locus of ν, as well as 108 points off the ramification locus. The image $\nu\big(\sigma(\overline{\mathcal{C}}_Q(5))\big) = \nu\big(-\sigma(\overline{\mathcal{C}}_Q(5))\big) \subset \mathbb{P}(2,1,1)$ is a curve of degree 24 that intersects the branch locus of ν at 36 points, being tangent at each, that has multiplicity 10 at $\nu(Q) = (x_0 : 0 : 1)$, multiplicity 2 at $\nu(-5Q)$, and that is singular at 74 more points (namely the 20 images under ν of the remaining singular points of $\sigma(\overline{\mathcal{C}}_Q(5))$, and the 54 images of the intersection points of $\sigma(\overline{\mathcal{C}}_Q(5))$ and $-\sigma(\overline{\mathcal{C}}_Q(5))$). These properties narrow down the 168-dimensional family of curves in $\mathbb{P}(2,1,1)$ of degree 24 to only finitely many curves.

Remark 5.8. Given that for all pairs (S, Q) described in the examples in the previous section there are at least six (-1)-curves on S going through Q, whenever we want to exclude any of these examples, it suffices to assume that Q does not lie on six (-1)-curves.

We will now explain in terms of the image T of $\gamma\colon\Gamma \to \mathbb{P}$ (cf. Remarks 3.8 and 5.7) why it is not surprising that the existence of many (-1)-curves on S through Q is related to the existence of a component of $\overline{\mathcal{C}}_Q(5)$ that maps under $\sigma\colon\overline{\mathcal{C}}_Q(5) \dashrightarrow S$ to a fiber of $\varphi\colon S \dashrightarrow \mathbb{P}^1$.

In Example 4.1, the scheme-theoretic intersection $D' = T \cap S$ consists of the ten (-1)-curves going through Q and the cuspidal fiber S_∞ with multiplicity 2. As a divisor on S, we have that D' is linearly equivalent to $-12K_S$ (cf. Remark 5.7).

In general, the pull-back of $S \cap T$ under the blow-up $\mathcal{E} \to S$ is a divisor D on $\mathcal{E}$ that consists of

(i) the components of the strict transform D_0 of the image $\sigma(\overline{\mathcal{C}}_Q(5))$,
(ii) the strict transforms of the (-1)-curves on S through Q,
(iii) the fiber $\mathcal{E}_0$, and
(iv) the zero section $\mathcal{O}$

with certain multiplicities. The degree of the restriction of $\pi\colon\mathcal{E} \to \mathbb{P}^1$ to D equals the intersection number of D with any fiber of π. Generically, this degree equals $\deg T = 12$, the multiplicities of $\mathcal{E}_0$ and $\mathcal{O}$ are 0, and the multiplicities of the components in (i) and (ii) are 1; the (-1)-curves on S intersect fibers with multiplicity 1, so the restriction of π to D_0 has degree $12-s$, where s is the number of (-1)-curves on S through Q. Note that Example 4.1 is not generic in the sense that the multiplicity of the zero section $\mathcal{O}$ is 2, thus reducing the degree of the restriction of π to D_0 to 0.

In general, the multiplicities of the components in (iii) and (iv) seem to depend only on the order of Q in $\mathcal{E}_0^{\mathrm{ns}}(k)$ and the singularity type of $\mathcal{E}_0$, but in any case we find that the more (-1)-curves there are on S that go through Q, the smaller the degree of the restriction of π to D_0, forcing all components of D_0 to be vertical in extreme cases.

In fact, a thorough investigation of the degree of T (which may itself be nonreduced) as well as all multiplicities might yield another proof of Corollary 5.6 under the assumption that Q not lie on six (-1)-curves of S, but it is not clear that this will require less computational effort than the given proof, especially given that even in the generic case, the intersection $S \cap T$ does not appear to admit a very elegant description (cf. Remark 5.7).

6. Torsion in a base change

In this section, k is still a field of characteristic not equal to 2 or 3.

Lemma 6.1. *Let B be a smooth curve over k and $\pi\colon\mathcal{E} \to B$ a minimal nonsingular elliptic fibration. Let C be a smooth curve over k and $\tau\colon C \to B$ a nonconstant morphism. Let $\pi'\colon\mathcal{E}' \to C$ be the minimal nonsingular model of the base change $\mathcal{E} \times_B C \to C$ of π by τ. Let $c \in C(\overline{k})$ be a point and set $b = \tau(c)$. Let $\mathcal{E}_b$ and $\mathcal{E}'_c$ be the fibers of π and π' over b and c, respectively. Let $e = e_c(\tau)$ be the ramification index of τ at c. Then the following statements hold.*

1. *If $\mathcal{E}_b$ has type I_d for some integer d, then $\mathcal{E}'_c$ has type I_{de}.*
2. *If $\mathcal{E}_b$ has type I_d^* for some integer d, then $\mathcal{E}'_c$ has type I_{de} for even e and type I_{de}^* for odd e.*
3. *If $\mathcal{E}_b$ has type IV^*, then $\mathcal{E}'_c$ has type I_0, IV^*, IV for $e \equiv 0,1,2 \pmod 3$, respectively.*
4. *If $\mathcal{E}_b$ has type II, then $\mathcal{E}'_c$ has type $I_0, II, IV, I_0^*, IV^*, II^*$ for $e \equiv 0,1,2,3,4,5 \pmod 6$, respectively.*
5. *If $\mathcal{E}_b$ has type III, then $\mathcal{E}'_c$ has type I_0, III, I_0^*, III^* for $e \equiv 0,1,2,3 \pmod 4$, respectively.*

Proof. This follows directly from Tate's algorithm (see [35] and [34, IV.9.4]). See also [21, Table VI.4.1], which is stated for characteristic zero. □

Lemma 6.2. *Suppose k is algebraically closed. Let S be a del Pezzo surface of degree 1 over k and $\pi\colon\mathcal{E} \to \mathbb{P}^1$ the associated elliptic fibration. Let M and N denote the number of singular fibers of π of type I_1 and type II, respectively. Then we have $M + 2N = 12$. If π is not isotrivial, then we have $M \geq 4$.*

Proof. Let $f, g \in k[z,w]$ be such that S is given by (1). The surface $\mathcal{E}$ is a minimal nonsingular elliptic surface with fibers that are either nodal (type I_1) or cuspidal (type II). The discriminant $\Delta = 4f^3 + 27g^2$ vanishes at points of $\mathbb{P}^1$ corresponding to nodal and cuspidal fibers to order 1 and 2, respectively, so we get $M + 2N = \deg\Delta = 12$. For any $t \in \mathbb{P}^1(\overline{k})$ for which $\pi^{-1}(t)$ has type II, both Δ and the j-invariant $j = 2^8 3^3 f^3/\Delta$ vanish at t (see [34, Tate's Algorithm, IV.9.4]), which implies that f vanishes at t. It follows that f vanishes at at least N points, so if $M < 4$, i.e, $N \geq 5$, then $f = 0$, so π is isotrivial. □

Remark 6.3. In characteristic zero, the identity $M + 2N = 12$ follows from the more general fact that the Euler number of $\mathcal{E}$, which is 12, equals the sum of the local Euler numbers, which are 1 and 2 for fibers of type I_1 and II, respectively (see [21, Table IV.3.1] and [21, Lemma IV.3.3]). The inequality $M < 4$ implies $N \geq 5$, which implies $N = 6$ by [26, Lemma 1.2], which in turn implies that π is isotrivial.

For $n \geq 3$, let $E(n) \to Y_1(n)$ be the universal elliptic curve over the usual modular curve $Y_1(n)$ over $\mathbb{Z}[1/n]$ with a section P that has order n in every fiber. Then every elliptic curve E over a scheme S over $\mathbb{Z}[1/n]$ –with nowhere vanishing j-invariant if $n = 3$– with a section that has order n in every fiber, is the base change of $E(n)/Y_1(n)$ by a unique morphism $S \to Y_1(n)$. Let $X_1(n)$ be the usual projective closure of $Y_1(n)$, and let $\upsilon(n) : \mathbb{E}(n) \to X_1(n)$ be the minimal nonsingular elliptic fibration over $X_1(n)$ associated to $E(n)/Y_1(n)$. From Ogg's description of the cusps of $X_1(n)$ in [25], we conclude that for each $n \geq 5$ and each divisor d of n, the number of fibers of $\upsilon(n)$ of type I_d is $\frac{1}{2}\varphi(d)\varphi(n/d)$ (see also [17, p. 219], or [17, Table 3] for explicit models for small n, which also show the types of the singular fibers). Table 1 gives the genus $g(X_1(n))$ of $X_1(n)$ (see [24, p. 109]) and describes the singular fibers of $\upsilon(n)$ for several n (see [31, Proposition 4.2]).

n	$g(X_1(n))$	sing. fibers of $\upsilon(n)$
3	0	$IV^* + I_3 + I_1$
5	0	$2I_5 + 2I_1$
7	0	$3I_7 + 3I_1$
11	1	$5I_{11} + 5I_1$

TABLE 1. Singular fibers of $\upsilon(n)$

To parametrize elliptic curves over a field of characteristic p with a point of order p, we use Igusa curves instead of the modular curves above. For an extensive treatise of the subject, we refer the reader to [11] and [15, Chapter 12]. For any prime $p \geq 3$, the smooth affine Igusa curve $\mathrm{Ig}(p)^{\mathrm{ord}}$ over $\mathbb{F}_p$ parametrizes ordinary elliptic curves E with a point that generates the kernel of the Verschiebung map in the following sense (see [15, Section 12.3 and Corollary 12.6.3]). For every scheme S over $\mathbb{F}_p$, the absolute Frobenius $S \to S$ is the map that corresponds on affine rings to the map $x \to x^p$. For every elliptic curve $E \to S$, we let $E^{(p)} \to S$ denote the base change of $E \to S$ by the absolute Frobenius $S \to S$. By the universal property of the fibered product, the absolute Frobenius $E \to E$ factors as the composition of the projection $E^{(p)} \to E$ and a map $F = F_{E/S} : E \to E^{(p)}$ that we call the *relative Frobenius*. The dual isogeny $V = V_{E/S} : E^{(p)} \to E$ of $F_{E/S}$ is called the *Verschiebung*. There exists an elliptic curve $\mathfrak{E}(p)^\circ$ over the Igusa curve $\mathrm{Ig}(p)^{\mathrm{ord}}$, as well a section $\mathfrak{P}$ of the associated elliptic curve $\mathfrak{E}(p)^{\circ(p)} \to \mathrm{Ig}(p)^{\mathrm{ord}}$, such that all fibers of both fibrations are ordinary and $\mathfrak{P}$ generates the kernel of the Verschiebung $V : \mathfrak{E}(p)^{\circ(p)} \to \mathfrak{E}(p)^\circ$, and such that for every elliptic curve E over a scheme S over $\mathbb{F}_p$ of which all fibers are ordinary, with a section P of the associated curve $E^{(p)} \to S$ that generates the kernel of the Verschiebung $V : E^{(p)} \to E$, there is a unique morphism $\alpha : S \to \mathrm{Ig}(p)^{\mathrm{ord}}$ such that $E, E^{(p)}$, and P are the base change of $\mathfrak{E}^\circ$, $\mathfrak{E}(p)^{\circ(p)}$, and $\mathfrak{P}$, respectively, by α.

If k is a field of characteristic p and E' is an elliptic curve over $S = \operatorname{Spec} k$ with a point P of order p, then there is an elliptic curve $E \to S$ such that $E^{(p)} \to S$ is isomorphic to $E' \to S$ and P generates the kernel of Verschiebung; hence $E' \to S$ is a base change of the universal curve $\mathfrak{E}(p)^{\circ(p)} \to \mathrm{Ig}(p)^{\mathrm{ord}}$.

Let $\overline{\mathrm{Ig}(p)^{\mathrm{ord}}}$ denote the nonsingular projective completion of $\mathrm{Ig}(p)^{\mathrm{ord}}$, and let $\omega(p) : \mathfrak{E}(p)^{(p)} \to \overline{\mathrm{Ig}(p)^{\mathrm{ord}}}$ denote the minimal nonsingular projective model of $\mathfrak{E}(p)^{\circ(p)} \to \mathrm{Ig}(p)^{\mathrm{ord}}$.

Table 2 gives the genus $g(\overline{\mathrm{Ig}(p)^{\mathrm{ord}}})$ of $\overline{\mathrm{Ig}(p)^{\mathrm{ord}}}$ (see [11, p. 96 and 99]) and the fiber types of the singular fibers of $\omega(p)$ for several primes p. The fibers at the $(p-1)/2$ cusps have type I_p [19, Theorem 10.3] and the type of the fibers above the supersingular points can be deduced from [19, Theorem 10.1]; for $p = 13$ it suffices to note that the only supersingular j-value modulo 13 is 5, while for $p \in \{5, 7, 11\}$, the fibers are also given in [13, Proposition 1.3]. This will be used in the proof of Theorem 6.4.

p	$g(\overline{\mathrm{Ig}(p)^{\mathrm{ord}}})$	sing. fibers of $\omega(p)$
5	0	$2I_5 + II$
7	0	$3I_7 + III$
11	0	$5I_{11} + II + III$
13	1	$6I_{13} + I_0^*$

TABLE 2. Singular fibers of $\omega(p)$

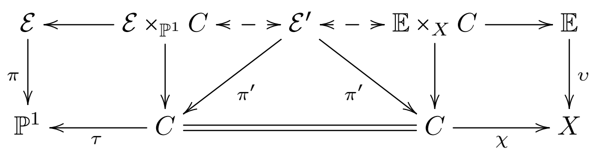
Theorem 6.4. *Let S be a del Pezzo surface of degree 1 over k and $\pi\colon \mathcal{E} \to \mathbb{P}^1$ the associated elliptic fibration. Let C be a smooth, connected curve over k of genus at most 1, and $\tau\colon C \to \mathbb{P}^1$ a nonconstant morphism. Then the base change $\mathcal{E} \times_{\mathbb{P}^1} C \to C$ of π by τ has no nonzero section of finite order.*

Proof. Without loss of generality, we assume that C is projective and that k is algebraically closed. As the curve C is smooth and connected, it is integral, so it has a unique generic point η that is dense in C. The curve $\mathcal{E} \times_{\mathbb{P}^1} \eta$ is an elliptic curve over the function field $\kappa(C)$ of C, which is an extension of the function field $k(t)$ of $\mathbb{P}^1$ with $t = z/w$. Let $j \in k(t)$ be the j-invariant of the generic fiber of π. Assume that the elliptic fibration $\mathcal{E} \times_{\mathbb{P}^1} C \to C$ has a nonzero section of finite order, say order $n > 1$. Then the curve $\mathcal{E} \times_{\mathbb{P}^1} \eta$ has a point of order n over $\kappa(C)$. Without loss of generality, we assume that n is prime. Let $f, g \in k[z, w]$ be homogeneous polynomials such that S is isomorphic to the surface in $\mathbb{P}$ given by (1). Let M and N denote the number of fibers of π of type I_1 (nodal) and II (cuspidal), respectively. Then $M + 2N = 12$ by Lemma 6.2. We will show that the genus of C is at least 2 by considering several cases, thus deriving the contradiction that proves the statement.

I) We first consider the case $n = 2$. Note that π itself has no section of order 2, for if it did, it would be given by $y = 0$ and $x = h(z, w)$ for some homogeneous polynomial $h \in k[z, w]$ of degree 2, and then S would be singular at the point on this section in the fibers given by $3h^2 + f = 0$. Since the locus $L \subset S$ of the 2-torsion points has degree 3 over $\mathbb{P}^1(z, w)$, it follows that L is irreducible. To compute its genus, note that the map $\lambda\colon L \to \mathbb{P}^1$ ramifies whenever $D = 4f^3 + 27g^2$ vanishes. Moreover, if D vanishes to order 1 at $t \in \mathbb{P}^1$, which happens if and only if the fiber $\mathcal{E}_t$ of π has type I_1, then there are two points on L above t with ramification indices 1 and 2, while if D vanishes to order 2, which happens if and only if the fiber $\mathcal{E}_t$ has type II, then there is a unique point on L above t with ramification index 3. It follows that the degree of the ramification divisor of λ equals $M + 2N = 12$, so the Riemann-Hurwitz formula applied to λ shows that the genus of L equals $1 + \frac{1}{2}(-2(\deg \lambda) + 12) = 4$. If the base change of π by τ has a section of order 2, then this section would map nontrivially to L, so we get $g(C) \geq 4$.

II) We now consider the case that j is constant, that is $j \in k$, and may assume $n \neq 2$. Then there are $a, b \in k$ and $h \in \overline{k(t)}$, where $\overline{k(t)}$ denotes an algebraic closure of $k(t)$, such that $f(t, 1) = ah^2$ and $g(t, 1) = bh^3$. If $f, g \neq 0$, then $h = ab^{-1}g(t, 1)f(t, 1)^{-1}$ is contained in $k(t)$ and one checks that S is not smooth. If $g = 0$, then S is not smooth either, so we find $f = 0$ and again from smoothness of S, we find that $g(t, 1)$ is separable and has degree 5 or 6 in t (cf. [40, Proposition 3.1]). Suppose $P = (x_1, y_1) \in \mathcal{E} \times_{\mathbb{P}^1} \eta$ is a point over $\kappa(C)$ of order n. Let $\overline{\kappa(C)}$ be an algebraic closure of $\kappa(C)$ and $\beta \in \overline{\kappa(C)}$ an element satisfying $\beta^6 = g(t, 1)$. Then $(x_1\beta^{-2}, y_1\beta^{-3})$ is a point of order n on the curve given by $y^2 = x^3 + 1$, so there are $x_2, y_2 \in k$ such that $x_1 = x_2\beta^2$ and $y_1 = y_2\beta^3$. From $n \neq 2$, we get $y_1 \neq 0$. If $x_1 \neq 0$, then $\beta = x_2 y_2^{-1} y_1 x_1^{-1}$ is contained in $\kappa(C)$; if $x_1 = 0$, then $y_1^2 = g(t, 1)$, so in any case, $g(t,1)$ is a square in $\kappa(C)$, which implies that $\kappa(C)$ contains a subfield of genus 2, so C has at least genus 2 itself.

III) The case $n \geq 3$ and $j \notin k$. If the characteristic of k is not equal to n, then we set $Y = Y_1(n)$ and $X = X_1(n)$ and $\mathbb{E} = \mathbb{E}(n)$ and $\upsilon = \upsilon(n)$; otherwise, we set $Y = \mathrm{Ig}(n)^{\mathrm{ord}}$ and $X = \overline{\mathrm{Ig}(n)^{\mathrm{ord}}}$ and $\mathbb{E} = \mathfrak{E}(n)^{(n)}$ and $\upsilon = \omega(n)$. In either case, there is a morphism $\eta \to Y \subset X$ such that the elliptic curve $\mathcal{E} \times_{\mathbb{P}^1} \eta$ over η is the base change of $\mathbb{E}$ over X. This morphism extends to a morphism $\chi\colon C \to X$, which is nonconstant because j is not constant. The elliptic surfaces $\mathcal{E} \times_{\mathbb{P}^1} C$ and $\mathbb{E} \times_X C$ have isomorphic generic fibers $\mathcal{E} \times_{\mathbb{P}^1} \eta \cong \mathbb{E} \times_X \eta$, so their minimal nonsingular models are isomorphic as well by [21, Proposition II.1.2 and Corollary II.1.3]. Let $\pi'\colon \mathcal{E}' \to C$ be this minimal nonsingular elliptic fibration.



Set $d = \deg \tau$ and let $R \in \operatorname{Div} C$ denote the ramification divisor of τ. Then the degree of R is at least

$$\sum_{c \in C} (e_c(\tau) - 1) \geq \sum_{\substack{b \in \mathbb{P}^1 \\ \mathcal{E}_b \text{ type } I_1}} \left(\sum_{\substack{c \in C \\ \tau(c) = b}} (e_c(\tau) - 1) \right) + \sum_{\substack{b \in \mathbb{P}^1 \\ \mathcal{E}_b \text{ type } II}} \left(\sum_{\substack{c \in C \\ \tau(c) = b}} (e_c(\tau) - 1) \right), \tag{18}$$

where $e_c(\tau)$ denotes the ramification index of τ at c.

Lemma 6.1 relates the types of the singular fibers of π' to those of π and the ramification of τ on one hand, and to those of υ and the ramification of χ on the other hand. The remainder of the proof consists of a largely combinatorial argument to give a lower bound for the degree of R, which then, by the Riemann-Hurwitz formula, yields a lower bound for the genus of C.

Lemma 6.1 implies that the points $c \in C$ for which the fiber $\mathcal{E}_{\tau(c)}$ of π above $\tau(c)$ has type I_1 are exactly the points for which the fiber $\mathcal{E}'_c$ of π' above c has type I_m for some integer $m \geq 1$, and exactly the points for which the fiber $\mathbb{E}_{\chi(c)}$ of υ above $\chi(c)$ has type I_j or I_j^* for some integer $j \geq 1$; for such points c, and integers m and j, the quotient $\ell = m/j$ is a positive integer and we have $e_c(\tau) = j\ell$ and $e_c(\chi) = \ell$. For each $j \geq 1$, let r_j denote the number of fibers of υ of type I_j or I_j^*; for each $\ell \geq 1$, let $s_{j,\ell}$ denote the number of fibers of π' of type $I_{j\ell}$ that lie above a point $c \in C$ for which the fiber of υ above $\chi(c)$ has type I_j or I_j^*. For every $x \in X$ we have $\sum_{c \in \chi^{-1}(x)} e_c(\chi) = \deg \chi$. Summing over all $x \in X$ for which the fiber $\mathbb{E}_x$ has type I_j, we find $\sum_{\ell \geq 1} \ell s_{j,\ell} = (\deg \chi) \cdot r_j$ for all $j \geq 1$. The same argument applied to τ yields

$$Md = \sum_{j,\ell \geq 1} j\ell s_{j,\ell} = (\deg \chi) \cdot \sum_{j \geq 1} j r_j.$$

It follows that the first term of the right-hand side of (18) equals

$$\sum_{j,\ell \geq 1} (j\ell - 1) s_{j,\ell} \geq \sum_{j,\ell \geq 1} (j-1)\ell s_{j,\ell} = (\deg \chi) \cdot \sum_{j \geq 1} (j-1) r_j = \frac{\sum_{j \geq 1} (j-1) r_j}{\sum_{j \geq 1} j r_j} \cdot Md. \tag{19}$$

We consider two subcases.

A) The characteristic of k is not equal to n. From $g(X) \leq g(C) \leq 1$ we conclude $n \leq 12$ or $n = 14$ or $n = 15$ (see [24, p. 109]), and since $n \geq 3$ is prime, we have $n \in \{3,5,7,11\}$. From Table 1 above, we find that the fraction in the right-most expression of (19) is at least $\frac{1}{2}$. Since $\upsilon = \upsilon(n)$ has only fibers of type IV^*, I_1, and I_n, Lemma 6.1 implies that π' does not have fibers of type II, II^*, or I_0^*. Again from Lemma 6.1, this time viewing π' as the minimal model of the base change of π by τ, we find that for every $c \in C$ for which the fiber of π above $\tau(c)$ has type II, the ramification index $e_c(\tau)$ is even, so we have $e_c(\tau) - 1 \geq \frac{1}{2} e_c(\tau)$. Therefore, the second term of the right-hand side of (18) is at least $\frac{1}{2}Nd$, so the degree of R is at least $\frac{1}{2}Md + \frac{1}{2}Nd \geq \frac{1}{4}d(M + 2N) = 3d$. The Riemann-Hurwitz formula applied to τ then yields $2g - 2 = -2d + \deg R \geq d > 0$, so $g > 1$.

B) The characteristic of k is equal to n. From $g(X) \leq g(C) \leq 1$ we conclude $n \in \{5,7,11,13\}$ (for a formula for the genus of X, see [11, p. 96 and 99]). From Table 1 above, we find that the fraction in the right-most expression of (19) is at least $\frac{4}{5}$. Also, from the fact that π is not isotrivial, we get $M \geq 4$ by Lemma 6.2, so the degree of R is at least $\frac{4}{5}Md > 3d$. As before, the Riemann-Hurwitz formula yields $g > 1$. □

7. Proof of the main theorems

In this section, the field k is still of characteristic different from 2 and 3.

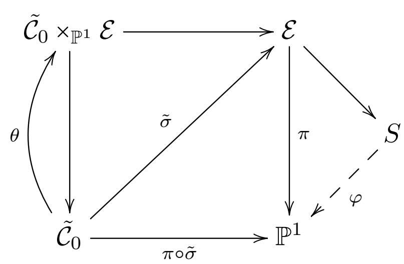
Theorem 7.1. *Suppose k is infinite. Let $S \subset \mathbb{P}$ be a del Pezzo surface of degree 1 over k, given by (1) for some homogeneous $f, g \in k[z,w]$ of degree 4 and 6, respectively. Let $Q = (x_0 : y_0 : 0 : 1) \in S(k)$ be a rational point with $y_0 \neq 0$. Suppose that the following statements hold.*

- *If the order of Q in $S_0^{\mathrm{ns}}(k)$ is at least 4, then $\mathcal{C}_Q(5)$ has infinitely many k-points.*
- *If the characteristic of k equals 5, then the order of Q in $S_0^{\mathrm{ns}}(k)$ is not 5.*
- *The pair (S,Q) is not isomorphic to a pair described in Examples 4.1, 4.3(iii), or 4.4(iii).*
- *If the pair (S,Q) is isomorphic to a pair described in Subexamples 4.3(i) or 4.4(i), then the set of k-points on $\mathcal{C}_Q(5)$ is Zariski dense in $\mathcal{C}_Q(5)$.*

Then the set $S(k)$ of k-points on S is Zariski dense in S.

Proof. Given S and Q, we let the curve $\mathcal{C}_Q(5)$, its completion $\overline{\mathcal{C}}_Q(5)$, the rational map $\sigma: \overline{\mathcal{C}}_Q(5) \dashrightarrow S$, the elliptic fibration $\pi: \mathcal{E} \to \mathbb{P}^1$, and the element $\phi_3 \in k$ be as in Sections 2 and 3. We claim that there exists an irreducible component $\mathcal{C}_0$ of $\overline{\mathcal{C}}_Q(5)$ for which $\sigma(\mathcal{C}_0)$ is a horizontal curve on S and $\mathcal{C}_0(k)$ is infinite. Indeed, if the order of Q in $S_0^{\mathrm{ns}}(k)$ is at least 4, then $\phi_3 \neq 0$, so $\mathcal{C}_Q(5)$ is a double cover of $\mathbb{A}^1(p)$, the curve $\mathcal{C}_Q(5)$ has at most two irreducible components, and if there are two, then there is an involution that switches them, so the first assumption of the theorem implies that $\mathcal{C}_Q(5)(k)$ is Zariski dense in $\mathcal{C}_Q(5)$; thus, there exists an irreducible component $\mathcal{C}_0$ of $\mathcal{C}_Q(5)$ that satisfies the claim by Corollary 5.6. Suppose, for the remainder of this paragraph and the proof of the claim, that the order of Q is 3. Then for any pair (S,Q) that is not isomorphic to one of the pairs described in Examples 4.3 and 4.4, the unique component of $\mathcal{C}_Q(5)$ that projects birationally to $\mathbb{A}^1(p)$ satisfies the claim by Proposition 5.5. For any pair (S,Q) that *is* isomorphic to one of the pairs described in those examples, the curve $\mathcal{C}_Q(5)$ contains a component $\mathcal{C}_0$ whose projection to $\mathbb{A}^1(p)$ is constant; this component $\mathcal{C}_0$ satisfies the claim, as its image is horizontal by Proposition 5.5 (by assumption we are not in the Subexample 4.3(iii) or 4.4(iii)) and density of $\mathcal{C}_0(k)$ follows either automatically in the case of Subexample (ii) or by assumption in the case of Subexample (i).

Let $\mathcal{C}_0$ be a component of $\overline{\mathcal{C}}_Q(5)$ as in the claim, and let $\tilde{\mathcal{C}}_0$ be a normalization of $\mathcal{C}_0$. Then the rational map $\sigma: \overline{\mathcal{C}}_Q(5) \dashrightarrow S$ induces a morphism $\tilde{\sigma}: \tilde{\mathcal{C}}_0 \to \mathcal{E}$. The composition $\pi \circ \tilde{\sigma}: \tilde{\mathcal{C}}_0 \to \mathbb{P}^1$ corresponds on an open subset to the rational map $\varphi \circ \sigma: \overline{\mathcal{C}}_Q(5) \dashrightarrow \mathbb{P}^1$, so it is surjective by the claim. Let θ denote the section $\mathrm{id} \times \tilde{\sigma}: \tilde{\mathcal{C}}_0 \to \tilde{\mathcal{C}}_0 \times_{\mathbb{P}^1} \mathcal{E}$ of the elliptic surface $\tilde{\mathcal{C}}_0 \times_{\mathbb{P}^1} \mathcal{E} \to \tilde{\mathcal{C}}_0$.



The section θ is not the zero section because $\sigma: \overline{\mathcal{C}}_Q(5) \dashrightarrow S$ sends points $P \in \mathcal{C}_Q(5)$ whose associated curve C is not contained in S to a point unequal to $\mathcal{O} \in S$. The genus of $\tilde{\mathcal{C}}_0$ is at most 1, so by Theorem 6.4, the section θ has infinite order. Since $\tilde{\mathcal{C}}_0(k)$ is Zariski dense in $\tilde{\mathcal{C}}_0$, it follows that the rational points are dense on the images of all infinitely many multiples of θ. Thus, the k-rational points are dense in the surface $\tilde{\mathcal{C}}_0 \times_{\mathbb{P}^1} \mathcal{E}$ and as this surface maps dominantly to S, we conclude that $S(k)$ is Zariski dense in S. □

Obviously, for any point $Q' \in S(k) - \{\mathcal{O}\}$, we may apply an automorphism of $\mathbb{P}^1(z,w)$ to ensure that we have $\varphi(Q') = (0:1)$, so the implicit assumption in Theorem 7.1 and related statements that $\varphi(Q) = (0:1)$ is not a restriction.

Note that if the order of Q in $S_0^{\mathrm{ns}}(k)$ is 3 and the pair (S,Q) is not isomorphic to a pair described in Examples 4.3(i) and 4.4(i), then the hypotheses of Theorem 7.1 are automatically satisfied, without further assumptions on $\mathcal{C}_Q(5)$.

Example 7.2. We took a small sample of approximately a hundred randomly chosen del Pezzo surfaces over $\mathbb{Q}$ given by (1) with f and g having only coefficients 0, 1, and -1. For nearly half of the cases, a short point search revealed a rational point Q for which we could show that it satisfies all conditions of Theorem 7.1, thus proving the rational points are Zariski dense. For the remaining cases, we could still find points Q, but the coefficients of $\mathcal{C}_Q(5)$ were too large to show that $\mathcal{C}_Q(5)$ has infinitely many rational points.

Example 7.3. As mentioned in Section 1, A. Várilly-Alvarado proves in [40, Theorem 2.1] that if we have $k = \mathbb{Q}$ and $f = 0$, and some technical conditions on g, as well as a finiteness conjecture hold, then the set of rational points is Zariski dense on the surface given by (1). He also mentions the surface S with $f = 0$ and $g = 243z^6 + 16w^6$ as an example that would not succumb to his methods, so we took S as a test example for our method. Unfortunately, the point $(0:4:0:1)$ of order 3 on $S_0 \subset S$ lies on nine (-1)-curves (cf. Example 4.4(iii)). It is not hard to find more rational points on this surface, but we did not succeeded in finding any points on the curve $\mathcal{C}_Q(5)$ associated to any of these points Q as the coefficients are rather large: for the second-smallest point $Q = (-63:14:1:5)$, the conductor of the Jacobian of $\mathcal{C}_Q(5)$ has 62 digits. N. Elkies did prove that the points on S are dense with a different method [7].

Proof of Theorem 1.2. The fact that Q is not fixed by the automorphism that changes the sign of y implies $Q \neq \mathcal{O}$. Without loss of generality, we assume $\varphi(Q) = (0:1)$, say $Q = (x_0 : y_0 : 0 : 1)$, with $y_0 \neq 0$. Hence, we may apply Theorem 7.1. The last hypothesis of Theorem 1.2 implies the last two of Theorem 7.1, which shows that $S(k)$ is indeed Zariski dense in S. □

Proof of Theorem 1.3. Note that any point (x_0, y_0) on an elliptic curve given by $y^2 = x^3 + ax + b$ has order 3 if and only if $(a + 3x_0^2)^2 = 12x_0y_0^2$. Define the polynomials $f = \sum_{i=0}^4 f_i u^i$ and $g = \sum_{j=0}^6 g_j u^j$. Suppose we have $\ell \in \{0,\dots,4\}$, $m \in \{0,\dots,6\}$, and $\varepsilon > 0$. Since every elliptic curve over the real numbers $\mathbb{R}$ has a nontrivial 3-torsion point, we may choose a nonzero rational number $t \in \mathbb{Q}^*$ and a point $Q = (x_0 : y_0 : t : 1) \in S(\mathbb{R})$ such that the fiber S_t given by $y^2 = x^3 + f(t)x + g(t)$ is smooth, the point Q has order 3 in $S_t(\mathbb{R})$, and Q does not lie on six (-1)-curves on S. Set $\xi_0 = \frac{1}{6}y_0^{-1}(f(t) + 3x_0^2)$, so that Q being 3-torsion implies $3\xi_0^2 = x_0$. Choose $\xi_1, y_1 \in \mathbb{Q}^*$ close to ξ_0 and y_0, respectively, and set $x_1 = 3\xi_1^2$ and $Q' = (x_1 : y_1 : t : 1)$. Also set

$$\begin{aligned}
\lambda &= f_\ell + t^{-\ell}(6\xi_1 y_1 - 3x_1^2 - f(t)),\\
\mu &= g_m + t^{-m}(y_1^2 - x_1^3 - (6\xi_1 y_1 - 3x_1^2)x_1 - g(t)),\\
f' &= f - f_\ell u^\ell + \lambda u^\ell,\\
g' &= g - g_m u^m + \mu u^m,
\end{aligned}$$

so that f' and g' are the polynomials obtained from f and g after replacing f_ℓ and g_m by λ and μ, respectively. Then we have $f'(t) = 6\xi_1 y_1 - 3x_1^2$ and $g'(t) = y_1^2 - x_1^3 - f'(t)x_1$, so Q lies on the surface S' given by (2) with the two values f_ℓ and g_m replaced by λ and μ, respectively. If we choose ξ_1 and y_1 arbitrarily close to ξ_0 and y_0, then λ and μ will be arbitrarily close to f_ℓ and g_m. By choosing them close enough, we also guarantee that S' and S'_t are smooth, and that Q' does not lie on six (-1)-curves on S'. From the identity $(f'(t) + 3x_1^2)^2 = 36\xi^2 y_1^2 = 12x_1y_1^2$ we conclude that Q' has order 3 in $S'_t(\mathbb{Q})$, so we may apply Theorem 1.2, which yields that $S'(\mathbb{Q})$ is Zariski dense in S'. □

Lemma 7.4. *Let k be an infinite field and $X \to \mathbb{P}^1$ an elliptic fibration over k with a nontorsion section. Then there are infinitely many points $t \in \mathbb{P}^1(k)$ for which the fiber X_t contains infinitely many k-rational points.*

Proof. If k is algebraic over a finite field, then this follows from the Weil conjectures. Otherwise, we replace k without loss of generality by an infinite subfield that is finitely generated over its prime subfield, over which everything is defined. Then k is either a number field or a transcendental extension of its prime field and in all cases, k is Hilbertian (see [9] for number fields and [8, Theorem 13.4.2] for a modern treatment of the general case). The lemma now follows immediately from Néron's Specialization Theorem [23, Théorème IV.6] (see also [18, Theorem 7.2] and [27, Remark 3.7(1)]). □

Proof of Theorem 1.4. Without loss of generality, we assume that the nodal fiber lies above $(0:1)$. When Q runs over the nodal curve S_0, the curves $\overline{\mathcal{C}}_Q(5)$ form a family of genus-one curves. More precisely, the equation in (9) describes a surface $X \subset \mathbb{A}^1(x_0) \times \mathbb{P}(1,2,1)$, and if $Q = (x_0 : y_0 : 0 : 1)$ is a point on S_0 with $y_0 \neq 0$ and not of order 3 in $S_0^{\text{ns}}(k)$, then the fiber of the projection $\mu: X \to \mathbb{A}^1$ above x_0 is isomorphic to $\overline{\mathcal{C}}_Q(5)$. The fibered product $(S_0 - \{\mathcal{O}\}) \times_{\mathbb{A}^1} X$ is the family of curves $\overline{\mathcal{C}}_Q(5)$, at least outside finitely many points $Q \in S_0$. Let $d \in k^*$ be such that $f_0 = -3d^2$ and $g_0 = 2d^3$. Note that we have two rational maps $\chi_i: \mathbb{A}^1 \dashrightarrow X$, for $i = 1,2$, that are rational sections of μ, namely given by $x_0 \mapsto (x_0, (1 : \alpha_i : 0))$ with $\alpha_1 = \frac{1}{4}(x_0 + 2d)^{-1}$ and $\alpha_2 = \frac{1}{4}(x_0 + 7d)(x_0 + 2d)^{-1}(x_0 + 3d)^{-1}$ as in Lemma 3.4. These maps extend to morphisms and we choose χ_1 to be the zero section, making μ a Jacobian elliptic fibration.

We claim that the section χ_2 has infinite order. The model $X \to \mathbb{A}^1$ is highly singular, so instead we consider the surface $X' \subset \mathbb{A}^1(x_0) \times \mathbb{P}(1,2,1)$ that is the image of X under the birational map

$$\begin{aligned} \mathbb{A}^1 \times \mathbb{P}(1,2,1) &\dashrightarrow \mathbb{A}^1 \times \mathbb{P}(1,2,1) \\ (x_0, (\overline{p} : \overline{q} : \overline{r})) &\mapsto (x_0, (\overline{p}' : \overline{q}' : \overline{r}')) \end{aligned}$$

with

$$\begin{aligned} \overline{p}' &= 8(x_0 - d)^2 \overline{p} + (x_0 - d)(f_1 d + g_1)\overline{r}, \\ \overline{q}' &= 2\varphi_2^{-1}(2c_1\overline{q} + c_2\overline{p}^2 + c_3\overline{pr} + c_4\overline{r}^2), \\ \overline{r}' &= 8\overline{r}, \end{aligned}$$

where $\phi_2, c_1, c_2, c_3, c_4$ depend on x_0 (which is now variable instead of fixed) as they did before. Note that $f_1 d + g_1$ is nonzero because S is smooth. For $x_0 \notin \{d, -2d, -3d\}$, the fibers of $\mu: X \to \mathbb{A}^1$ and $\mu': X' \to \mathbb{A}^1$ are isomorphic. The model X' is given by $\overline{q}^2 = H(\overline{p}, \overline{r})$, where $H \in k[x_0][\overline{p}, \overline{r}]$ is homogeneous of degree 4. The fiber X'_d of μ' above $x_0 = d$ is given by

$$\overline{q}^2 = 81d^4(f_1 d + g_1)\overline{p}^2\overline{r}(\overline{p} + (f_1 d + g_1)\overline{r}). \tag{20}$$

This fiber X'_d is singular at the point $(d, (0:0:1))$, and in fact so is X', but the fiber is smooth everywhere else. The sections χ_1 and χ_2 correspond to the sections $\chi'_1: x_0 \mapsto (x_0, (4 : 6d(d - x_0) : 0))$ and $\chi'_2: x_0 \mapsto (x_0, (4 : 6d(x_0 - d) : 0))$ of μ', respectively. These sections intersect in the point $(d, (1:0:0))$, which is smooth in its fiber. Therefore, in a minimal nonsingular projective model $\overline{\mu}: \overline{X} \to \mathbb{P}^1$ of the fibration μ, the two sections intersect as well. Hence, χ'_2 is in the kernel of the reduction $\overline{X}(\mathbb{P}^1) \to \overline{X}_d(k)$, where $\overline{X}_d$ is the fiber of $\overline{\mu}$ above $x_0 = d$. This kernel is isomorphic to a subgroup of the formal group associated to $\overline{\mu}$ (or μ') and the completion of $k[x_0]$ at the maximal ideal $(x_0 - d)$, cf. [33, Proposition VII.2.2]. By [33, Proposition IV.3.2(b)], all torsion elements of the formal group have p-power order, where p is the characteristic of k. This proves the claim for $p = 0$ (cf. [22, Theorem 1.1(a)]). We now assume $p > 0$ and determine the Kodaira type of the singular fiber $\overline{X}_d$ of $\overline{\mu}$. One checks that the discriminant of H equals

$$\Delta = (x_0 - d)^3(x_0 + 2d)^8(x_0 + 3d)^2 D(x_0),$$

where D is a polynomial of degree 35 satisfying $2^{11}D(d) = -3^{13}d^{11}(f_1 d + g_1)^{12} \neq 0$. Hence, the valuation of Δ at $x_0 = d$ equals 3; the fiber X'_d described in (20) is nodal, so the reduction is multiplicative and we conclude from [34, Tate's Algorithm IV.9.4] that $\overline{X}_d$ has type I_3. It follows that the j-invariant of μ is not constant. Suppose that χ'_2 is torsion. Then $\overline{\mu}$ admits a section of order p, so there is a surjective morphism $\psi: \mathbb{P}^1 \to \overline{\mathrm{Ig}(p)^{\mathrm{ord}}}$ such that the generic fiber of $\overline{\mu}$ is isomorphic to the generic fiber of the base change of the fibration $.(p): \mathfrak{E}(p)^{(p)} \to \overline{\mathrm{Ig}(p)^{\mathrm{ord}}}$ by ψ

(cf. part III of the proof of Theorem 6.4). This implies that the minimal nonsingular model of this base change is isomorphic to $\overline{\mu}$. However, the existence of ψ implies that $\overline{\mathrm{Ig}(p)^{\mathrm{ord}}}$ has genus 0, so $p \leq 11$, and from Lemma 6.1 and Table 2 we find that no base change of $\omega(p)$ has a minimal nonsingular model with fibers of type I_3. This contradicts the fact that $\overline{X}_d$ has type I_3 and the claim follows.

It follows that the section $(S_0 - \{\mathcal{O}\}) \dashrightarrow (S_0 - \{\mathcal{O}\}) \times_{\mathbb{A}^1} X$ induced by χ_2 also has infinite order on the elliptic fibration $(S_0 - \{\mathcal{O}\}) \times_{\mathbb{A}^1} X \to S_0 - \{\mathcal{O}\}$ with the section induced by χ_1 as zero section. After replacing S_0 by its normalization we may apply Lemma 7.4, which implies that the curve $\overline{\mathcal{C}}_Q(5)$ has infinitely many rational points for infinitely many $Q \in S_0^{\mathrm{ns}}(k)$, in particular for some Q of order larger than 5 in $S_0^{\mathrm{ns}}(k)$. Theorem 1.2 then shows that $S(k)$ is Zariski dense in S. □

References

[1] Bogomolov, F. A. and Tschinkel, Yu., *On the density of rational points on elliptic fibrations*, J. Reine Angew. Math. **511** (1999), 87–93.

[2] Bosma, W., Cannon, J., and Playoust, C., *The Magma algebra system. I. The user language*, J. Symbolic Comput., **24** (1997), 235–265.

[3] Colliot-Thélène, J.-L., *L'arithmétique des variétés rationnelles*, Ann. Fac. Sci. Toulouse Math. (6), Vol. **1** (1992), no. 3, 295–336.

[4] Colliot-Thélène, J.-L., *Points rationnels sur les fibrations*, Higher dimensional varieties and rational points (Budapest, 2001), Bolyai Soc. Math. Stud. **12**, Springer, Berlin, 2003, 171–221.

[5] Coombes, K.R., *Every rational surface is separably split*, Comment. Math. Helvetici, **63** (1988), 305–311.

[6] Couveignes, J.-M., and Edixhoven, S., *Computational Aspects of Modular Forms and Galois Representations*, Ed. Bas Edixhoven & Jean-Marc Couveignes, with Robin de Jong, Franz Merkl & Johan Bosman, Ann. of Math. Stud., No. **176**, Princeton Univ. Press, Princeton, NJ, 2011.

[7] Elkies, N., private communication.

[8] Fired, M.D. and Jarden, M., Field arithmetic, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Vol. **11**, Second edition, Springer-Verlag, Berlin, 2005.

[9] Hilbert, D., *Ueber die Irreduzibilität ganzer rationalen Funktionen mit ganzzahligen Koeffizienten*, J. Crelle **110** (1892), 105–129 (=Ges. Abh. II, 264–286).

[10] Igusa, J., *Fibre Systems of Jacobian Varieties*, American Journal of Mathematics **81** (1959), No. 2, 453–476.

[11] Igusa, J., *On the algebraic theory of elliptic modular functions*, J. Math. Soc. Japan, Vol. **20** (1968), Nos. 1-2, 96–106.

[12] Iskovskikh, V.A., *Minimal models of rational surfaces over arbitrary fields*, Izv. Akad. Nauk SSSR Ser. Mat., **43** (1979), no. 1, 19–43, 237, English translation: Math. USSR-Izv., **14** (1980), no. 1, 17–39.

[13] Ito, H. and Liedtke, C., *Elliptic K3 surfaces with p^n-torsion sections*, J. Algebraic Geom., **22** (2013), 105–139.

[14] Jabara, E., *Rational points on some elliptic surfaces*, Acta Arith., **153** (2012), no. 1, 93–108.

[15] Katz, N. and Mazur, B., Arithmetic moduli of elliptic curves, Annals of Mathematics Studies, No. **108**, Princeton University Press, Princeton, NJ, 1985.

[16] Kollár, J., *Unirationality of cubic hypersurfaces*, J. Inst. Math. Jussieu **1** (2002), no. 3, 467–476.

[17] Kubert, D.S., *Universal bounds on the torsion of elliptic curves*, Proc. London Math. Soc. (3) **33** (1976), no. 2, 193–237.

[18] Lang, S, *Survey of Diophantine geometry*, Corrected second printing, Springer-Verlag, Berlin Heidelberg, 1997.

[19] Liedtke, C. and Schröer, S., *The Néron model over the Igusa curves*, J. Number Theory, **130** (2010), no. 10, 2157–2197.

[20] Manin, Yu. I., *Cubic Forms*, North-Holland, 1986.

[21] Miranda, R., *The basic theory of elliptic surfaces*, Dottorato di Ricerca in Matematica, Dipartimento di Matematica dell' Universita di Pisa, ETS Editrice Pisa (1989).

[22] Miranda, R. and Persson, U., *Torsion groups of elliptic surfaces*, Compositio Math. **72** (1989), no. 3, 249–267.

[23] Néron, A., *Problèmes arithmétiques et géométriques rattachés à la notion de rang d'une courbe algébrique dans un corps*, Bull. Soc. Math. France, Vol. **80** (1952), 101–166.

[24] Ogg, A.P., *Rational points of finite order on elliptic curves,* Invent. Math., **12** (1971), 105-111.

[25] Ogg, A.P., *Rational points on certain elliptic modular curves*, Analytic number theory (Proc. Sympos. Pure Math., Vol XXIV, St. Louis Univ., St. Louis, Mo., 1972), Amer. Math. Soc. (1973), 221–231.

[26] Persson, U., *Configurations of Kodaira fibers on rational elliptic surfaces*, Mathematische Zeitschrift, **205** (1990), 1–47.

[27] Petersen, S. *On a question of Frey and Jarden about the rank of abelian varieties*, J. Number Theory, Vol. **120** (2006) no. 2, 287–302.

[28] Pieropan, M., *On the unirationality of Del Pezzo surfaces over an arbitrary field*, Algant Master thesis, `http://www.algant.eu/documents/theses/pieropan.pdf`.

[29] Segre, B., *A note on arithmetical properties of cubic surfaces*, J. London Math. Soc. **18** (1943), 24–31.

[30] Segre, B., *On the rational solutions of homogeneous cubic equations in four variables*, Math. Notae **11** (1951), 1–68.
[31] Shioda, T., *On elliptic modular surfaces*, J. Math. Soc. Japan, **24** (1972), 20–59.
[32] Shioda, T., *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Paul., **39** (1990), no. 2, 211–240.
[33] Silverman, J. The arithmetic of elliptic curves, Graduate Texts in Mathematics, No. **106**, Springer-Verlag, New York, 1986.
[34] Silverman, J., Advanced topics in the arithmetic of elliptic curves, Graduate Texts in Mathematics, No. **151**, Springer-Verlag, New York, 1994.
[35] Tate, J., *Algorithm for determining the type of a singular fiber in an elliptic pencil*, Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math., Vol. **476** (1975), Springer, Berlin, 33–52.
[36] Ulas, M., *Rational points on certain elliptic surfaces*, Acta Arith., **129** (2007), no. 2, 167–185.
[37] Ulas, M., *Rational points on certain del Pezzo surfaces of degree one*, Glasg. Math. J., **50** (2008), no. 3, 557–564.
[38] Ulas, M. and Togbé, A., *On the Diophantine equation* $z^2 = f(x)^2 \pm f(y)^2$, Publ. Math. Debrecen, **76** (2010), 1–2, 183–201.
[39] Várilly-Alvarado, A., *Weak approximation on del Pezzo surfaces of degree 1*, Advances in Mathematics, **219** (2008), 2123–2145.
[40] Várilly-Alvarado, A., *Density of rational points on isotrivial rational elliptic surfaces*, Algebra and Number Theory, **5** (2011), 659–690.

Instituto de Matemática, Universidade Federal do Rio de Janeiro, Ilha do Fundão, 21941-909 Rio de Janeiro, Brasil
E-mail address: salgado@im.ufrj.br
URL: www.im.ufrj.br/~salgado

Mathematisch Instituut, Universiteit Leiden, Postbus 9512, 2300 RA, Leiden, Netherlands

Section de mathématiques EPFL, FSB-SMA, Station 8 - Bâtiment MA, CH-1015 Lausanne
E-mail address: rvl@math.leidenuniv.nl
URL: http://www.math.leidenuniv.nl/~rvl/