You only have to do two of the problems of your choice.
You are allowed to refer to results from the notes, but not to the exercises.

Problem 1. Let $K$ be a field of characteristic 0.

(a) Consider the polynomial ring $K[X_1, X_2]$. For all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \GL_2(K)$, we define $g \cdot X_1 = aX_1 + cX_2$, $g \cdot X_2 = bX_1 + dX_2$. Show that these formulas can be extended to define a representation of $\GL_2(K)$ on the vector space of polynomials $K[X_1, X_2]$.

(b) Consider the vector space $V$ of polynomials $f$ in two variables $X_1, X_2$ with coefficients in $K$ which are homogeneous of degree 2. Show that $V \subset K[X_1, X_2]$ is an irreducible subrepresentation of dimension 3.

Hint: Show that the only subspaces of $V$ that are stable under the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \GL_2(K)$ are $\{0\}$, $C X_1^2$, $C X_1 X_2$ and $V$; similarly, determine the subspaces of $V$ that are stable under the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

From the $K$-basis $(X_1^2, X_1 X_2, X_2^2)$ of $V$, we obtain an isomorphism $V \cong K^3$ and a representation

$$\Sym^2: \GL_2(K) \to \GL_3(K).$$

This representation is called the symmetric square. From now on, we will take $K = \overline{\Q}_l$. Let $F$ be a number field, and let $\ell$ be a prime number. Consider a semi-simple Galois representation

$$\rho: \Gal(F/F) \to \GL_2(\overline{\Q}_\ell).$$

We write

$$r = \Sym^2(\rho): \Gal(F/F) \to \GL_3(\overline{\Q}_\ell)$$

for the composition of $\rho$ with the representation $\Sym^2: \GL_2(\overline{\Q}_\ell) \to \GL_3(\overline{\Q}_\ell)$.

(c) Show that at every $F$-place $v$ where the representation $\rho$ is unramified, the representation $r$ is unramified as well, and we have

$$\text{charpol}(r(\Frob_v)) = X^3 - (t_v^2 - d_v)X^2 + d_v(t_v^2 - d_v)X - d_v^3 \in \overline{\Q}_\ell[X],$$

where $t_v = \Tr \rho(\Frob_v)$ and $d_v = \det(\rho(\Frob_v))$ in $\overline{\Q}_\ell$.

(d) Consider another semi-simple Galois representation

$$r': \Gal(F/F) \to \GL_3(\overline{\Q}_\ell),$$

such that for almost all $F$-places $v$ where $r'$ is unramified, the characteristic polynomial of $r'(\Frob_v) \in \GL_3(\overline{\Q}_\ell)$ is given by equation [1]. Show that $r'$ is isomorphic to $r$.

Date: 23 December 2016.
Problem 2. Let $F$ be a number field, and let $\chi : \mathbb{A}_F^\infty \to \mathbb{C}^\times$ be a Hecke character, i.e. a continuous morphism which is trivial on $F^\times$ embedded diagonally in the idèles $\mathbb{A}_F^\times = \prod'_v (F_v^\times : \mathcal{O}_{F_v}^\times)$. Assume that $F$ is totally real, i.e. all Archimedean places are real. Let $S = \{v_1, \ldots, v_r\}$ be the set of Archimedean places of $F$, all of which are real by assumption; here $r = [F : \mathbb{Q}]$. By a version of Dirichlet’s unit theorem from algebraic number theory, the abelian group $\mathcal{O}_F^\times$ is isomorphic to $\mathbb{Z}^{r-1}$ times a finite group, and the image of the group homomorphism $\mathcal{O}_F^\times \to \mathbb{R}^{r-1}$ is a discrete subgroup of rank $r-1$ in $\mathbb{R}^{r-1}$. In particular, the $\mathbb{R}$-vector space spanned by this subgroup has dimension $r-1$.

In this exercise we will show that there exists a real number $w \in \mathbb{R}$ such that the character $\chi \cdot |\cdot|^{-w} : \mathbb{A}_F^\times \to \mathbb{C}^\times$ has finite image.

(a) Let $\chi_\infty : F_\infty^\times \to \mathbb{C}^\times$ be the restriction of $\chi$ to $F_\infty^\times := (F \otimes_{\mathbb{Q}} \mathbb{R})^\times \cong \prod_{v|\infty} F_v^\times$ via the inclusion of $F_\infty^\times$ into the infinite part of the idèles $\mathbb{A}_F^\times$. Show that $\chi_\infty$ is trivial on a subgroup of $\mathcal{O}_F^\times$ which is of finite index.

(b) Let $H$ be a subgroup of finite index in $\mathcal{O}_F^\times$. Show that the additive group $\text{Hom}(F_\infty^\times/H, \mathbb{R})$ of continuous group homomorphisms $F_\infty^\times/H \to \mathbb{R}$ has a natural structure of a real vector space of dimension 1.

(c) Deduce that there exists a real number $w$ satisfying
\[
\log |\chi_\infty(x)| = w \log \left( \prod_{v|\infty} |x_v|_{F_v} \right)
\]
for all $x = (x_v)_{v|\infty} \in F_\infty^\times$.

(d) Identify $\mathbb{A}_F^\infty \times / F^\times \widehat{\mathcal{O}}_F^\times$ with the class group of $F$, and deduce that this quotient is finite. Then show that for any compact open subgroup $U \subset \mathbb{A}_F^{\infty, \times}$ the quotient $\mathbb{A}_F^{\infty, \times} / F^\times U$ is finite.

(e) Show that the character $\mathbb{A}_F^\infty / F^\times \to \mathbb{C}^\times$, $x \mapsto |x|_{\mathbb{A}_F^\times}^{-w} \cdot \chi(x)$ has finite image.

Problem 3. In this problem we assume that the global Langlands conjecture is true and investigate some of its consequences. Let $F$ be a number field, and let $F'$ be a quadratic extension of $F$.

(a) Let $V$ be a two-dimensional $\mathbb{C}$-vector space, and let $\phi$ be an endomorphism of $V$. Write the characteristic polynomial of $\phi$ as $X^2 - tX + d$. Show that the characteristic polynomial of $\phi \circ \phi$ equals $X^2 - (t^2 - 2d)X + d^2$.

(b) Let $\pi$ be a cuspidal algebraic automorphic representation of $\text{GL}_2(\mathbb{A}_F)$, and let $S$ be the set of all finite places $v$ of $F$ such that both the smooth representation $\pi_v$ of $\text{GL}_2(F_v)$ is unramified at $v$ and the extension $F'/F$ is unramified at $v$. Show that there exists a real number $w$ such that the character $\mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$, $x \mapsto |x|_{\mathbb{A}_F^\times}^{-w} \cdot \chi(x)$ has finite image.
at \( v \). For each \( v \in S \), recall that the Satake parameter of \( \pi_v \) is a semi-simple conjugacy class in \( \text{GL}_2(\mathbb{C}) \); we write its characteristic polynomial as \( X^2 - t_v X + d_v \in \mathbb{C}[X] \).

Assuming the global Langlands conjecture, prove that there exists a unique automorphic representation \( \Pi \) of \( \text{GL}_2(\mathbb{A}_F) \) with the following properties: \( \Pi \) is unramified at all places \( w \) of \( F' \) lying above a place \( v \in S \), and for every such place \( w \), the Satake parameter of \( \Pi_w \) is the unique semi-simple conjugacy class in \( \text{GL}_2(\mathbb{C}) \) whose characteristic polynomial is given by

\[
\begin{cases}
X^2 - t_v X + d_v & \text{if } v \text{ is split in } F', \\
X^2 - (t_v^2 - 2d_v) X + d_v^2 & \text{if } v \text{ is inert in } F'.
\end{cases}
\]

(Problem 4) Let \( p \) and \( \ell \) be distinct prime numbers. Let \( \langle p \rangle \) be the subgroup of \( \mathbb{Q}_p^\times \) generated by \( p \), and let \( G_{Q_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \). For all integers \( r \geq 0 \), let \( A_r \) be the Abelian group defined by

\[
A_r = (\overline{\mathbb{Q}}_p^\times/\langle p \rangle)[\ell^r] = \{ x \in \overline{\mathbb{Q}}_p^\times \mid x^{\ell^r} \in \langle p \rangle \}/\langle p \rangle
\]

with the natural action of \( G_{Q_p} \).

(a) Show that \( A_r \) is (non-canonically) isomorphic to \( \mathbb{Z}/\ell^r \mathbb{Z} \times \mathbb{Z}/\ell^r \mathbb{Z} \).

(b) Show that there exists a Galois-equivariant short exact sequence

\[
1 \longrightarrow \mu_{\ell^r}(\overline{\mathbb{Q}}_p) \longrightarrow A_r \longrightarrow B_r \longrightarrow 1
\]

where \( B_r \) is a cyclic group of order \( \ell^r \) with trivial action of \( G_{Q_p} \).

(c) Define \( Q_\ell(1) = Q_\ell \otimes_{\mathbb{Z}_\ell} \lim_{\longleftarrow r} \mu_{\ell^r}(\overline{\mathbb{Q}}_p) \) and

\[
V = Q_\ell \otimes_{\mathbb{Z}_\ell} \lim_{\longleftarrow r} A_r.
\]

Let \( I_{Q_p} \subset G_{Q_p} \) be the inertia subgroup, and let \( V^{I_{Q_p}} \subseteq V \) be the subspace of inertia invariants. Show that there is an isomorphism \( Q_\ell(1) \simto V^{I_{Q_p}} \).

(d) Show that the \( L \)-function of the representation \( V \) of \( G_{Q_p} \) equals \( (1 - p \cdot p^{-s})^{-1} \).