Question 1.  

(a) Let $\hat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$, and let $\mathbb{A}$ be the adèlé ring of $\mathbb{Q}$. Consider the quotients $(\mathbb{R} \times \hat{\mathbb{Z}})/\mathbb{Z}$ and $A/\mathbb{Q} = (\mathbb{R} \times A^\infty)/\mathbb{Q}$, where $\mathbb{Z}$ and $\mathbb{Q}$ are diagonally embedded in $\mathbb{R} \times \hat{\mathbb{Z}}$ and $\mathbb{R} \times A^\infty$, respectively. Show that the map 

$$(x, y) \mod \mathbb{Z} \mapsto (x, y) \mod \mathbb{Q}$$

is an isomorphism of groups.

(b) Let $\mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\} \subset \mathbb{C}$ be the field of Gauss numbers, and let $\mathbb{A}_{\mathbb{Q}(i)}^\times = \mathbb{C}^\times \times A_{\mathbb{Q}(i)}^\times$ be the idèlé group of $\mathbb{Q}(i)$. The ring of integers $\mathbb{Z}[i]$ of $\mathbb{Q}(i)$ is known to be a principal ideal domain with unit group $\langle i \rangle = \{\pm 1, \pm i\}$. Use this to prove that the map

$$\langle i \rangle \backslash (\mathbb{C}^\times \times \hat{\mathbb{Z}}[i]^\times) \longrightarrow \mathbb{Q}(i)^\times \backslash A_{\mathbb{Q}(i)}^\times$$

$$\langle i \rangle \cdot (x, y) \longmapsto \mathbb{Q}(i)^\times \cdot (x, y)$$

is an isomorphism of groups, where $\langle i \rangle$ and $\mathbb{Q}(i)^\times$ are embedded diagonally in $\mathbb{C}^\times \times \hat{\mathbb{Z}}[i]^\times$ and $A_{\mathbb{Q}(i)}^\times$, respectively.

Question 2. Let $F$ be a number field, and let $E$ be an elliptic curve over $F$.

(a) The group law on $E$ can be given by rational functions with coefficients in $F$, and the $n$-torsion subgroup $E[n](F)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$. Use these facts to show that the action of the absolute Galois group $\text{Gal}(F/F)$ on $E[n](F)$ induces a group homomorphism $\text{Gal}(F/F) \rightarrow \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$ that is well-defined up to conjugacy by elements of $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$.

(b) Let $\ell$ be a prime number. Define the $\ell$-adic Tate module $T_\ell(E)$, including the action of $\text{Gal}(F/F)$ on $T_\ell(E)$.

(c) Let $\ell$ be a prime number, and let $v$ be a finite place of $F$ with $v \nmid \ell$ and such that $E$ has good reduction at $v$. Express the number of points on $E$ over the residue field $\kappa(v)$ in terms of the Galois representation $T_\ell(E)$. 

Question 3. Let $F$ be a number field, and let $E$ be an elliptic curve over $F$. For every finite place $v$ of $F$ at which $E$ has good reduction, let $\kappa(v)$ be the residue field, and let $a_v(E) = 1 - \#E(\kappa(v)) + \#\kappa(v)$.

Consider the set $S$ of finite places $v$ of $F$ with $v \nmid 3$ and such that $E$ has good reduction at $v$ and the integer $a_v(E)$ is divisible by $3$. In this question we investigate the density $\delta_S$ of the set $S$ in the set of all places of $F$.

(a) Let $Z$ be the normal subgroup of $\text{GL}_2(\mathbb{F}_3)$ consisting of the matrices $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ with $x \in \mathbb{F}_3 \times$. Let $\text{PGL}_2(\mathbb{F}_3)$ be the quotient $\text{GL}_2(\mathbb{F}_3)/Z$. Show that $\text{PGL}_2(\mathbb{F}_3)$ is isomorphic to the fourth symmetric group $S_4$.

Hint: consider the action of the group $\text{PGL}_2(\mathbb{F}_3)$ on the set $\mathbb{P}^1(\mathbb{F}_3)$ of lines in the 2-dimensional space $(\mathbb{F}_3)^2$ passing through the origin.

(b) Assume that $F$ and $E$ are such that the Galois representation $\rho_{E,3} : \text{Gal}(\overline{F}/F) \to \text{Aut}_{\mathbb{F}_3}(E[3](\overline{F})) \simeq \text{GL}_2(\mathbb{F}_3)$ on the 3-torsion of $E$ is surjective. Show that the density $\delta_S$ equals $3/8$.

(c) It is known that the group $S_4$ has 11 subgroups up to conjugacy. Show that as $F$ and $E$ vary (and $\rho_{E,3}$ is not necessarily surjective), there are at most 11 possible values for the density $\delta_S$.

Question 4. Let $F$ be a $p$-adic local field, and let $\mathcal{O}_F$ be its ring of integers. We write $G = \text{GL}_2(F)$ and $K = \text{GL}_2(\mathcal{O}_F)$.

(a) State what it means for an irreducible admissible smooth representation $(\pi, V)$ of $G$ to be unramified.

(b) Give the definition of the (unramified or spherical) Hecke algebra $\mathcal{H}(G, K)$, including the algebra structure.

(c) Give the definition of the Satake transform $S : \mathcal{H}(G, K) \sim \to \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}]^{S_2}$.

(Here $S_2$ acts on $\mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}]$ by permuting the variables.)

(d) Explain that using $S$ one can attach to any unramified irreducible smooth representation $\pi$ of $\text{GL}_2(F)$ a semi-simple conjugacy class $\phi_\pi$ in $\text{GL}_2(\mathbb{C})$ (the Satake parameter of $\pi$).

Hint: For (b) and (c), you will need the subgroups $B, N, T \subset G$, where $B$ consists of the upper triangular matrices, $N$ consists of the upper triangular matrices with ones on the diagonal, and $T$ consists of the diagonal matrices. If $H$ is one of the groups $G, B, N, T$, write $\mu_H$ for the left Haar measure on $H$ such that the compact open subgroup $H \cap K$ of $H$ has measure 1. It is known that $\mu_G, \mu_T$ and $\mu_N$ are also right Haar measures, but $\mu_B$ is not; you will need the modular function $\delta_B : B \to \mathbb{R}_{>0}$. 

Question 5. In this question we investigate some consequences of the local Langlands theorem and the global Langlands conjecture. Let $F$ be a number field, let $\ell$ be a prime number, and fix an isomorphism $\nu: \mathbb{Q}_\ell \rightarrow C$. The symmetric square representation

$$\text{Sym}^2: \text{GL}_2(\mathbb{Q}_\ell) \rightarrow \text{GL}_3(\mathbb{Q}_\ell)$$

is a 3-dimensional irreducible representation obtained from the action of $\text{GL}_2(\mathbb{Q}_\ell)$ on the space of homogenous polynomials $f \in \mathbb{Q}_\ell[X_1, X_2]$ of degree 2 defined by

$$g \cdot f := f(aX_1 + cX_2, bX_1 + dX_2) \text{ for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Q}_\ell), f \in \mathbb{Q}_\ell[X_1, X_2].$$

We have seen in the practice exam that for all $g \in \text{GL}_2(\mathbb{Q}_\ell)$, writing $t = \text{Tr}(g)$ and $d = \det(g)$, we have

$$\text{charpol}(\text{Sym}^2(g)) = X^3 - (t^2 - d)X^2 + d(t^2 - d)X - d^3 \in \mathbb{Q}_\ell[X].$$

In the solutions below, you may use this as a fact that you do not need to justify.

(a) We first consider the local case. Let $v$ be a finite place of $F$, let $F_v$ be the completion of $F$ at $v$, and let $q_v$ be the cardinality of the residue field of $F_v$. Let $\pi_v$ be an unramified smooth admissible irreducible representation of $\text{GL}_2(F_v)$, let $\phi_{\pi_v}$ be the Satake parameter of $\pi_v$ (a conjugacy class in $\text{GL}_2(C)$), and write $d_v = \det(\phi_{\pi_v})$ and $t_v = \text{Tr}(\phi_{\pi_v})$. Show that to $\pi_v$ we may attach a smooth admissible irreducible representation $\Pi_v$ of $\text{GL}_3(F_v)$, in such a way that the $L$-function of $\Pi_v$ equals

$$\frac{1}{1 - (t_v^2 - d_v)q_v^{-s} + d_v(t_v^2 - d_v)q_v^{-2s} - d_v^3q_v^{-3s}}.$$

A Galois representation $\rho: \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_n(\mathbb{Q}_\ell)$ that is isomorphic to a subquotient of the $\ell$-adic étale cohomology of an algebraic variety $X$ over $F$ is called geometric. In number theory and algebraic geometry, it is the geometric Galois representations that we are interested in, and a great deal of research is done on these representations. In particular, there is a famous conjecture of Fontaine and Mazur that characterizes the irreducible Galois representations $\rho_\pi$ arising from some algebraic cuspidal automorphic representation $\pi$ as those which are geometric.

In the questions below you may use that the class of geometric Galois representations is stable under all possible linear algebra constructions, meaning that if $V$ and $W$ are geometric $\ell$-adic Galois representations, then representations such as $V \oplus W$, $V \otimes W$, $V^{\otimes 2} \otimes W^{\vee}$, $\Lambda^3 V \otimes W^{\otimes 2}$, and so on, are all geometric as well. In particular, if $\rho: \text{Gal}(\bar{F}/F) \rightarrow \mathbb{Q}_\ell^3$ is a 2-dimensional geometric Galois representation, then the symmetric square of $\rho$, i.e. the representation

$$\text{Sym}^2(\rho) := \text{Sym}^2 \circ \rho: \text{Gal}(\bar{F}/F) \rightarrow \mathbb{Q}_\ell^3,$$

is geometric as well.

In the rest of this question, let $\pi$ be an algebraic cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$. We call $\pi$ non-special if there exists no strict algebraic

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1 Students who choose this question will receive a bonus point on their exam score.

2 "Algebraic" means that $H$ is equal to the vanishing locus of a collection of polynomials.
subgroup $H \subset \text{GL}_2(\mathbb{Q}_\ell)$ such that $\rho_\pi(\text{Gal}(\overline{F}/F)) \subset H$. For these non-special $\pi$, the composition $r \circ \rho_\pi$ is irreducible for any irreducible algebraic representation $r: \text{GL}_2(\mathbb{Q}_\ell) \to \text{GL}_N(\mathbb{Q}_\ell)$. Specializing to $r = \text{Sym}^2$, the representation $\text{Sym}^2(\rho_\pi)$ is irreducible as well. We will assume from now on that $\pi$ is non-special.

(b) Assuming the global Langlands conjecture and the Fontaine-Mazur conjecture, prove that there exists an automorphic representation $\Pi$ of $\text{GL}_3(\mathbb{A}_F)$ such that at all $F$-places $v$ where $\pi_v$ is unramified, $\Pi_v$ is unramified as well, and the Satake parameter $\phi_{\Pi_v}$ of $\Pi_v$ equals $\text{Sym}^2(\phi_{\pi_v})$, where $\phi_{\pi_v}$ is the Satake parameter of $\pi_v$.

(c) Prove that the automorphic representation $\Pi$ is uniquely characterized (up to isomorphism) by this property.

(d) Let $E$ be an elliptic curve over $F$ without complex multiplication, so that the Galois representation on its Tate module $V_\ell(E) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(E)$ is non-special in the sense that the image of $\text{Gal}(\overline{F}/F)$ is not contained in any strict algebraic subgroup of $\text{GL}_2(\mathbb{Q}_\ell) \cong \text{GL}_2(\mathbb{Q}_\ell)$. Let $S$ be the set of finite places of $F$ where $E$ has good reduction. For all $v \in S$, let $\kappa(v)$ be the residue field of $F$ at $v$, let $q_v = \# \kappa(v)$, and let $a_v(E) = 1 - \# E(\kappa(v)) + q_v$. Prove that the Euler product

$$\prod_{v \in S} \frac{1}{1 - (a_v(E)^2 - q_v)q_v^{-s} + q_v(a_v(E)^2 - q_v)q_v^{-2s} - q_v^3q_v^{-3s}}$$

converges for $\Re s$ sufficiently large and has an analytic continuation to the whole complex plane that satisfies a functional equation (which you do not need to specify).

Good luck!

Thank you for following our course, it has been a pleasure.