Goss L-functions

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1. Introduction

1.1. Motivation

The purpose of this talk is to introduce *Goss L-functions*. These L-functions are associated to strictly compatible families of Galois representations in the setting of function fields. These strictly compatible families arise for example from Drinfeld modules.

An essential difference with classical L-functions of global fields is that not only the base field has positive characteristic, but also the fields containing the coefficients of the Galois representations and L-functions that we consider. This means that we have to "translate" many methods and results from complex analysis to our setting.

The reference for this talk is [1, Chapter 8].

1.2. Notation

Let $\mathbf{F}_q$ be a finite field of q elements. Let C be a smooth, projective, geometrically connected curve over $\mathbf{F}_q$. Let K be the function field of C. Let ∞ be a fixed closed point of C, and let $\mathbf{F}_\infty$ be its residue field. We put $d_\infty = [\mathbf{F}_\infty : \mathbf{F}_q]$. (We will assume for simplicity that ∞ is a $\mathbf{F}_q$-rational point, so that $\mathbf{F}_\infty = \mathbf{F}_q$ and $d_\infty = 1$.) Let A be the coordinate ring of $C \setminus \{\infty\}$. Our assumption that C is geometrically connected implies $A^\times = \mathbf{F}_q^\times$.

Let K_∞ be the completion of K at ∞. Let $\mathbf{C}_\infty$ be the completion of an algebraic closure of K_∞. Let $|\ | = |\ |_\infty$ denote the standard absolute value. Furthermore, we recall that A is discrete in K_∞ (e.g. $\mathbf{F}_q[t]$ is discrete in $\mathbf{F}_q((t^{-1}))$).

2. A variant of a theorem of Mahler

Let p be a prime number. Let L be a field of characteristic p that is complete with respect to a non-trivial absolute value $|\ |$. Let $C(\mathbf{Z}_p, L)$ be the L-vector space of all continuous functions $\mathbf{Z}_p \to L$. The goal of this section is to give an explicit description of $C(\mathbf{Z}_p, L)$.

For every integer $k \geq 0$, we have a function

$$\begin{aligned} \mathbf{Z}_p &\to \mathbf{F}_p \\ y &\mapsto \binom{y}{k} \end{aligned}$$

defined as follows: if $a \in \mathbf{Z}_p$, then the element $\binom{a}{k} = \frac{a(a-1)\ldots(a-k+1)}{k!} \in \mathbf{Q}_p$ is actually in $\mathbf{Z}_p$, so we can reduce it modulo p. (To see this, note for example that $\mathbf{Z}$ is dense in $\mathbf{Z}_p$ and $\binom{a}{k}$ is in $\mathbf{Z}$ for all $a \in \mathbf{Z}$.)

The function $y \mapsto \binom{y}{k}$ is locally constant; more precisely, if N is sufficiently large so that $p^N > k$, then $\binom{y}{k}$ only depends on the class of y modulo p^N.

Example 2.1. *The function $\mathbf{Z}_2 \to \mathbf{F}_2, y \mapsto \binom{y}{2}$ is given explicitly by*

$$\binom{y}{2} = \begin{cases} 0 & \text{if } y \equiv 0, 1 \pmod 4, \\ 1 & \text{if } y \equiv 2, 3 \pmod 4. \end{cases}$$

Let $(a_k)_{k\geq 0}$ be a sequence in L such that $a_k \to 0$ as $k \to \infty$. We consider the *Newton series* with coefficients $(a_k)_{k\geq 0}$; this is the function

$$\begin{aligned} \phi\colon \mathbf{Z}_p &\longrightarrow L \\ x &\longmapsto \sum_{k\geq 0} a_k \binom{x}{k}. \end{aligned}$$

It is clear that the series converges uniformly on $\mathbf{Z}_p$, so ϕ is a continuous function.

One can recover the coefficients a_k from ϕ by

$$a_k = (\Delta^k \phi)(0) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \phi(i),$$

where $\Delta\phi(x) = \phi(x+1) - \phi(x)$.

Theorem 2.2. *Let L be a field of characteristic p that is complete with respect to a non-trivial absolute value. Then the L-linear map*

$$\{(a_k)_{k\geq 0} \mid a_k \in L, a_k \to 0 \text{ as } k \to \infty\} \longrightarrow C(\mathbf{Z}_p, L)$$
$$(a_k)_{k\geq 0} \longmapsto \left(x \mapsto \sum_{k\geq 0} a_k \binom{x}{k} \right)$$

is an isomorphism. The inverse is given by

$$\phi \longmapsto ((\Delta^k \phi)(0))_{k\geq 0}.$$

The proof is omitted.

3. The ζ-function of A

In this section, we introduce (for the moment without considering questions of convergence and analyticity) the prototypical example of a Goss L-function, namely the ζ-function of the base ring A.

3.1. A decomposition of $K_\infty^\times$

Let $U_1 \subset K_\infty^\times$ denote the subgroup of 1-units in $K_\infty^\times$, i.e. the group of elements $x \in K_\infty^\times$ with $v_\infty(x) = 0$ and such that the image of x in the residue field $\mathbf{F}_\infty$ equals 1.

Because the field K_∞ is complete, it contains a subfield mapping isomorphically to the residue field $\mathbf{F}_\infty$; we will denote this subfield by $\mathbf{F}_\infty$ as well.

Definition. A *sign function* on $K_\infty^\times$ is a group homomorphism

$$\operatorname{sgn}\colon K_\infty^\times \to \mathbf{F}_\infty^\times$$

that is the identity on the subgroup $\mathbf{F}_\infty^\times$ of $K_\infty^\times$. It is known (since U_1 is a pro-p-group and $\#\mathbf{F}_\infty^\times$ is not divisible by p) that every sign function is trivial on U_1.

From now on, we fix a sign function sgn on $K_\infty^\times$, and we extend sgn to a map $\operatorname{sgn}\colon K_\infty \to \mathbf{F}_\infty$ by setting $\operatorname{sgn}(0) = 0$. Furthermore, we fix a positive uniformiser of K_∞, i.e. an element $\pi \in K_\infty^\times$ satisfying $v_\infty(\pi) = 1$ and $\operatorname{sgn}(\pi) = 1$. For every $\alpha \in K_\infty^\times$, these choices determine a decomposition

$$\alpha = \operatorname{sgn}(\alpha)\pi^j \langle a \rangle,$$

where $j = v_\infty(\alpha)$ and where $\langle a \rangle \in U_1$ (recall that sgn is trivial on U_1). In other words, the choice of sgn and π gives us a (non-canonical) decomposition

$$K_\infty^\times \simeq \mathbf{F}_\infty^\times \times \mathbf{Z} \times U_1.$$

3.2. The 1-unit part of a fractional ideal

Let $\widehat{U}_1$ denote the subgroup of 1-units in $\mathbf{C}_\infty^\times$. There is a natural continuous group action of $\mathbf{Z}_p$ on $\widehat{U}_1$, which can be defined by the formula

$$\alpha^y = \sum_{j\geq 0} \binom{y}{j} (\alpha - 1)^j.$$

Furthermore, one can show that every element in $\widehat{U}_1$ has a *unique* p-th root in $\widehat{U}_1$. This shows that the action of $\mathbf{Z}_p$ extends uniquely to an action of $\mathbf{Q}_p$.

Let I_A be the group of fractional ideals of A, and let P_A^+ be the subgroup of fractional ideals generated by elements $\alpha \in K^\times$ with $\mathrm{sgn}(\alpha) = 1$. There exists a unique group homomorphism

$$\langle\ \rangle\colon P_A^+ \to U_1 \subset \widehat{U}_1$$

sending αA to $\langle \alpha \rangle$ for every positive element $\alpha \in K^\times$. It is known that the "narrow class group" I_A/P_A^+ is finite. Together with the fact that the Abelian group $\widehat{U}_1$ is uniquely divisible, this implies that $\langle\ \rangle$ extends uniquely to a group homomorphism

$$\langle\ \rangle\colon I_A \to \widehat{U}_1.$$

If $\mathfrak{a}$ is a fractional ideal of A, we call $\langle \mathfrak{a} \rangle$ the 1-*unit part* of $\mathfrak{a}$.

We define the degree of a fractional ideal as follows. If $\mathfrak{p}$ is a prime ideal of A, then we let $\deg \mathfrak{p}$ denote the degree of the residue field of $\mathfrak{p}$ over $\mathbf{F}_q$. We extend this by multiplicativity to a group homomorphism

$$\deg = \deg_A\colon I_A \to q^{\mathbf{Z}} \subset \mathbf{Q}^\times$$

The product formula implies the identity

$$\deg_A(aA) = -v_\infty(a) d_\infty \quad \text{for all } a \in K^\times.$$

In particular, we have

$$\deg_A \pi = -d_\infty.$$

3.3. Exponentiation of ideals

We define the ∞-*plane* (or *character space at infinity*) as

$$S_\infty = \mathbf{C}_\infty^\times \times \mathbf{Z}_p.$$

This is a topological group, which we will write additively, i.e.

$$(x, y) + (x', y') = (xx', y + y').$$

In addition, we choose an element $\pi_* \in \mathbf{C}_\infty^\times$ with $\pi_*^{d_\infty} = \pi$. Using this, we fix an embedding

$$\begin{aligned} \mathbf{Z} &\longrightarrow S_\infty \\ j &\longmapsto s_j = (\pi_*^{-j}, j). \end{aligned}$$

Let $\mathfrak{a} \in I_A$ be a fractional ideal of A, and let $s = (x, y) \in S_\infty$. We define

$$\mathfrak{a}^s = x^{\deg \mathfrak{a}} \langle \mathfrak{a} \rangle^y \in \mathbf{C}_\infty^\times.$$

One immediately verifies the identities

$$(\mathfrak{a}\mathfrak{b})^s = \mathfrak{a}^s \cdot \mathfrak{b}^s \quad \text{and} \quad \mathfrak{a}^{s+t} = \mathfrak{a}^s \cdot \mathfrak{a}^t,$$

i.e. we get a bilinear map

$$I_A \times S_\infty \to \mathbf{C}_\infty^\times.$$

Lemma 3.1. *For all $j \in \mathbf{Z}$ and all principal ideals $\mathfrak{a} = (\alpha) \in I_A$, we have*

$$\mathfrak{a}^{s_j} = (\alpha / \mathrm{sgn}\, \alpha)^j.$$

Proof. Exercise. □

Example. The ζ-function of A is the function

$$\begin{aligned}\zeta_A\colon S_\infty &\longrightarrow \mathbf{C}_\infty\\ s &\longmapsto \sum_{\mathfrak{a}\subseteq A} \mathfrak{a}^{-s},\end{aligned}$$

where $\mathfrak{a}$ runs over all non-zero ideals of A.

We will see later that this series indeed defines a function with good analytic and arithmetic properties.

4. Analytic functions on S_∞

In analogy with complex L-functions, it is important to have a good notion of "analytic functions" on S_∞. Because S_∞ is more complicated than the usual complex plane, the definition of entire functions is correspondingly more involved.

4.1. Entire functions

We let $\mathbf{C}_\infty[[x^{-1}]]$ denote the ring of formal power series in one variable (which we denote by x^{-1}). Let $\mathbf{C}_\infty[[x^{-1}]]_{\mathrm{entire}}$ denote the subring of $\mathbf{C}_\infty[[x^{-1}]]$ consisting of power series that converge on all of $\mathbf{C}_\infty$.

Lemma 4.1. *Let $f = \sum_{n\ge0} a_n x^{-n} \in \mathbf{C}_\infty[[x^{-1}]]$ be a power series. Then the following are equivalent:*

(1) The series f converges on all of $\mathbf{C}_\infty$.

(2) For every positive real number r, we have $|a_n|r^n \to 0$ as $n\to\infty$.

(3) We have $|a_n|^{1/n} \to 0$ as $n \to \infty$.

Proof. Exercise. □

We identify $\mathbf{C}_\infty[[x^{-1}]]$ with a subring of the ring of all continuous functions $\mathbf{C}_\infty \to \mathbf{C}_\infty$: to a power series $f = \sum_{j\ge0} a_j x^{-j} \in \mathbf{C}_\infty[[x^{-1}]]_{\mathrm{entire}}$ we associate the function sending x to $\sum_{j\ge0} a_j x^{-j} \in \mathbf{C}_\infty$. We equip $\mathbf{C}_\infty[[x^{-1}]]$ with the topology of uniform convergence on bounded subsets.

Let $g = \sum_{j\ge0} a_j x^{-j} \in \mathbf{C}_\infty[[x^{-1}]]_{\mathrm{entire}}$. For all $r > 0$, we define

$$\|\hat{g}\|_r = \max_{j\ge0} |a_j| r^j.$$

This maximum exists (in $\mathbf{R}$) because of Lemma 4.1.

Definition. An *entire function* on S_∞ is a function

$$f\colon S_\infty \to \mathbf{C}_\infty$$

such that there exists a continuous function

$$g\colon \mathbf{Z}_p \to \mathbf{C}_\infty[[x^{-1}]]_{\mathrm{entire}}$$

(where $\mathbf{C}_\infty[[x^{-1}]]_{\mathrm{entire}}$ is equipped with the topology described above) such that

$$f(x,y) = g(y)(x) \quad \text{for all } (x,y) \in S_\infty = \mathbf{C}_\infty^\times \times \mathbf{Z}_p.$$

Remark. Any continuous function g as above is automatically uniformly continuous. This is a special case of the *Heine–Cantor theorem*, which states that if $f\colon X \to Y$ is a continuous function between metric spaces with X compact, then f is uniformly continuous.

Example. Let $a \in A$ with $\operatorname{sgn}(a) = 1$. We consider the function

$$\begin{aligned} f\colon S_\infty &\to \mathbf{C}_\infty \\ s &\mapsto a^{-s}. \end{aligned}$$

This can be written as

$$\begin{aligned} f(x, y) &= x^{-\deg a}\langle a\rangle^{-y} \\ &= g(y)(x) \end{aligned}$$

where

$$g(y) = \langle a\rangle^{-y} x^{-\deg a} \in \mathbf{C}_\infty[[x^{-1}]].$$

Since the function $y \mapsto \langle a\rangle^{-y}$ is continuous, we conclude that g is continuous and hence f is an entire function.

By the above definition and the fact that for every $j \ge 0$ taking the coefficient of x^{-j} defines a continuous map $\mathbf{C}_\infty[[x^{-1}]]_{\text{entire}} \to \mathbf{C}_\infty$, every entire function $f\colon S_\infty \to \mathbf{C}_\infty$ can be written as

$$f(x, y) = \sum_{j\ge 0} f_j(y) x^{-j},$$

where $f_j\colon \mathbf{Z}_p \to \mathbf{C}_\infty$ is a continuous function.

For every continuous function $g\colon \mathbf{Z}_p \to \mathbf{C}_\infty$, we define

$$\|g\| = \max_{\mathbf{Z}_p} |g|.$$

Theorem 4.2. *Let $f\colon S_\infty \to \mathbf{C}_\infty$ be an entire function, and define f_j as above. Then for every $r > 0$ we have*

$$\|f_j\| r^j \to 0 \quad \text{as } j \to \infty.$$

Conversely, given a collection of continuous functions $f_j\colon \mathbf{Z}_p \to \mathbf{C}_\infty$ satisfying the above growth condition, the function $f(x, y) = \sum_{j\ge 0} f_j(y)x^{-j}$ is entire.

The proof is omitted.

Another useful representation of entire functions is by means of *Newton series*. Every entire function $f\colon S_\infty \to \mathbf{C}_\infty$ can be written as

$$f(x, y) = \sum_{k\ge 0} \hat f_k(x) \binom{y}{k}$$

for some power series $\hat f_k \in \mathbf{C}_\infty[[x^{-1}]]$.

Theorem 4.3. *Let $f\colon S_\infty \to \mathbf{C}_\infty$ be an entire function, and define $\hat f_k$ as above. Then for every $r > 0$ we have $\|\hat f_k\|_r \to 0$ as $k \to \infty$. Conversely, given a collection of power series $\hat f_k$ satisfying this growth condition, the function $f(x, y) = \sum_{k\ge 0} \hat f_k(x)\binom{y}{k}$ is entire.*

The proof is omitted.

4.2. Essentially algebraic functions

The following definition encapsulates a deep and important property of the L-functions that we will define.

Definition. An entire function $f\colon S_\infty \to \mathbf{C}_\infty$ is *essentially algebraic* if there exists a finite extension L of K inside $\mathbf{C}_\infty$ and a family of polynomials $h_{f,j} \in L[x^{-1}]$ for all $j \ge 0$ such that

$$f(x\pi_*^j, -j) = h_{f,j}(x) \quad \text{for all } x \in \mathbf{C}_\infty^\times.$$

Proposition 4.4. *Let $f\colon S_\infty \to \mathbf{C}_\infty$ be an entire function such that for all $j \ge 0$ the power series defining the function $x \mapsto f(x\pi_*^j, -j)$ has coefficients in A. Then f is essentially algebraic.*

Proof. Let $j \ge 0$. By the definition of entire functions, we have

$$f(x\pi_*^j, -j) = g(-j)(\pi_*^j x)$$

with $g(-j) \in \mathbf{C}_\infty[[x^{-1}]]_{\text{entire}}$. This implies that the right-hand side is in $\mathbf{C}_\infty[[x^{-1}]]_{\text{entire}}$; on the other hand, it is in $A[[x^{-1}]]$ by assumption. We note that

$$\mathbf{C}_\infty[[x^{-1}]]_{\text{entire}} \cap A[[x^{-1}]] = A[x^{-1}],$$

because the coefficients of an entire power series tend to 0 and A is discrete in $\mathbf{C}_\infty$. This shows that the function $x \mapsto f(x\pi_*^j, -j)$ is a polynomial in x^{-1} with coefficients in A. We conclude that f is essentially algebraic. □

Example. Let $a \in A$ with $\operatorname{sgn}(a) = 1$. We again consider the function

$$\begin{aligned} f\colon S_\infty &\to \mathbf{C}_\infty \\ s &\mapsto a^{-s}. \end{aligned}$$

Letting $\langle a \rangle$ denote the 1-unit part of a as before, we get

$$a = \pi_*^{-\deg a} \langle a \rangle$$

and

$$f(x, y) = x^{-\deg a} \langle a \rangle^{-y}.$$

This implies that for every integer $j \geq 0$ we have

$$\begin{aligned} f(x\pi_*^j, -j) &= (x\pi_*^j)^{-\deg a} \langle a \rangle^j \\ &= (\pi_*^{-\deg a} \langle a \rangle)^j x^{-\deg a} \\ &= a^j x^{-\deg a}. \end{aligned}$$

The assumption $a \in A$ implies $\deg a \geq 0$, and hence $f(x\pi_*^j, -j)$ is a polynomial in x^{-1} with coefficients in A. We conclude that f is essentially algebraic.

Example. We take $f(s) = \zeta_A(s)$. We will see later that ζ_A is an entire function. Let us work out what this means in the following situation:

$$\begin{aligned} A &= \mathbf{F}_q[t] \\ K &= \mathbf{F}_q(t) \\ K_\infty &= \mathbf{F}_q((t^{-1})) \\ \pi &= t^{-1} \end{aligned}$$

We (have to) take $\pi_* = \pi = t^{-1}$, and we define the sign function sgn by requiring that if $a \in A$ is non-zero, then $\operatorname{sgn}(a)$ is the leading term of a. Then the set of positive elements of A is simply the set of monic polynomials. If $a \in A$ is monic, we have

$$\langle a \rangle = t^{-\deg a} a.$$

We write M_d for the set of monic polynomials of degree d in A. We have

$$\begin{aligned} \zeta_A(x, y) &= \sum_{a \in A \text{ monic}} x^{-\deg a} \langle a \rangle^{-y} \\ &= \sum_{d \geq 0} \left(\sum_{a \in M_d} (t^{-d} a)^{-y} \right) x^{-d}. \end{aligned}$$

Hence we can rewrite $\zeta_A(x\pi_*^j, -j)$ as

$$\begin{aligned} \zeta_A(xt^{-j}, -j) &= \sum_{d \geq 0} \left(\sum_{a \in M_d} (t^{-d} a)^j \right) (xt^{-j})^{-d} \\ &= \sum_{d \geq 0} \left(\sum_{a \in M_d} a^j \right) x^{-d}. \end{aligned}$$

This is a power series in x^{-1} with coefficients in A. Assuming that ζ_A is indeed an entire function, we deduce from Proposition 4.4 that ζ_A is essentially algebraic and that the above power series is actually a polynomial. This statement is equivalent to the following:

Theorem 4.5. *Let $\mathbf{F}_q$ be a finite field of q elements, and for every $d \geq 0$, let M_d be the set of monic polynomials of degree d over $\mathbf{F}_q$. Then for every $j \geq 0$, there exists $D_j \geq 0$ such that*

$$\sum_{a \in M_d} a^j = 0 \quad \textit{for all } d > D_j.$$

5. $\mathfrak{p}$-adic representations and L-functions

Just as in the classical setting, L-functions can be defined starting from *strictly compatible families of Galois representations*. We will now introduce these.

5.1. Notation and preliminaries

Let L be a finite (not necessarily separable) extension of K. Let $\mathcal{O}_L$ be the ring of "A-integers" in L; this is the integral closure of A in L. It is known that the A-algebra $\mathcal{O}_L$ is locally free of finite rank as an A-module. We fix a separable closure L^{sep} of L, and we define

$$G_L = \mathrm{Gal}(L^{\mathrm{sep}}/L).$$

For every place w of L, let $\deg w$ denote the degree of L; this is just the degree of the residue field $\mathbf{F}_w$ of w over $\mathbf{F}_q$. If w is a place of L lying over a place v of A, we write

$$f_w = (\deg w)/(\deg v) = [\mathbf{F}_w : \mathbf{F}_v].$$

Let $I_{\mathcal{O}_L}$ denote the group of fractional $\mathcal{O}_L$-ideals; this is a free Abelian group generated by the prime ideals of $\mathcal{O}_L$. We define a group homomorphism

$$\mathrm{n} = \mathrm{n}_{L/K}\colon I_{\mathcal{O}_L} \to I_A$$

by requiring that for every prime ideal w of $\mathcal{O}_L$ we have $\mathrm{n}_{L/K} w = v^{f_w}$.

5.2. Strictly compatible families of $\mathfrak{p}$-adic representations

Definition. Let $\mathfrak{p}$ be a prime of $\operatorname{Spec} A$. A $\mathfrak{p}$-*adic representation* of G_L is a continuous homomorphism

$$\rho\colon G_L \to \mathrm{Aut}_{K_\mathfrak{p}}(V)$$

where V is a finite-dimensional vector space over the local field $K_\mathfrak{p}$.

Definition. Let ρ be a $\mathfrak{p}$-adic representation of G_L, and let w be a place of L. We say that ρ is *unramified* at w if the restriction of ρ to some (hence any) inertia group at w (in G_L) is trivial.

Let w be a place of L such that ρ is unramified at w. We write Frob_w for the conjugacy class of *geometric* Frobenius elements at w. Then we define

$$P_{\rho,w} = \det(\mathrm{id} - \rho(\mathrm{Frob}_w)u \mid V) \in K_\mathfrak{p}[u].$$

Note that $\rho(\mathrm{Frob}_w)$ is defined up to conjugacy in $\rho(G_L) \subset \mathrm{Aut}_{K_\mathfrak{p}}(V)$ because ρ is unramified at w. This implies that the definition of $P_{\rho,w}$ is independent of the choice of Frob_w in its conjugacy class.

Definition. For each prime $\mathfrak{p} \in \operatorname{Spec} A$, let a $\mathfrak{p}$-adic representation $\rho_\mathfrak{p}$ of G_L be given. We say that the family $(\rho_\mathfrak{p})_{\mathfrak{p}\in\operatorname{Spec} A}$ is *strictly compatible* if there exists a finite subset B of places of L, containing all the places of L above ∞, such that the following conditions hold:

(1) Let $\mathfrak{p}$ be a prime of $\operatorname{Spec} A$, and let w be a place of L satisfying $w \notin B$ and $w \nmid \mathfrak{p}$. Then $\rho_\mathfrak{p}$ is unramified at w, and $P_{\rho_\mathfrak{p},w}$ lies in $K[u]$.

(2) Let $\mathfrak{p}, \mathfrak{p}'$ be two primes of $\operatorname{Spec} A$, and let w be a place of L satisfying $w \notin B$, $w \nmid \mathfrak{p}$ and $w \nmid \mathfrak{p}'$. Then the two polynomials $P_{\rho_\mathfrak{p},w}$ and $P_{\rho_{\mathfrak{p}'},w}$ (which are both in $K[u]$ by (1)) are equal.

If $\hat\rho = (\rho_\mathfrak{p})_{\mathfrak{p}\in\operatorname{Spec} A}$ is a strictly compatible family of $\mathfrak{p}$-adic representations of G_L, then there exists a unique smallest set B as in the above definition. This is called the set of *bad places* of $(\rho_\mathfrak{p})_{\mathfrak{p}\in\operatorname{Spec} A}$. For w not in B, we write $P_{\hat\rho,w}$ for the polynomial $P_{\hat\rho,w}$ such that $P_{\hat\rho,w} = P_{\rho_\mathfrak{p},w}$ for all $\mathfrak{p}$ with $w \nmid \mathfrak{p}$.

5.3. L-functions

Definition. Let $\hat{\rho} = (\rho_\mathfrak{p})_{\mathfrak{p}\in\operatorname{Spec} A}$ be a strictly compatible family of $\mathfrak{p}$-adic representations, and let B be its set of bad places. The *L-function* of $\hat{\rho}$ is the function

$$\begin{aligned} L(\hat{\rho},\ \ )\colon S_\infty &\longrightarrow \mathbf{C}_\infty \\ s &\longmapsto \prod_{w\notin B} P_{\hat{\rho},w}((\mathrm{n}_{L/K}w)^{-s})^{-1}. \end{aligned}$$

where w runs over all places of L that are not in B.

Remark. Of course, one would also like to define L-factors at the places in B. However, this turns out to be quite difficult, and we will not say anything about this.

Remark. One can also define $\mathfrak{p}$-adic L-functions, in analogy with p-adic L-functions in the classical theory. We will not say anything about these either.

Proposition 5.1. *(1) Let $\mathbf{1}$ be the strictly compatible family of one-dimensional $\mathfrak{p}$-adic representations of G_L where each $\mathfrak{p}$-adic representation is the identity. Then $\zeta_{\mathcal{O}_L}(s) = L(\mathbf{1}, s)$ converges on the half-plane*

$$\{(x, y) \in S_\infty \mid |x|_\infty > 1\}.$$

(2) Let ϕ be an A-Drinfeld module of rank d over L. Then $L(\phi, s)$ converges on the half-plane

$$\{(x, y) \in S_\infty \mid |x|_\infty > q^{1/d}\}.$$

We omit the proof. The first part can be proved directly from the definitions. The second part can be proved using the Riemann hypothesis for Drinfeld modules over finite fields.

5.4. Strictly compatible families arising from ($\mathfrak{p}$-adic Tate modules of) Drinfeld modules

Let $\phi\colon A \to \operatorname{End}_{\mathbf{F}_q}(\mathbf{G}_{\mathrm{m},L}) \cong L\{\tau\}$ be an A-Drinfeld module of rank d over L. For every prime ideal $\mathfrak{p}$ of A, let $A_\mathfrak{p}$ denote the completion of A at $\mathfrak{p}$; note that this is (non-canonically) isomorphic to the power series ring over the residue field of $\mathfrak{p}$.

We put

$$T_\mathfrak{p}\phi = \varprojlim_n \phi[\mathfrak{p}^n].$$

This is in a natural way an $A_\mathfrak{p}[\operatorname{Gal}(L^{\mathrm{sep}}/L)]$-module, and it gives rise to a $\mathfrak{p}$-adic representation of G_L. One can show that when $\mathfrak{p}$ varies, this gives a strictly compatible family of $\mathfrak{p}$-adic representations.

6. L-functions of finite characters

As before, let L be a finite (not necessarily separable) extension of K, let $\mathcal{O}_L$ be the ring of A-integers in L, and let $G_L = \operatorname{Gal}(L^{\mathrm{sep}}/L)$.

We consider a group homomorphism

$$\chi\colon G_L \to \mathbf{C}_\infty^\times.$$

We assume that χ has finite image, or equivalently that there exists a finite Abelian extension L_1 of L such that χ factors as

$$\chi\colon G_L \to \operatorname{Gal}(L_1/L) \to \mathbf{C}_\infty^\times.$$

Let $\mathcal{B}$ be the conductor of L_1/L, as defined in class field theory; this is a product of places of L with multiplicities. We write $\mathcal{B} = \mathcal{B}_\infty\mathcal{B}_{\mathrm{fin}}$, where $\mathcal{B}_\infty$ has support at the places of L above the place ∞ of K and $\mathcal{B}_{\mathrm{fin}}$ is a product of finite places; we note that $\mathcal{B}_{\mathrm{fin}}$ can be regarded as an (integral) ideal of $\mathcal{O}_L$.

If $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_L$, we write $\chi(\mathfrak{p}) = 0$ if $\mathfrak{p} \mid \mathcal{B}_{\mathrm{fin}}$, and $\chi(\mathfrak{p}) = \chi(\operatorname{Frob}_\mathfrak{p})$ if $\mathfrak{p} \nmid \mathcal{B}_{\mathrm{fin}}$, where $\operatorname{Frob}_\mathfrak{p}$ is again a *geometric* Frobenius element at $\mathfrak{p}$.

Definition. The *L-function* of χ is the function

$$\begin{aligned} L(\chi,\ ) \colon S_\infty &\longrightarrow \mathbf{C}_\infty \\ s &\longmapsto \prod_{\mathfrak{p}\subset\mathcal{O}_L \text{ prime}} \bigl(1-\chi(\mathfrak{p})(\mathrm{n}_{L/K}\mathfrak{p})^{-s}\bigr)^{-1} \end{aligned}$$

When χ is the trivial character, we can take $L_1 = L$; then we get $L(\chi, s) = \zeta_{\mathcal{O}_L}(s)$.

Theorem 6.1. *The function $L(\chi,\ )$ is an entire function on S_∞.*

Proof. For the sake of exposition, we only give the full proof in the case $L = K = \mathbf{F}_q(t)$, but we will only start making this restriction later on.

First, we expand the Euler product defining $L(\chi, s)$ as

$$\begin{aligned} L(\chi,s) &= \prod_{\mathfrak{p}\in I_L(\mathcal{B}) \text{ prime}} \bigl(1-\chi(\mathfrak{p})(\mathrm{n}_{L/K}\mathfrak{p})^{-s}\bigr)^{-1} \\ &= \sum_{\mathfrak{a}\in I_L(\mathcal{B}) \text{ integral}} \chi(\mathfrak{a})(\mathrm{n}_{L/K}\mathfrak{a})^{-s}. \end{aligned}$$

Let $P_L(\mathcal{B})$ be the group of principal fractional ideals of L that are generated by elements that are congruent to 1 modulo $\mathcal{B}$. Let S be a set of coset representatives for the ray class group $I_L(\mathcal{B})/P_L(\mathcal{B})$; this is a finite set. Then we can split up the above sum as

$$L(\chi,s) = \sum_{\mathfrak{b}\in S} \chi(\mathfrak{b})(\mathrm{n}\mathfrak{b})^{-s} \left(\sum_{\substack{\mathfrak{a}\in P_L(\mathcal{B}) \\ \mathfrak{a}\mathfrak{b} \text{ integral}}} (\mathrm{n}\mathfrak{a})^{-s} \right)$$

It clearly suffices to show that for each $\mathfrak{b}\in S$ the function

$$L_{\mathfrak{b}}(\chi,s) = \left(\sum_{\substack{\mathfrak{a}\in P_L(\mathcal{B}) \\ \mathfrak{a}\mathfrak{b} \text{ integral}}} (\mathrm{n}\mathfrak{a})^{-s} \right) (\mathrm{n}\mathfrak{b})^{-s}$$

is entire. We write this as

$$L_{\mathfrak{b}}(\chi,s) = \left(\sum_{\substack{a\in(\mathfrak{b}^{-1}\setminus\{0\})/\mathcal{O}_L^\times \\ (a)\in P_L(\mathcal{B})}} (\mathrm{n}(a))^{-s} \right) (\mathrm{n}\mathfrak{b})^{-s}.$$

The idea is now as follows. If we write

$$L_{\mathfrak{b}}(\chi,s) = \sum_{d\ge 0} f_d(y) x^{-d},$$

then we want to show that there exists $c > 1$ such that

$$\|f_d\| \le c^{-d^2} \text{ for all } d \gg 0.$$

This will imply in particular that

$$\|f_d\|^{1/d} \le c^{-d} \text{ for all } j \gg 0,$$

and hence that $L_{\mathfrak{b}}(\chi,s)$ is entire. In other words, we want to show that

$$v_\infty(f_d(y)) \gg d^2 \text{ as } d\to\infty,$$

uniformly for $y \in \mathbf{Z}_p$. For this, because of continuity we may restrict to the case where $y = -j$ with j a non-negative integer.

For the sake of exposition, we now assume $L_1 = L = K$. Then we can take $\mathcal{B} = \emptyset$ and $S = \{\mathcal{O}_L\}$. Writing $s = (x, y)$ and again denoting the set of monic polynomials of degree d in $\mathbf{F}_q[t]$ by M_d, we obtain

$$\begin{aligned} L(\mathbf{1}, s) &= \sum_{a \in \mathbf{F}_q[t] \text{ monic}} a^{-s} \\ &= \sum_{d \geq 0} \left(\sum_{a \in M_d} \langle a \rangle^{-y} \right) x^{-d}, \end{aligned}$$

as we saw before. Hence

$$f_d(y) = \sum_{a \in M_d} \langle a \rangle^{-y}.$$

We now take $y = -j$ with $j \geq 0$. Since $\langle a \rangle = t^{-d} a$ for all $a \in M_d$, we obtain

$$\begin{aligned} f_d(-j) &= \sum_{a \in M_d} (t^{-d} a)^j \\ &= \sum_{c_0, \ldots, c_{d-1} \in \mathbf{F}_q} \left(1 + \sum_{i=0}^{d-1} c_i t^{-(d-i)} \right)^j \\ &= \sum_{c_0, \ldots, c_{d-1} \in \mathbf{F}_q} \sum_{r=0}^{j} \binom{j}{r} \left(\sum_{i=0}^{d-1} c_i t^{-(d-i)} \right)^r \\ &= \sum_{r=0}^{j} \binom{j}{r} \sum_{c_0, \ldots, c_{d-1} \in \mathbf{F}_q} \left(\sum_{i=0}^{d-1} c_i t^{-(d-i)} \right)^r \end{aligned}$$

By the strong triangle inequality and the fact that binomial coefficients are integers, we get

$$|f_d(-j)| \leq \max_{0 \leq r \leq j} \left| \sum_{c_0, \ldots, c_{d-1} \in \mathbf{F}_q} \left(\sum_{i=0}^{d-1} c_i t^{-(d-i)} \right)^r \right|$$

Hence we obtain

$$\begin{aligned} \|f_d\| &= \sup_{j \geq 0} |f_d(-j)| \\ &\leq \sup_{r \geq 0} \left| \sum_{c_0, \ldots, c_{d-1} \in \mathbf{F}_q} \left(\sum_{i=0}^{d-1} c_i t^{-(d-i)} \right)^r \right| \end{aligned}$$

We have now reduced the problem to an elementary problem about polynomials. It follows from a general result [1, Lemma 8.8.1(b)] that the t^{-1}-adic valuation of each of the elements of $\mathbf{F}_q[t^{-1}]$ between the absolute value signs above is at least $(q-1)d(d+1)/2$, which grows quadratically with d. This implies the required bound on $\|f_d\|$. □

Remark. Let ϕ be an A-Drinfeld module over L with complex multiplication (i.e. if M is the A-lattice in $\mathbf{C}_\infty$ attached to ϕ, then the A-algebra $\{\alpha \in \mathbf{C}_\infty \mid \alpha M \subseteq M\}$ has rank d over A). Then $L(\phi, s)$ factors as a product of L-functions attached to characters of G_L. These are entire, and we conclude that $L(\phi, s)$ is entire.

References

[1] D. Goss, *Basic Structures of Function Field Arithmetic*. Springer-Verlag, Berlin/Heidelberg, 1996.