NOTES ON NILPOTENT ENDOMORPHISMS AND THE JORDAN NORMAL FORM

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1. Nilpotent endomorphisms

In class we have seen one of the proofs of Theorem 4.3 from the book. Here is the same proof, but split up into parts, as well as a more elaborate example.

Lemma 1. *Let V be a vector space and $f\colon V \to V$ an endomorphism. Suppose $m > 0$ is an integer such that $f^m = 0$. If for each $j \in \{0, 1, \ldots, m-1\}$ we have a complementary subspace X_j of $\ker f^j$ inside $\ker f^{j+1}$, then we have*

$$V = X_0 \oplus X_1 \oplus X_2 \oplus \ldots \oplus X_{m-1}.$$

Proof. Note that we have $\ker f^m = V$ and $\ker f^0 = \{0\}$. For all $j \in \{0, 1, \ldots, m-1\}$, we have $\ker f^{j+1} = \ker f^j \oplus X_j$, so we find

$$\begin{aligned} V &= \ker f^m = \ker f^{m-1} \oplus X_{m-1} = (\ker f^{m-2} \oplus X_{m-2}) \oplus X_{m-1} = \\ &= \ker f^{m-2} \oplus (X_{m-2} \oplus X_{m-1}) = \cdots = \ker f^0 \oplus X_0 \oplus X_1 \oplus \ldots \oplus X_{m-1} = \\ &= X_0 \oplus X_1 \oplus \ldots \oplus X_{m-1}. \end{aligned}$$

□

Lemma 2. *Let V be a vector space and $f\colon V \to V$ an endomorphism. Let $j \geq 0$ be an integer. If $(x_1, x_2, \ldots, x_k)$ is a basis of a complementary space of $\ker f^j$ inside $\ker f^{j+1}$, then the sequence $(f(x_1), f(x_2), \ldots, f(x_k))$ can be extended to a basis $(f(x_1), f(x_2), \ldots, f(x_k), x_{k+1}, \ldots, x_l)$ of a complementary space of $\ker f^{j-1}$ inside $\ker f^j$.*

Proof. Let X denote the subspace generated by $(x_1, x_2, \ldots, x_k)$. Then the subspace generated by $(f(x_1), f(x_2), \ldots, f(x_k))$ is $f(X)$. For every element $z \in X$ we have $z \in \ker f^{j+1}$, so we have $f(z) \in \ker f^j$, and therefore $f(X) \subset \ker f^j$.

We claim that if any scalars $\lambda_1, \ldots, \lambda_k$ satisfy $\sum_{i=1}^k \lambda_i f(x_i) \in \ker f^{j-1}$, then we have $\lambda_1 = \ldots = \lambda_k = 0$. Indeed, set $z = \sum_{i=1}^k \lambda_i x_i \in X$. Then the assumption of the claim states $f(z) \in \ker f^{j-1}$, so $z \in \ker f^j$. From $z \in X \cap \ker f^j = \{0\}$ we conclude $z = 0$. Since the elements $x_1, \ldots, x_k$ are linearly independent, we conclude $\lambda_1 = \ldots = \lambda_k = 0$.

The claim implies in particular that the elements $f(x_1), f(x_2), \ldots, f(x_k)$ are linearly independent, so they form a basis for $f(X)$. The claim also implies $f(X) \cap \ker f^{j-1} = \{0\}$, so, by Lemma 2.6 from the book, $f(X)$ can be extended to a complementary space X' of $\ker f^{j-1}$ inside $\ker f^j$, and the basis for $f(X)$ can be extended to a basis $(f(x_1), f(x_2), \ldots, f(x_k), x_{k+1}, \ldots, x_l)$ for X'. □

Proposition 1. *Let V be a finite-dimensional vector space and $f\colon V \to V$ a nilpotent endomorphism. Then there exist elements $w_1, w_2, \dots, w_s \in V$ and nonnegative integers $e_1, e_2, \dots, e_s$ such that*

$$\big(w_1, f(w_1), \dots, f^{e_1}(w_1), w_2, f(w_2), \dots, f^{e_2}(w_2), \dots, w_s, f(w_s), \dots, f^{e_s}(w_s)\big) \tag{1}$$

is a basis for V and $f^{e_i+1}(w_i) = 0$ for all $1 \le i \le s$.

Proof. Let m be a positive integer such that $f^m = 0$. We start by picking a basis for some complementary subspace X_{m-1} of $\ker f^{m-1}$ inside $\ker f^m = V$. We use Lemma 2 recursively for $j = m-1, m-2, \dots, 2, 1$, to obtain a complementary space X_j of $\ker f^j$ inside $\ker f^{j+1}$ for each such j, together with bases satisfying that if $(x_1, \dots, x_k)$ is a basis for X_j, then the basis for X_{j-1} starts with $(f(x_1), f(x_2), \dots, f(x_k))$.

By Lemma 1, we have $V = X_0 \oplus X_1 \oplus X_2 \oplus \ldots \oplus X_{m-1}$, so the union of the bases for the X_j together form a basis for V. If we let $w_1, \dots, w_r$ be the elements of this union that are not the image of another element in the union, then we can rearrange the elements of the union as in (1) for some integers $e_1, \dots, e_r$. In fact, the integer e_i equals the index j for which $w_i \in X_j$. Since we have $f^{e_i}(w_i) \in X_0 = \ker f$, we also find $f^{e_i+1}(w_i) = 0$, which finishes the proof. □

Remark 1. Suppose we are in the setting of the proposition. If we reverse the order of the elements in the basis in 1, then we obtain a basis B for V, with respect to which the matrix $[f]^B_B$ is a block matrix as in Remark 4.4 of the book.

Remark 2. Note that the blocks have sizes $e_1 + 1, e_2 + 1, \dots, e_r + 1$ (though in opposite order, if we are precise). To see how many blocks of each size there are, we note the following. For each integer $n \ge 0$, we set $r_n = \dim \ker f^n$. Furthermore, we set $s_n = r_n - r_{n-1}$ and $t_n = s_n - s_{n+1}$. Note that $\ker f^j$ is spanned by the union of the first j of the elements corresponding to each block, i.e., by

$$f^{e_1}(w_1), f^{e_1-1}(w_1), \dots, f^{e_1-j+1}(w_1), \dots, f^{e_r}(w_r), f^{e_r-1}(w_r), \dots, f^{e_r-j+1}(w_r),$$

except that in this sequence we have to leave out those expressions where the exponent of f is negative. This implies that $s_n = \dim \ker f^n - \dim \ker f^{n-1}$ is equal to the number blocks of size at least n. (Roughly said, each block of size at least n contributes one more element to a basis for $\ker f^n$, compared to a basis for $\ker f^{n-1}$.) We conclude that the number of blocks of size exactly equal to n is $s_n - s_{n+1} = t_n$.

Remark 3. Note that in terms of the proof of Proposition 1, we have

$$\dim X_j = \dim \ker f^{j+1} - \dim \ker f^j = r_{j+1} - r_j = s_{j+1}.$$

Remark 4. In terms of the algorithm in Remark 4.8 in the book, we have $U_j = X_j \oplus X_{j+1} \oplus \ldots \oplus X_{m-1}$. The elements in step (4) of that algorithm form a basis for X_j.

Example 1. Consider the real matrix

$$A = \begin{pmatrix} -5 & 10 & -8 & 4 & 1 \\ -4 & 8 & -10 & 8 & 2 \\ -3 & 6 & -12 & 12 & 3 \\ -2 & 4 & -8 & 4 & 10 \\ -1 & 2 & -4 & 2 & 5 \end{pmatrix}$$

We compute

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & -18 & 36 \\ 0 & 0 & 0 & -36 & 72 \\ 0 & 0 & 0 & -54 & 108 \\ 0 & 0 & 0 & -36 & 72 \\ 0 & 0 & 0 & -18 & 36 \end{pmatrix} \qquad \text{and} \qquad A^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so we can start the algorithm of Remark 4.8 in the book (or, equivalently, the proces suggested by the proof of Proposition 1 above) with $m = 3$. The kernel $\ker A$ is generated by

$$x = (-3, 0, 3, 2, 1) \qquad \text{and} \qquad x' = (2, 1, 0, 0, 0).$$

The kernel $\ker A^2$ is generated by

$$e_1 = (1, 0, 0, 0, 0), \quad e_2 = (0, 1, 0, 0, 0), \quad e_3 = (0, 0, 1, 0, 0), \quad \text{and} \quad y = (0, 0, 0, 2, 1).$$

Clearly, we have $\ker A^3 = \mathbb{R}^5$. In terms of Remark 2, we find $r_0 = 0$ and $r_1 = 2$ and $r_2 = 4$ and $r_n = 5$ for $n \geq 3$; this yields $s_1 = 2$ and $s_2 = 2$ and $s_3 = 1$ and $s_4 = 0$. Finally, we obtain $t_1 = 0$ and $t_2 = 1$ and $t_3 = 1$, so we already find that the standard nilpotent form consists of one block of size 2 and one block of size 3.

To find an appropriate basis, we start with picking a complementary space X_2 of $\ker A^2$ inside $\ker A^3 = \mathbb{R}^5$. Since $\dim \ker A^3 - \dim \ker A^2 = 3 - 2 = 1$, it suffices to pick any element of $\mathbb{R}^5$ that is not contained in $\ker A$. We choose $w_1 = e_5 = (0, 0, 0, 0, 1)$, which gives $Aw_1 = (1, 2, 3, 10, 5)$ and $A^2 w_1 = 36(1, 2, 3, 2, 1)$ en $A^3 w_1 = 0$. This gives $X_2 = \langle w_1 \rangle$. In the next step, we are looking for a complementary space X_1 of $\ker A$ inside $\ker A^2$ such that $f(X_2) \subset X_1$. In other words, we want to extend $f(X_2) = \langle Aw_1 \rangle$ to a complementary space of $\ker A$ inside $\ker A^2$. In order to do this, we follow the proof of Lemma 2.6 in the book: take a basis for $\ker A$ and for $f(X_2)$ and put the elements of these two bases as columns in a matrix; we also take generators for $\ker A^2$ and add these as columns to the matrix. We obtain

$$\left(\begin{array}{cc|c|cccc} -3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 & 1 & 0 \\ 2 & 0 & 10 & 0 & 0 & 0 & 2 \\ 1 & 0 & 5 & 0 & 0 & 0 & 1 \end{array}\right).$$

A row echelon form for this matrix is

$$\left(\begin{array}{cc|c|cccc} 1 & 0 & 5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

which has pivots in the first three columns as expected. Of the last four columns, only the first contains a pivot, so in order to extend $f(X_2)$ to a complementary space X_1 as mentioned, it suffices to add the first generator for $\ker A^2$, so we take $w_2 = (1, 0, 0, 0, 0)$, which gives $Aw_2 = -(5, 4, 3, 2, 1)$. The last step, namely finding a complementary space X_0 for $\ker A^0 = \{0\}$ inside $\ker A$ which contains $f(X_1)$, is trivial, as $f(X_1)$ is generated by $A^2 w_1$ and Aw_2, so $\dim f(X_1) = 2 = \dim \ker A$, so we have $X_0 = f(X_1)$ and we do not need to extend.

Hence, we obtain a basis $B = (A^2w_1, Aw_1, w_1, Aw_2, w_2)$ (note the order of the elements). If we denote the standard basis for $\mathbb{R}^5$ by E, the basis transformation matrix

$$P = [\mathrm{id}]_E^B = \begin{pmatrix} 36 & 1 & 0 & -5 & 1 \\ 72 & 2 & 0 & -4 & 0 \\ 108 & 3 & 0 & -3 & 0 \\ 72 & 10 & 0 & -2 & 0 \\ 36 & 5 & 1 & -1 & 0 \end{pmatrix}$$

satisfies

$$P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. Jordan Normal Form

Small example never give a good idea what is going on, because either you have very few blocks or very small blocks. So here we present a 10×10 matrix of which we will find the Jordan Normal Form, together with a corresponding basis transformation. We consider the real matrix

$$M = \begin{pmatrix} -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & -1 & 3 & -3 & 3 & -3 & 3 & -3 & 3 & -3 \\ 0 & 0 & 2 & 0 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

which has characteristic polynomial $(x+1)^2(x-2)^8$. Therefore, we have to deal with the two generalised eigenspaces

$$U_1 = \ker(M+I)^2 \qquad \text{and} \qquad U_2 = \ker(M-2I)^8$$

of dimensions 2 and 8, respectively. Indeed, by Theorem 5.1 of the book, we have $\mathbb{R}^{10} = U_1 \oplus U_2$. Let $e_1, \ldots, e_{10} \in \mathbb{R}^{10}$ denote the standard basis vectors.

We start with the hardest case, namely U_2. By definition of U_2, the restriction of $M - 2I$ to U_2 is nilpotent, as $(M-2I)^8$ restricts to 0 on U_2. By finding a row echelon form for $(M-2I)^n$ for $1 \le n \le 3$, we find $r_1 = \dim\ker(M-2I) = 4$ and $r_2 = \dim\ker(M-2I)^2 = 7$ and $r_3 = \dim\ker(M-2I)^3 = 8$. For $n > 3$ we have

$$8 = \dim\ker(M-2I)^3 \le \dim\ker(M-2I)^n \le \dim U_2 = 8,$$

so we conclude $\ker(M-2I)^3 = U_2$ and $r_n = \dim\ker(M-2I)^n = 8$ for $n > 3$. This yields the following table for $s_n = r_n - r_{n-1}$ and $t_n = s_n - s_{n+1}$.

n	r_n	s_n	t_n
0	0		
1	4	4	1
2	7	3	2
3	8	1	1
4	8	0	0
5	8	0	0

We conclude that in any Jordan Normal Form for M, there is one Jordan block for eigenvalue 2 of size 1, there are two of size 2, and there is one of size 3.

To find a corresponding basis for U_2, we consider the filtration

$$\{0\} \subset \ker(M-2I) \subset \ker(M-2I)^2 \subset \ker(M-2I)^3 = U_2$$

and we will construct subspaces X_0, X_1, X_2 with explicit bases C_0, C_1, C_2, respectively, such that

(1) X_j is a complementary space of $\ker(M-2I)^j$ inside $\ker(M-2I)^{j+1}$;
(2) $(M-2I)(X_j) \subset X_{j-1}$;
(3) if $C_j = (u_1, u_2, \ldots, u_k)$, then C_{j-1} starts with the sequence $((M-2I)u_1, (M-2I)u_2, \ldots, (M-2I)u_k)$.

We had already brought $(M-2I)^n$ into row echelon form before and we can use that to find explicit bases for $\ker(M-2I)^n$ for $1 \le n \le 3$. We find

$$\begin{aligned}\ker(M-2I) &= \langle x_1, x_2, x_3, x_4\rangle,\\ \ker(M-2I)^2 &= \langle y_1, y_2, y_3, y_4, y_5, y_6, y_7\rangle,\\ \ker(M-2I)^3 &= \langle z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\rangle,\end{aligned}$$

with

$$\begin{aligned}x_1 &= (0,1,0,-1,0,0,0,0,0,0),\\ x_2 &= (0,0,1,1,0,0,0,0,0,0),\\ x_3 &= (0,0,0,0,1,1,0,0,0,0),\\ x_4 &= (0,0,0,0,0,0,1,1,0,0),\end{aligned}
\quad
\begin{aligned}y_1 &= (0,1,0,0,0,0,0,0,1,0),\\ y_2 &= (0,0,1,0,0,0,0,0,-1,0),\\ y_3 &= (0,0,0,1,0,0,0,0,1,0),\\ y_4 &= (0,0,0,0,1,0,0,0,-1,0),\\ y_5 &= (0,0,0,0,0,1,0,0,1,0),\\ y_6 &= (0,0,0,0,0,0,1,0,-1,0),\\ y_7 &= (0,0,0,0,0,0,0,1,1,0),\end{aligned}
\quad
\begin{aligned}z_1 &= (0,1,0,0,0,0,0,0,0,-1),\\ z_2 &= (0,0,1,0,0,0,0,0,0,1),\\ z_3 &= (0,0,0,1,0,0,0,0,0,-1),\\ z_4 &= (0,0,0,0,1,0,0,0,0,1),\\ z_5 &= (0,0,0,0,0,1,0,0,0,-1),\\ z_6 &= (0,0,0,0,0,0,1,0,0,1),\\ z_7 &= (0,0,0,0,0,0,0,1,0,-1),\\ z_8 &= (0,0,0,0,0,0,0,0,1,1).\end{aligned}$$

In step 1, we want a complementary subspace X_2 of $\ker(M-2I)^2$ inside $\ker(M-2I)^3$. One way to do this is to put the basis elements $y_1, \ldots, y_7$ for $\ker(M-2I)^2$ as columns in a matrix, and add the generators $z_1, \ldots, z_8$ for $\ker(M-2I)^3$ as more columns to the right:

$$\left(\begin{array}{ccccccc|cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{array}\right).$$

The reduced row echelon form for this matrix is

$$\left(\begin{array}{ccccccc|cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

Of the added columns to the right, only the first has a pivot. This implies that the first of the added generators, namely z_1, generates a complementary space of $\ker(M-2I)^2$ inside $\ker(M-2I)^3$. [Of course, we could have seen this without any computation. From the last coordinate, we see that no z_i is contained in $\ker(M-2I)^2$, as the last coordinate of all the y_i is 0; since $\ker(M-2I)^2$ has codimension 1 inside $\ker(M-2I)^3$ (meaning the difference of their dimensions is 1), any element in $\ker(M-2I)^3$ that is not contained in $\ker(M-2I)^2$ generates a complementary space of $\ker(M-2I)^2$ inside $\ker(M-2I)^3$.] So, we take $w_1 = z_1$ and $X_2 = \langle w_1 \rangle$ and $C_2 = (w_1)$.

In step 2, we want to extend $(M-2I)(X_2)$, that is, the image of X_2 under multiplication by $M-2I$, to a complementary subspace X_1 of $\ker(M-2I)$ inside $\ker(M-2I)^2$. We follow the proof of Lemma 2.6 from the book. First, note that $(M-2I)(X_2)$ has basis $(M-2I)w_1 = (0,0,1,1,1,1,1,0,-1,0)$. We put the basis elements $x_1,\dots,x_4$ for $\ker(M-2I)$ as columns in a matrix, we add $(M-2I)w_1$ as a column to the right, and we finally add the generators $y_1,\dots,y_7$ for $\ker(M-2I)^2$

as columns to the far right:

$$\left(\begin{array}{cccc|c|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

The reduced row echelon form for this matrix is

$$\left(\begin{array}{cccc|c|ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

So of the last seven columns, the first and the fourth contain a pivot. This means that if we add y_1 and y_4 to $(M-2I)w_1$, then we obtain a basis for a complementary space of $\ker(M-2I)$ inside $\ker(M-2I)^2$. Hence, we set $w_2 = y_1$ and $w_3 = y_4$, and $C_1 = ((M-2I)w_1, w_2, w_3)$ and we denote the space $\langle C_1 \rangle$ by X_1.

In step 3, we construct a complementary space of $\ker(M-2I)^0$ inside $\ker(M-2I)$. Since we have $(M-2I)^0 = I$, we find $\ker(M-2I)^0 = \{0\}$, so $X_0 = \ker(M-2I)$. We already have the elements $f(u)$ in X_0 for $u \in C_1$; these equal $(M-2I)^2 w_1 = (0,0,0,0,0,0,-1,-1,0,0)$ and $(M-2I)w_2 = (0,0,1,1,1,1,1,1,0,0)$ and $(M-2I)w_3 = (0,0,0,0,-1,-1,-1,-1,0,0)$. We put these as columns in a matrix and add columns for the generators $x_1, \ldots, x_4$ for $\ker(M-2I)$.

$$\left(\begin{array}{ccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

The reduced row echelon form for this matrix is

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since only the first of the right-most four columns has a pivot, it suffices to add x_1 to the elements we already had in order to get a basis for $\ker(M-2I)$. In other words, we set $w_4 = x_1$ and $C_0 = ((M-2I)^2 w_1, (M-2I)w_2, (M-2I)w_3, w_4)$. Then C_0 is a basis for $X_0 = \ker(M-2I)$. We now reorder the elements of the bases C_0, C_1, C_2 for X_0, X_1, X_2 to get a basis

$$C = \big((M-2I)^2 w_1, (M-2I)w_1, w_1, (M-2I)w_2, w_2, (M-2I)w_3, w_3, w_4\big)$$

for $X_0 \oplus X_1 \oplus X_2 = U_2$.

We continue with U_1. By definition of U_1, the restriction of $M+I$ to U_1 is nilpotent, as $(M+I)^2$ restricts to 0 on U_1. It is easy to verify that $\ker(M+I)$ is generated by e_1, while $\ker(M+I)^2$ is generated by e_1 and e_2. We proceed exactly the same as for U_2, but everything is so much easier in this case, that we leave it to the reader to identify the analogues of X_j and C_j. The vector e_2 generates a complementary space of $\ker(M+I)$ inside $\ker(M+I)^2$, so we set $w_5 = e_2$. Its image under $M+I$ is $(M+I)w_5 = e_1$, which, as we said, generates $\ker(M+I)$. Together, w_5 and $(M+I)w_5 = e_1$ form a basis D for the generalised eigenspace U_1.

The bases C and D together yield the basis

$$B = \big((B-2I)^2 w_1, (B-2I)w_1, w_1, (B-2I)w_2, w_2, (B-2I)w_3, w_3, w_4, (B+I)w_5, w_5\big)$$

for $U_1 \oplus U_2 = \mathbb{R}^{10}$. If we let E denote the standard basis for $\mathbb{R}^{10}$, then the matrix $P = [\mathrm{id}]_E^B$ (written as $P = {}_E[\mathrm{id}]_B$ in Delft), has the elements of B as columns, that is,

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now already know that $P^{-1}MP$ is a matrix in Jordan Normal Form, with Jordan blocks $B(2,3), B(2,2), B(2,2), B(2,1)$ and $B(-1,2)$ in this order along the diagonal (for this notation, see Theorem 5.2 from the book). Indeed, a simple but tedious calculation shows

$$\begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$