Linear algebra 2: exercises for Section 9 (first part)

Ex. 9.1. Let V be the vector space of continuous complex-valued functions defined on the interval $[0,1]$, with the inner product $\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$. Show that the set $\{x \mapsto e^{2\pi i k x} : k \in \mathbb{Z}\} \subset V$ is orthonormal. Is it a basis of V?

Ex. 9.2. Give an orthonormal basis for the 2-dimensional complex subspace V_3 of $\mathbb{C}^3$ given by the equation $x_1 - ix_2 + ix_3 = 0$.

Ex. 9.3. For the real vector space V of polynomial functions $[-1,1] \to \mathbb{R}$ with inner product given by

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx,$$

apply the Gram-Schmidt procedure to the elements $1, x, x^2, x^3$.

Ex. 9.4. For the real vector space V of continuous functions $[-\pi,\pi] \to \mathbb{R}$ with inner product given by

$$\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$$

show that the functions

$$1/\sqrt{2}, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$$

form an orthonormal set. [Note: for any function f the inner products with this list of functions is the sequence of Fourier coefficients of f.]