Linear algebra 2: exercises for Chapter 8 (second part)

Ex. 8.7. Let V be the 3-dimensional vector space of polynomials of degree at most 2 with coefficients in $\mathbb{R}$. For $f, g \in V$ define the bilinear form ϕ: $V \times V \to \mathbb{R}$ by

$$\phi(f, g) = \int_{-1}^{1} x f(x) g(x) dx.$$

1. Is ϕ non-degenerate or degenerate?
2. Give a basis of V for which the matrix associated to ϕ is diagonal.
3. Show that V has a 2-dimensional subspace U for which $U \subset U^{\perp}$.

Ex. 8.8. Let $e_1, \ldots, e_n$ be the standard basis of $V = \mathbb{R}^n$, and define a symmetric bilinear form ϕ on V by $\phi(e_i, e_j) = 2$ for all $i, j \in \{1, \ldots, n\}$. Give the signature of ϕ and a diagonalizing basis for ϕ.

Ex. 8.9. Suppose V is a vector space over $\mathbb{R}$ of finite dimension n with a symmetric non-degenerate bilinear form ϕ: $V \times V \to \mathbb{R}$, and suppose that U is a subspace of V with $U \subset U^{\perp}$. Then show that the dimension of U is at most $n/2$.

Ex. 8.10. For $x \in \mathbb{R}$ consider the matrix

$$A_x = \begin{pmatrix} x & -1 \\ -1 & x \end{pmatrix}$$

1. What is the signature of A_1 and A_{-1}?
2. For which x is A_x positive definite?
3. For which x is $\begin{pmatrix} x & -1 & 1 \\ -1 & x & 1 \\ 1 & 1 & 1 \end{pmatrix}$ positive definite?

Ex. 8.11. Let V be a vector space over $\mathbb{R}$, let $b : V \times V \to \mathbb{R}$ be an skew-symmetric bilinear form, and let $x \in V$ be an element that is not in the left kernel of b.

1. Show that there exist $y \in V$ such that $b(x, y) = 1$ and a linear subspace $U \subset V$ such that $V = \langle x, y \rangle \oplus U$ is an orthogonal direct sum with respect to b.
   Remark. The notation $\langle x, y \rangle$ denotes the subspace spanned by x and y, and of course has nothing to do with an inner product.
   Hint. Take $U = \langle x, y \rangle^{\perp} = \{v \in V : b(x, v) = b(y, v) = 0\}$.

2. Conclude that if $\dim V < \infty$, then then there exists a basis of V such that the matrix representing b with respect to this basis is a block diagonal matrix with blocks $B_1, \ldots, B_l$ of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and zero blocks $B_{l+1}, \ldots, B_k$.