Linear algebra 2: exercises for Section 5 (part 2)

Ex. 5.9. Let $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\phi(x) = Ax$ where A is the matrix

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We proved in class that generalized eigenspaces for ϕ are ϕ-invariant. What are these spaces in this case? Give all other ϕ-invariant subspaces of $\mathbb{R}^3$.

Ex. 5.10. Compute the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 & -2 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Does A have a Jordan normal form as 4×4 matrix over $\mathbb{R}$? What is the Jordan normal form of A as a 4×4 matrix over $\mathbb{C}$?

Ex. 5.11. Suppose that for a 20×20 matrix A the rank of A^i for $i = 0, 1, \ldots 9$ is given by the sequence $20, 15, 11, 7, 5, 3, 1, 0, 0, 0$. What sizes are the Jordan-blocks in the Jordan normal form of A?

Linear algebra 2: exercises for Section 6

Ex. 6.1. Define $\phi_i\colon \mathbb{R}^n \to \mathbb{R}$ by $\phi_i(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_i$ for $i = 1, 2, \ldots n$. Show that $\phi_1, \ldots, \phi_n$ is a basis of $(\mathbb{R}^n)^*$, and compute its dual basis of $\mathbb{R}^n$.

Ex. 6.2. Let V be an n-dimensional vector space, let $v_1, \ldots, v_n \in V$ and let $\phi_1, \ldots, \phi_n \in V^*$. Show that $\det((\phi_i(v_j))_{i,j})$ is non-zero if and only if $v_1, \ldots, v_n$ is a basis of V and $\phi_1, \ldots, \phi_n$ is a basis of V^*.

Ex. 6.3. Let V be the 3-dimensional vector space of polynomial functions $\mathbb{R} \to \mathbb{R}$ of degree at most 2. In each of the following cases, we define $\phi_i \in V^*$ for $i = 0, 1, 2$. In each case, indicate whether ϕ_0, ϕ_1, ϕ_2 is a basis of V^*, and if so, give the dual basis of V.

1. $\phi_i(f) = f(i)$
2. $\phi_i(f) = f^{(i)}(0)$, i.e., the ith derivative of f evaluated at 0.
3. $\phi_i(f) = f^{(i)}(1)$
4. $\phi_i(f) = \int_{-1}^{i} f(x)dx$

Ex. 6.4. For each positive integer n show that there are constants $a_1, a_2, \ldots, a_n$ so that

$$\int_0^1 f(x)e^x dx = \sum_{i=1}^{n} a_i f(i)$$

for all polynomial functions $f\colon \mathbb{R} \to \mathbb{R}$ of degree less than n.

Ex. 6.5. Suppose V is a finite dimensional vector space and W is a subspace. Let $f\colon V \to V$ be a linear map so that $f(w) = w$ for $w \in W$. Show that $f^T(v^*) - v^* \in W^o$ for all $v^* \in V^*$.

Conversely, if you assume that $f^T(v^*) - v^* \in W^o$ for all $v^* \in V^*$, can you show that $f(w) = w$ for $w \in W$?

* **Ex. 6.6.** Let V be a finite-dimensional vector space and let $U \subset V$ and $W \subset V^*$ be subspaces. We identify V and V^{**} via α_V (so $W^\circ \subset V$). Show that

$$\dim(U^\circ \cap W) + \dim U = \dim(U \cap W^\circ) + \dim W\,.$$

Ex. 6.7. Let $\phi_1, \ldots, \phi_n \in (\mathbb{R}^n)^*$. Prove that the solution set C of the linear inequalities $\phi_1(x) \geq 0, \ldots, \phi_n(x) \geq 0$ has the following properties:

1. $\alpha, \beta \in C \implies \alpha + \beta \in C$.
2. $\alpha \in C,\ t \in \mathbb{R}_{\geq 0} \implies t\alpha \in C$.
3. If $\phi_1, \ldots, \phi_n$ form a basis of $(\mathbb{R}^n)^*$, then

   $$C = \{t_1\alpha_1 + \ldots + t_n\alpha_n : t_i \in \mathbb{R}_{\geq 0}, \forall i \in \{1, \ldots, n\}\}\,,$$

   where $\alpha_1, \ldots, \alpha_n$ is the basis of $\mathbb{R}^n$ dual to $\phi_1, \ldots, \phi_n$.