Linear algebra 2: exercises for Section 9

Ex. 9.1. For which values of $\alpha \in \mathbb{C}$ is the matrix $\begin{pmatrix} \alpha & \frac{1}{2} \\ \frac{1}{2} & \alpha \end{pmatrix}$ unitary?

Ex. 9.2. Let V be the vector space of continuous complex-valued functions defined on the interval $[0,1]$, with the inner product $\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\, dx$. Show that the set $\{x \mapsto e^{2\pi i k x} : k \in \mathbb{Z}\} \subset V$ is orthonormal. Is it a basis of V?

Ex. 9.3. Give an orthonormal basis for the 2-dimensional complex subspace V_3 of $\mathbb{C}^3$ given by the equation $x_1 - ix_2 + ix_3 = 0$.

Ex. 9.4. Let A be an orthogonal $n \times n$ matrix with entries in $\mathbb{R}$. Show that $\det A = \pm 1$. If A is be an orthogonal 2×2 matrix with entries in $\mathbb{R}$ and $\det A = 1$, show that A is a rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in \mathbb{R}$.

Ex. 9.5. For the real vector space V of polynomial functions $[-1,1] \to \mathbb{R}$ with inner product given by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx,$$

apply the Gram-Schmidt procedure to the elements $1, x, x^2, x^3$.

Ex. 9.6. For the real vector space V of continuous functions $[-\pi, \pi] \to \mathbb{R}$ with inner product given by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-1}^{1} f(x)g(x)dx$$

show that for any $n \geq 0$ the functions

$$1/\sqrt{2}, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin nx, \cos nx$$

form an orthonormal set.