Opgaven lineaire algebra, vrijdag 9 oktober, 2009

(1) Opgaven van vorige week

(2) Let V be a vector space with $\dim V = n$, and take $v_1, \ldots, v_n \in V$. Prove that the following statements are equivalent.
  (i) $\{v_1, \ldots, v_n\}$ is a basis of V
  (ii) $v_1, \ldots, v_n$ are linearly independent
  (iii) $L(v_1, \ldots, v_n) = V$

(3) Let V be a vector space and $v_1, \ldots, v_n \in V$. Show that $\dim L(v_1, \ldots, v_n) \leq n$.

(4) Let V be a real vector space, and $a, b, c, d \in V$. Show that the following vectors are linearly dependent:

$$\begin{aligned} v_1 &= 2a + 9d \\ v_2 &= 5c \\ v_3 &= a + b + c + d \\ v_4 &= a + 2b + 3c + 4d \\ v_5 &= a \end{aligned}$$

Hint. There is a very short solution.

(5) Give a basis for each of the following $\mathbb{R}$-vector spaces.

$U_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0\}$

$U_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + 2x_2 + 3x_3 = 0,\ x_1 + x_2 - x_4 = 0\}$

Use the definition of 'basis' from the lecture to show your answer is correct.

(6) Let $P_n(\mathbb{Q})$ denote the vector space of polynomials with coefficients in $\mathbb{Q}$ of degree at most n. For each $k \in \mathbb{Z}_{\geq 0}$, define

$$\binom{x}{k} = \frac{1}{k!} x(x-1)(x-2)\cdots(x-k+1).$$

Show that

$$\left(\binom{x}{0}, \binom{x}{1}, \binom{x}{2}, \ldots \binom{x}{n-1}, \binom{x}{n}\right)$$

is a basis for $P_n(\mathbb{Q})$.

(7) Show that the vectors

$$v_1 = (1, 2, 3, 4), \quad v_2 = (1, 1, 1, 1), \text{ and } v_3 = (1, 1, 0, -1)$$

in $\mathbb{Q}^4$ are linearly independent and extend (v_1, v_2, v_3) to a basis for $\mathbb{Q}^4$.