Opgaven lineaire algebra, vrijdag 2 oktober, 2009

(zorg dat je elk onderwerp oefent, ook als dat betekent dat je sommige opgaven tot later bewaart)

(1) Which of the following are linear subspaces of the vector space $\mathbb{R}^2$? Explain your answers!
  (a) $U_1 = \{(x,y) \in \mathbb{R}^2 : y = -\sqrt{e^\pi}x\}$
  (b) $U_2 = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$
  (c) $U_3 = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$

(2) Which of the following are linear subspaces of the vector space V of all functions from $\mathbb{R}$ to $\mathbb{R}$?
  (a) $U_1 = \{f \in V \ : \ f \text{ is continuous}\}$
  (b) $U_2 = \{f \in V \ : \ f(3) = 0\}$
  (c) $U_3 = \{f \in V \ : \ f \text{ is continuous or } f(3) = 0\}$
  (d) $U_4 = \{f \in V \ : \ f \text{ is continuous and } f(3) = 0\}$
  (e) $U_5 = \{f \in V \ : \ f(0) = 3\}$
  (f) $U_6 = \{f \in V \ : \ f(0) \geq 0\}$

(3) Given a vector space V with subsets I and J of V, does the equality
$$L(I \cap J) = L(I) \cap L(J)$$
hold?

(4) Which of the following sequences of vectors in $\mathbb{R}^3$ are linear independent?
  (a) $\big((1,2,3),(2,1,-1),(-1,1,1)\big)$
  (b) $\big((1,3,2),(1,1,1),(-1,3,1)\big)$

(5) Let F be a field, $n > 0$ an integer, and set $V = F^n$. For $i \in \{1,\ldots,n\}$, let $e_i = (0,\ldots,0,1,0,\ldots,0)$ be the vector with all zeroes, except for a 1 at the i-th position. Show that $(e_1, e_2, \ldots, e_n)$ is a basis for F^n.

(6) For any positive integer $n > 0$, let P_n be the vector space of polynomials in x over the field F of degree at most n; show that $(1, x, x^2, \ldots, x^n)$ a basis is for P_n. Show that $(1, x-1, (x-1)^2, \ldots, (x-1)^n)$ also basis is (Hint: consider the degree).

(7) Give a basis for each of the following $\mathbb{R}$-vector spaces.

$U_1 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0\}$

$U_2 = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1 + 2x_2 + 3x_3 = 0,\ x_1 + x_2 - x_4 = 0\}$

Use the definition of 'basis' from the lecture to show your answer is correct.

(8) Let U_1 and U_2 be two linear subspaces of a vector space V.

Show that $U_1 \cup U_2$ is a linear subspace of V if and only if $U_1 \subset U_2$ or $U_2 \subset U_1$.

(9) Give examples of a vector space V and linear subspaces $U_1, U_2, U_3 \subset V$ that show that in general
  (a) $(U_1 \cap U_2) + U_3 \neq (U_1 + U_3) \cap (U_2 + U_3)$
  (b) $(U_1 + U_2) \cap U_3 \neq (U_1 \cap U_3) + (U_2 \cap U_3)$