1. Material covered

(Quick sketch)

On September 20, we went through some results in the first two chapters of Silverman's book. We will continue with the same next week.

We defined affine curves over a field k (set of $\overline{k}$-points in $\overline{k}^2$ that are the zeros of a polynomial $f \in k[x, y]$ that is irreducible over $\overline{k}$), and also saw a bad way of defining it (just taking the set of k-points).

We defined the coordinate ring $A[C]$ and the function field $\kappa(C)$ of an affine curve C.

For any field extension l of k, we defined the set of l-points on C as $\mathrm{Hom}_k(A[C], l)$, which is in bijection with $C \cap l^2$ if l is contained in $\overline{k}$.

We defined the maximal ideal and the local ring associated to a point P on C. We talked about valuation rings and discrete valuation rings, and stated the fact that the local ring associated to a smooth point is a discrete valuation ring.

We did several examples, including the computation of the order of a function at a point on a curve.

A projective curve is the set of zeros of some homogeneous polynomial $F \in k[X, Y, Z]$ that is irreducible over $\overline{k}$. Every point is contained in some standard affine part, through which we can define the local ring and maximal ideal at P (and thus order).

2. Homework

Choose four exercises from: Silverman's "The arithmetic of elliptic curves", exercises 1.1, 1.2, 1.3, 1.4, 2.1, and the exercises 15 and 16 below. Of the first two exercises (1.1 and 1.2) you can choose only one.

For exercise 2.1, the definition of discrete valuation ring to use is the following.

Definition. A noetherian local domain R with field of fractions K is a *discrete valuation ring* if there exists a nonzero group homomorphism $v\colon K^* \to \mathbb{Z}$ with $v(x + y) \geq \min(v(x), v(y))$ for all $x, y \in K^*$, such that R equals

$$R = \{0\} \cup \{x \in K^* \ : \ v(x) \geq 0\}.$$

You might want to read about Nakayama's Lemma for this problem 2.1.

Exercise 15 In each case below, there is some affine curve C over a field k and an element f of its function field. In each case, determine all $\overline{k}$-points P on the projective closure of the curve C for which the function f has order $v_P(f)$ not equal to 0, and determine for all these points a generator of the maximal ideal in the local ring at P, as well as the order $v_P(f)$ of f at P. Don't forget the points at infinity.

(1) The curve C over $\mathbb{Q}$ given by $x^2 + y^2 = 1$ with $f = x + 1$.
(2) The curve C over $\mathbb{Q}$ given by $y^2 = x^3 + x$ with $f = x/y$.
(3) The curve C over $\mathbb{Q}$ given by $x^3 + y^3 = 1$ with $f = 1/(x + y)$.
(4) The curve C over $\mathbb{Q}$ given by $x^3 + y^3 = 1$ with $f = x/(x + y)$.

Exercise 16 Let C be an affine curve over a field k, with coordinate ring $A[C]$. We have seen that if a k-point P corresponds to the k-algebra homomorphism $\phi\colon A[C] \to k$, then the kernel of ϕ is a maximal ideal of $A[C]$.

(1) Give an example of a curve C over a field k, and a maximal ideal $\mathfrak{m}$ of $A[C]$ that does not arise like this from a k-point of C. What is the degree of $A[C]/\mathfrak{m}$ over k in your example? Can you find an example where the localization $A[C]_{\mathfrak{m}}$ is a discrete valuation ring? Can you find an example where it isn't?
(2) Give an example of a curve C over a field k and a field extension l of k, as well as an l-point P of C, such that the kernel $\mathfrak{n}$ of the corresponding k-algebra homomorphism $\phi\colon A[C] \to l$ is not maximal. Describe in your case the local ring $A[C]_{\mathfrak{n}}$.