EXAM MATHEMATICAL METHODS OF PHYSICS.

TRACK ANALYSIS (Chapters I-V). Thursday, June 7th, 10.00-13.00.

Students who are entitled to a lighter version of the exam may skip problems 1, 8-11 and 16.

— Consider the differential equation

$$x^2y''(x) + \lambda y(x) = 0 \tag{1}$$

with $\lambda \in \mathbf{C}$.

1. Give the singular points of the differential equation in $\mathbf{C} \cup \{\infty\}$ and determine whether they are regular or irregular. Distinguish between $\lambda = 0$ and $\lambda \neq 0$. (6p)

2. Write the differential equation in self-adjoint form. (2p)

(1) has a solution $y_0(x) = x^\alpha$ for some $\alpha \in \mathbf{C}$.

3. Give a solution of (1) that is linearly independent with y_0. (6p)

Consider the Sturm-Liouville eigenvalue problem on $[1, e]$

$$x^2y''(x) + \lambda y(x) = 0, \qquad y(1) = y(e) = 0. \tag{2}$$

4. Give the eigenvalues and eigenfunctions of (2). Show explicitly that all eigenvalues $\lambda = \lambda_1, \lambda_2, \ldots$ are real numbers and that $\lambda_n \to \infty$ as $n \to \infty$. (Hint: remember that α is a complex number.) (8p)

5. State explicitly the orthogonality relation for the eigenfunctions. (2p)

— On the (complex) Hilbert space $H = L_2(0,1)$ the self-adjoint operator $\mathbf{K} : H \to H$ is given by $\mathbf{K}(f)(x) = \int_0^1 K(x,t)f(t)dt$ where $K(x,t) = 2xt - x - t$.

6. Show (by a calculation) that $\mathbf{K}$ is a bounded operator and give an upper bound for the norm of $\mathbf{K}$. (7p)

7. The kernel $K(x,t)$ is separable. Show this and explain how this implies that $\mathbf{K}$ is a compact operator. (3p)

8. Determine the spectrum $\sigma(\mathbf{K})$ of $\mathbf{K}$ and find the eigenspaces. (10p)

Consider the Fredholm integral equation

$$f(x) = x + \lambda \int_0^1 (2xt - x - t)f(t)dt \qquad \text{for } x \in [0,1]. \tag{3}$$

For small values of $|\lambda|$ the solution of (3) can be developed into a power series in the parameter λ, the Neumann series.

9. Determine the first three terms (up to order λ^2) of the Neumann series. (6p)

10. Determine the radius of convergence of the Neumann series. (2p)

11. For what values of λ does the equation (3) have a solution? Explain your answer. (6p)

— Consider the first-order partial differential equation

$$u_y + y u_x = 0, \qquad (x, y \in \mathbf{R}) \tag{4}$$

where $u : \mathbf{R}^2 \to \mathbf{R}$ is a function of x and y.

12. Give the characteristics of (4). (5p)

13. Solve (4) with the boundary equation $u(x, 0) = \phi(x)$ for some differentiable function $\phi : \mathbf{R} \to \mathbf{R}$. (5p)

14. If we replace the boundary condition given in (13) by the boundary condition $u(0, y) = \phi(y)$ for $y \in \mathbf{R}$, the problem is not well-posed. Explain what goes wrong. (3p)

— In this problem we consider the Helmholtz operator $\Delta + k^2$ in $\mathbf{R}^3$. Green's function $G(\mathbf{x}, \mathbf{y})$ for $\mathbf{R}^3$ is given by $G(\mathbf{x}, \mathbf{y}) = -\dfrac{e^{ik\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|}$, i.e. $(\Delta + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$.

Consider the following boundary value problem on the upper half space $H = \{\mathbf{x} = (x_1, x_2, x_3) : x_3 > 0\}$

$$(\Delta + k^2)u(\mathbf{x}) = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in H, \qquad u(x_1, x_2, 0) = 0 \tag{5}$$

where $f(\mathbf{x})$ is a continuous function on the closure $\{\mathbf{x} \in \mathbf{R}^3 : x_3 \geq 0\}$ of H which tends to zero sufficiently fast as $\|\mathbf{x}\| \to \infty$.

15. Show that a solution of (5) is given by $u(\mathbf{x}) = \int_H G_H(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d^3\mathbf{y}$ where $G_H(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}^*)$ for $\mathbf{x} \in \mathbf{R}^3$, $\mathbf{y} \in H$ and where $\mathbf{y}^*$ is the image of reflection of $\mathbf{y}$ in the plane $x_3 = 0$. (6p)

Now consider the following Dirichlet problem on H

$$(\Delta + k^2)u(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in H, \qquad u(x_1, x_2, 0) = g(x_1, x_2) \tag{6}$$

where g is a continuous function that goes to zero sufficiently fast as $x_1^2 + x_2^2 \to \infty$.

16. Use the function G_H (defined in problem 15) and a form of Green's identity to derive an integral expression for a solution of (6). (You do not have to work out expressions involving the function G_H.) (6p)

Finally, consider the eigenvalue problem on the interior of the unit sphere $S = \{\mathbf{x} \in \mathbf{R}^3 : \|\mathbf{x}\| < 1\}$ in $\mathbf{R}^3$:

$$(\Delta + k^2)u(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in S, \qquad u(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \partial S \tag{7}$$

where $\partial S = \{\mathbf{x} \in \mathbf{R}^3 : \|\mathbf{x}\| = 1\}$ is the boundary of S.

17. Determine the smallest value of k^2 such that the boundary value problem (7) has a non-trivial solution that is a function of only the radius $r = \|\mathbf{x}\|$. You may use that the Laplacian in spherical coordinates r, θ, ϕ is given by

$$\Delta u = \frac{1}{r}(ru)_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}.$$

(7p)

EXAM MATHEMATICAL METHODS OF PHYSICS.

COMPLETE COURSE (Chapters I-IX). Thursday, June 7th, 10.00-13.00.

Students who are entitled to a lighter version of the exam may skip problems 4, 5, 13 and 15-19.

— Consider the partial differential equation

$$u_y + yu_x = 0, \qquad (x, y \in \mathbf{R}) \tag{1}$$

where $u : \mathbf{R}^2 \to \mathbf{R}$ is a function of x and y.

1. Give the characteristics of (1). (5p)

2. Solve (1) with the boundary equation $u(x, 0) = \phi(x)$ for some differentiable function $\phi : \mathbf{R} \to \mathbf{R}$. (5p)

3. If we replace the boundary condition given in problem 2 by the boundary condition $u(0, y) = \phi(y)$ for $y \in \mathbf{R}$, the problem is not well-posed. Explain what goes wrong. (3p)

— In this problem we consider the upper half-plane $H = \{(x_1, x_2) \in \mathbf{R}^2 : x_2 > 0\}$ in two-dimensional Euclidian space $\mathbf{R}^2$, endowed with the metric $g_{ij} = (x_2)^2\delta_{ij}$, i.e. $ds^2 = (x_2)^2 d(x_1)^2 + (x_2)^2 d(x_2)^2$. The Christoffel symbols with respect to this metric are

$$\Gamma^2_{11} = -\frac{1}{x_2},\ \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{x_2},\ \Gamma^2_{22} = \frac{1}{x_2}$$

whereas the other Γ^k_{ij} are zero.

4. Show that the curves $x_1 = c$ (with c constant) are geodesics. (5p)

5. Are the curves $x_2 = c$ geodesics as well? (5p)

6. What is the result of parallel displacement of the vector field $\dfrac{\partial}{\partial x_2}$ from the point $(0, 1)$ to $(0, 2)$ along the geodesic $x_1 = 0$? (6p)

7. Calculate the length of the curve $x_2 = a$ $(a > 0)$ between the points $(0, a)$ and $(1, a)$. (3p)

— A tetrahedron is a regular solid in three dimensions, that is composed of four equilateral triangles. It has 4 vertices. The tetrahedron group $\mathcal{T}$ is defined as the (direct) symmetry group of the tetrahedron, consisting of all rotations that transform the tetrahedron into itself. The elements of $\mathcal{T}$ permute the 4 vertices of the tetrahedron and $\mathcal{T}$ is thus isomorphic to a subgroup of the symmetric group S_4. In fact, $\mathcal{T}$ is isomorphic to the subgroup A_4 of S_4 consisting of the even permutations of $\{1,2,3,4\}$.

8. How many elements does $\mathcal{T}$ have? (1p)

9. Show that the elements (123) and (12)(34) do indeed correspond to rotations of the tetrahedron. (4p)

The group A_4 has four conjugation classes which are represented by the group elements $e = (1)$, (12)(34), (123) and (132).

10. How many irreducible representations does A_4 have and what are their dimensions? (3p)

11. Below a character table for the irreducible representations $T^{(1)}, T^{(2)}, \ldots$ is given which has been partly filled in. Complete the table. (8p)

	(1)	(12)(34)	(123)	(132)
$T^{(1)}$	1	1	1	1
$T^{(2)}$	1	1	$e^{2\pi i/3}$	*
$T^{(3)}$	1	1	*	*
*	*	*	*	*
⋮	⋮	⋮	⋮	⋮

The fundamental representation of A_4 is defined as follows: a permutation $\pi \in A_4$ is represented by the 4×4-matrix $T(\pi) = \begin{pmatrix}\mathbf{e}_{\pi(1)} & \mathbf{e}_{\pi(2)} & \mathbf{e}_{\pi(3)} & \mathbf{e}_{\pi(4)}\end{pmatrix}$ (i.e. the i-th column of the matrix is the $\pi(i)$-th unit vector $\mathbf{e}_{\pi(i)}$).

12. Show that T is indeed a representation of A_4. (4p)

13. Decompose T as a direct sum of irreducible representations of A_4 (hint: look at the character). (8p)

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14. Show (by a calculation) that $\mathbf{K}$ is a bounded operator and give an upper bound for the norm of $\mathbf{K}$. (7p)

15. The kernel $K(x,t)$ is separable. Show this and explain how this implies that $\mathbf{K}$ is a compact operator. (3p)

16. Determine the spectrum $\sigma(\mathbf{K})$ of $\mathbf{K}$ and find the eigenspaces. (10p)

Consider the Fredholm integral equation

$$f(x) = x + \lambda \int_0^1 (2xt - x - t)f(t)dt \qquad \text{for } x \in [0,1]. \tag{2}$$

For small values of $|\lambda|$ the solution of (2) can be developed into a power series in the parameter λ, the Neumann series.

17. Determine the first three terms (up to order λ^2) of the Neumann series. (6p)

18. Determine the radius of convergence of the Neumann series. (2p)

19. For what values of λ does the equation (2) have a solution? Explain your answer. (6p)

— Consider the set A of 2×2-matrices $\{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix};\ x, y \in \mathbf{R},\ x > 0\}$.

20. Show that A is a subgroup of the group $GL(2,\mathbf{R})$ of invertible 2×2-matrices with real coefficients. (4p)

21. A is in fact a Lie group. Explain this. What is its dimension? (4p)

22. Give a basis of the Lie algebra $\mathcal{A}$ of A and give the structure constants of $\mathcal{A}$. (You may use a matrix representation of $\mathcal{A}$.) (5p)

23. Is $\mathcal{A}$ semisimple? Explain your answer. (3p)

24. Let $X \in \mathcal{A}$. Show that $\exp(tX)$ is of the form $\exp(tX) = \begin{pmatrix} e^{at} & b(e^{at} - 1) \\ 0 & 1 \end{pmatrix}$ for $a, b \in \mathbf{R}$ (do not calculate the matrix e^{tX} but use the definition of the exponential map $\exp : \mathcal{A} \to A$). (6p)