ON THE CONVERGENCE OF LOGARITHMIC FIRST RETURN TIMES

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ABSTRACT. Let T be an ergodic transformation on X and $\{\alpha_n\}$ a sequence of partitions on X. Define $K_n(x) = \min\{j \leq 1 : T^j x \in \alpha_n(x)\}$, where $\alpha_n(x)$ is the element of α_n containing x. In this paper we give conditions on T and α_n, for which $\lim_{n\to\infty} \frac{\log K_n(x)}{n}$ exists. We study the question in one and higher dimensions.

1. INTRODUCTION

Let $x \in [0,1)$ and let T be a transformation from $[0,1)$ to $[0,1)$. One can study the movement of x under iterations of T, and if B is a set of non-negative measure containing x, one can address the question of how many iterations of T are needed for the orbit of x to return to the set B for the first time. This number of iterations is called the first return time of x to B under T. The first and most famous result about this type of questions is the Poincaré Recurrence Theorem. It states that if B is a set of positive measure and if T is a transformation that preserves a finite measure, then almost all elements of B return to B eventually. An immediate consequence of this result is that almost all $x \in B$ return to B infinitely often. The Poincaré Recurrence Theorem is powerful, but it doesn't make a statement about the speed with which a point returns to a set it started in.

A great number of articles have been published answering this question in various ways. Usually the first return time of x to a partition element generated by the transformation T itself is studied. Given a partition $\mathcal{P}$ of $[0,1)$ and a transformation T, it is possible to construct a sequence of partitions $\{\mathcal{P}_n\}$ by looking at inverse images of $\mathcal{P}$ under iterations of T and taking intersections. More precisely,

$$\mathcal{P}_n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P} = \{A_0 \cap T^{-1}A_1 \cap \cdots \cap T^{-(n-1)}A_{n-1} : A_i \in \mathcal{P}\}.$$

Using the partition $\mathcal{P}$ and the transformation T, one can construct for almost every $x \in [0,1)$ an expansion. This is done by specifying for each n the partition element of $\mathcal{P}_n$ that the element $T^n x$ belongs to. The expansion that is obtained in this way is called the $(T,\mathcal{P})$-expansion of x. If $\mathcal{P}_n(x)$ denotes the partition element of $\mathcal{P}_n$ that contains x, then $\mathcal{P}_n(x)$ specifies the first n coordinates of the $(T,\mathcal{P})$-expansion of x.

Ornstein and Weiss proved an asymptotic result on the number of iterations of T that are needed before x returns to $\mathcal{P}_n(x)$ for the first time. Let $(X,\mathcal{F},\mu,T)$ be

1991 *Mathematics Subject Classification.* Primary 28D05, 37A35.

Key words and phrases. first return times, Shannon-McMillan-Breiman Theorem, Poincaré recurrence.

a dynamical system and let $E \subset X$. Then the *first return time* of an element x of E to the set E will be denoted by $R_E(x)$ and is given by

$$R_E(x) = \min\{j \geq 1 : T^j x \in E\}.$$

Then the following theorem states the result of Ornstein and Weiss. For the proof, consult [O].

Theorem 1.1. *Let $(X, \mathcal{F}, \mu, T)$ be an ergodic and measure preserving dynamical system with μ a probability measure and suppose $\mathcal{P}$ is a partition on X with $H(\mathcal{P}) < \infty$. Define the sequence of partitions $\{\mathcal{P}_n\}$ by $\mathcal{P}_n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$ for each $n \geq 1$. Then*

$$\lim_{n\to\infty} \frac{\log R_{\mathcal{P}_n(x)}(x)}{n} = h(T, \mathcal{P}) \quad \mu \text{ a.e.}$$

In other words, if we look at the $(T, \mathcal{P})$-expansion of x, the above theorem says that the first n coordinates of this expansion will repeat themselves for the first time after approximately 2^{nh} steps. Others proved similar results for more specific transformations or more general sets. Marton and Shields for example studied the asymptotic convergence of logarithmic waiting times. The *waiting time* of an element y (not necessarily in E) to E under T is denoted by $W_E(y)$ and is given by

$$W_E(y) = \min\{j \geq 1 : T^{j-1} y \in E\}.$$

Marton and Shield showed that also in this case we have to wait $2^{nh(T,\mathcal{P})}$ steps, before we see the first n coordinates again for the first time.

Theorem 1.2. *Let $(X, \mathcal{F}, \mu)$ be a probability space and T a measure preserving and ergodic transformation on X that is weak Bernoulli. Suppose that $\mathcal{P}$ is a partition on X with $H(\mathcal{P}) < \infty$. Let $x, y \in X$ with $\mathcal{P}(x) \neq \mathcal{P}(y)$ and define again the sequence of partitions $\{\mathcal{P}_n\}$ by $\mathcal{P}_n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$ for each $n \geq 1$. Then*

$$\lim_{n\to\infty} \frac{\log W_{\mathcal{P}_n(x)}(y)}{n} = h(T, \mathcal{P}) \quad \mu \text{ a.e.}$$

A proof of this theorem can be found in [M].

Seo studied the first return time of a point under a specific transformation, namely the irrational translation, to a dyadic interval centered at the point itself. The proof can be found in [S]. Recall that an irrational number $\theta \in (0, 1)$ is of type η if $\eta = \sup\{\beta : \liminf_{j\to\infty} j^\beta ||j\theta|| = 0\}$, where $||\theta||$ equals the distance to the nearest integer.

Theorem 1.3. *Let $\theta \in (0, 1)$ be an irrational number. Let furthermore $(I, \mathcal{B}, \lambda, T)$ be a dynamical system, with $I = [0, 1)$, $\mathcal{B}$ the Borel σ-algebra on I, λ the Lebesgue measure on $(I, \mathcal{B})$ and T the transformation on I defined by $Tx = x + \theta$ (mod 1). Then θ is of type 1 if and only if*

$$\lim_{n\to\infty} \frac{\log R_{B(x,2^{-n})}(x)}{n} = 1,$$

where $B(x, 2^{-n}) = \{y \in [0, 1) : |y - x| < 2^{-n}$.

In the proof of this theorem it is shown that

$$\limsup_{n\to\infty} \frac{\log R_{B(x,2^{-n})}(x)}{n} = 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{\log R_{B(x,2^{-n})}(x)}{n} = \frac{1}{\eta},$$

where η denotes the type of the irrational number θ.

A more general version of the theorem above is given by Barreira and Saussol. Let $X \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N}$. For each $x \in X$ and $r, \epsilon > 0$ we can define the set

$$A_\epsilon(x,r) = \{y \in B(x,r) : R_{B(x,r)}(y) \leq \mu(B(x,r))^{-1+\epsilon}\},$$

where $B(x,r)$ is the ball in $\mathbb{R}^d$ with centre x and radius r. A measure μ is said to have *long return time* with respect to a transformation T if

$$\liminf_{r\to 0} \frac{\log \mu(A_\epsilon(x,r))}{\log \mu(B(x,r))} > 1.$$

Barreira and Saussol proved the next theorem. The proof as well as examples of transformations having the above property can be found in [B].

Theorem 1.4. *Let $T : X \to X$ be a Borel measurable transformation on a measurable set $X \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N}$ and μ a T-invariant probability measure on X. If μ has long return time with respect to T and if*

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} > 0$$

for μ-almost every $x \in X$, then

$$\liminf_{r\to 0} \frac{\log R_{B(x,r)}(x)}{-\log r} = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

and

$$\limsup_{r\to 0} \frac{\log R_{B(x,r)}(x)}{-\log r} = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

for μ-almost every $x \in X$.

In this paper we will not be looking at return times to sets generated by the transformation itself, but at return times to sets generated by another transformation. Let S be a transformation and α a partition, together generating the sequence of partitions $\{\alpha_n\}$, where $\alpha_n = \bigvee_{i=0}^{n-1} S^{-i}\alpha$. We can then look at the (S,α)-expansion of an element x and if $\alpha_n(x)$ denotes the partition element of α_n containing x, then this partition element specifies the first n coordinates of this expansion. If T is another transformation defined on the same space and if we define the value $K_n(x)$ by

$$K_n(x) = \min\{j \geq 1 : T^j x \in \alpha_n(x)\},$$

then $K_n(x)$ denotes the first return time of x under T to the partition element of α_n it started in. In other words, $K_n(x)$ is the first j, such that the first n coordinates of the (S,α)-expansion of $T^j x$ equal those of the (S,α)-expansion of x itself. An important difference of this setup with respect to the results mentioned above, is that the partitions under consideration are independent of the transformation T and are in fact generated by the other transformation S. We will show that also the logarithm of the quantity $K_n(x)$ converges asymptotically, but surprisingly this limit doesn't depend on the transformation T at all. We will prove that for almost all elements x,

$$\lim_{n\to\infty} \frac{\log K_n(x)}{n} = h,$$

where h is the entropy of S relative to the partition α. This means that the convergence of this value depends only on the randomness of S and not on that of T. Thus, the same result holds for all transformations T satisfying the right conditions. This generalizes a theorem of Kim and Kim [K], where they proved the result for the case $\{\alpha_n\}$ is the collection of dyadic intervals of order n.

We will begin by stating and proving the one-dimensional version of the result mentioned above. In the last section we will give generalizations of this result to higher dimensions by using the Theorem of Barreira and Saussol. This is done for specific choices of S and α.

Throughout this text, if α is a partition of a non-empty set X and x is an element of X, then $\alpha(x)$ will denote the partition element of α containing x.

2. One-dimensional Log Return Times

In this section we will study the asymptotic behaviour of the log return time for certain one-dimensional partitions defined on the unit interval $I = [0, 1)$. Therefore, consider the dynamical systems $(I, \mathcal{B}, \mu_1, T)$ and $(I, \mathcal{B}, \mu_2, S)$, where I is the unit interval $[0, 1)$, $\mathcal{B}$ is the Borel σ-algebra on I, and μ_1 and μ_2 are probability measures on $(I, \mathcal{B})$ with $\mu_1 \ll \mu_2$. Suppose T and S are ergodic transformations from I to itself and μ_1-, respectively μ_2-invariant. Let α be an interval partition of I, with $H(\alpha) < \infty$ and such that each partition $\alpha_n = \bigvee_{i=0}^{n-1} S^{-i}\alpha$ is again an interval partition of I. Throughout this section, we let $h = h_{\mu_2}(S, \alpha)$ indicate the entropy of S with respect to α, and we suppose $h > 0$. Define for every $x \in [0, 1)$ the first return time of x under T to its cylinder set $\alpha_n(x)$ by

$$K_n(x) = \min\{j \geq 1 : T^j x \in \alpha_n(x)\}.$$

Theorem 2.1. *Let $\mathcal{P}$ be a finite partition of I, consisting of intervals. Define the sequence of partitions $\{\mathcal{P}_n\}_{n\geq 1}$ by $\mathcal{P}_n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$ and suppose that for each $n \geq 1$ the partition $\mathcal{P}_n$ also consists of intervals only. Suppose furthermore that $H(\mathcal{P}) < \infty$ and $h_{\mu_1}(T, \mathcal{P}) > 0$. Then*

$$\lim_{n\to\infty} \frac{\log K_n(x)}{n} = h \quad \mu_1 \text{ a.e.}$$

Remark. Notice that to guarantee that each of the partitions $\mathcal{P}_n$ is an interval partition, it is enough that the transformation T is monotone on each of the elements of the partition $\mathcal{P}$.

Proof. Let $h^* = h_{\mu_1}(T, \mathcal{P})$, and let $0 < \epsilon < h$ be given. Call an element A of $\mathcal{P}_n$ (n, ϵ)-typical if $2^{-n(h^*+\epsilon)} < \mu_1(A) < 2^{-n(h^*-\epsilon)}$, and a similar definition for members of α_n. Let $l_n^{(\mathcal{P})}$ be the number of (n, ϵ)-typical sets for the partition $\mathcal{P}_n$, and $l_n^{(\alpha)}$ be the number of (n, ϵ)-typical sets for the partition α_n. By the Shannon-McMillan-Breiman Theorem, $\mathcal{P}_n(x)$ and $\alpha_n(x)$ are (n, ϵ)-typical for all n sufficiently large, and for almost every x,. Furthermore, there exist a positive integer N such that $l_n^{(\mathcal{P})}(\epsilon) < 2^{n(h^*+\epsilon)}$ and $l_n^{(\alpha)}(\epsilon) < 2^{n(h+\epsilon)}$ for all $n > N$. For each $n > N$, define the integer $m = m(n) = \lfloor \frac{(h-\epsilon)n}{h^*+2\epsilon} \rfloor$ and let

$$D_n(\epsilon) = \left\{ x \in I : \begin{array}{l} 2^{-m(h^*+\epsilon)} < \mu_1(\mathcal{P}_m(x)) < 2^{-m(h^*-\epsilon)} \\ 2^{-n(h+\epsilon)} < \mu_2(\alpha_n(x)) < 2^{-n(h-\epsilon)} \\ \alpha_n(x) \not\subseteq \mathcal{P}_m(x) \end{array} \right\}.$$

If $x \in D_n(\epsilon)$, then $\alpha_n(x)$ will intersect at least two elements of the partition $\mathcal{P}_m$ and the elements with which it intersects include at least one (m, ϵ)-typical element of $\mathcal{P}_m$, namely $\mathcal{P}_m(x)$. Since both partitions $\mathcal{P}_m$ and α_n consist of intervals only, the measure of $D_n(\epsilon)$ can be estimated by

$$\begin{aligned}\mu_2(D_n(\epsilon)) &\leq 2\cdot \text{ the number of } (m,\epsilon)\text{-typical elements of } \mathcal{P}_m \\ &\quad \cdot\text{the maximal } \mu_2\text{-measure of an } (n,\epsilon)\text{-typical element of } \alpha_n \\ &\leq 2\cdot 2^{m(h^*+\epsilon)}\cdot 2^{-n(h-\epsilon)} \leq 2\cdot 2^{-n((h-\epsilon)-(h-\epsilon)\frac{h^*+\epsilon}{h^*+2\epsilon})}.\end{aligned}$$

By the Borel-Cantelli Lemma this means that $\mu_2(D_n(\epsilon)\text{ i.o.}) = 0$ and the Shannon-McMillan-Breiman Theorem gives that $\mu_2(\{x \in I : \alpha_n(x) \not\subseteq \mathcal{P}_m(x)\text{ i.o.}\}) = 0$. Since $\mu_1 \ll \mu_2$, it follows that $\mu_1(\{x \in I : \alpha_n(x) \not\subseteq \mathcal{P}_m(x)\text{ i.o.}\}) = 0$, so for almost all $x \in I$, for n big enough, $\alpha_n(x) \subseteq \mathcal{P}_m(x)$. Therefore $K_n(x) \geq R_{\mathcal{P}_m(x)}(x)$ and by Theorem 1.1 then

$$\begin{aligned}\liminf_{n\to\infty}\frac{\log K_n(x)}{n} &\geq \liminf_{n\to\infty}\frac{\log R_{\mathcal{P}_m(x)}(x)}{m}\cdot\frac{m}{n} \\ &= (h-\epsilon)\frac{h^*}{h^*+2\epsilon}\quad \mu_1 \text{ a.e.}\end{aligned}$$

Since this holds for all ϵ sufficiently small, we have that $\liminf_{n\to\infty}\frac{\log K_n(x)}{n} \geq h$ μ_1 a.e.

The fact that $\limsup_{n\to\infty}\frac{\log K_n(x)}{n} \leq h$ for μ_1-almost every x can be proven in a similar way by taking $\epsilon < h^*$, replacing the set $D_n(\epsilon)$ by

$$D'_n(\epsilon) = \left\{ x \in I : \begin{array}{l} 2^{-m(h^*+\epsilon)} < \mu_1(\mathcal{P}_m(x)) < 2^{-m(h^*-\epsilon)} \\ 2^{-n(h+\epsilon)} < \mu_2(\alpha_n(x)) < 2^{-n(h-\epsilon)} \\ \mathcal{P}_m(x) \not\subseteq \alpha_n(x) \end{array} \right\},$$

and taking $m = m(n) = \lceil\frac{(h+2\epsilon)n}{h^*-\epsilon}\rceil$. Notice that in this case we can immedeately give an estimate of the μ_1-measure of $D'_n(\epsilon)$, so that we do not have to impose that $\mu_2 \ll \mu_1$. □

Now consider the number

$$K_n(x,y) = \min\{j \geq 1 : T^{j-1}y \in \alpha_n(x)\},$$

which can be interpreted as the time we have to wait until an element $y \in X$ enters for the first time the partition element of α_n in which x lies. We are going to prove the equivalence of Theorem 2.1 for this waiting time.

Theorem 2.2. *Let $\mathcal{P}$ be a finite partition of I, consisting of intervals and such that each of the partitions $\mathcal{P}_n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$ is again an interval partition. Suppose furthermore that $H(\mathcal{P}) < \infty$ and $h_{\mu_1}(T,\mathcal{P}) > 0$. If T is a measure preserving, ergodic, weakly Bernoulli transformation, then*

$$\lim_{n\to\infty}\frac{\log K_n(x,y)}{n} = h \quad (\mu_1\times\mu_1) \text{ a.e.}$$

Proof. Fix $y \in I$, and let $\epsilon > 0$. Define

$$A_n = \left\{ x \in I : \begin{array}{l} 2^{-n(h+\epsilon)} < \mu_2(\alpha_n(x)) < 2^{-n(h-\epsilon)} \\ \\ K_n(x,y) < 2^{n(h-2\epsilon)} \end{array} \right\}.$$

Then A_n is the union of those (n,ϵ)-typical elements of α_n, that contain $T^{j-1}y$ for some $j < 2^{n(h-2\epsilon)}$. Since this number cannot exceed $2^{n(h-2\epsilon)}$, we have

$$\mu_2(A_n) < 2^{n(h-2\epsilon)}\cdot 2^{-n(h-\epsilon)} = 2^{-n\epsilon}.$$

Then by the Borel-Cantelli Lemma, the Shannon-McMillan-Breiman Theorem and the fact that $\mu_1 \ll \mu_2$, we have, as in the proof of Theorem 2.1, that

$$\liminf_{n\to\infty} \frac{\log K_n(x,y)}{n} \geq \liminf_{n\to\infty} \frac{\log 2^{n(h-2\epsilon)}}{n} = h - 2\epsilon \quad (\mu_1 \times \mu_1) \text{ a.e.}$$

To prove that $\limsup_{n\to\infty} \frac{\log K_n(x,y)}{n} \leq h$ for $(\mu_1 \times \mu_1)$-almost every (x, y), we let $h^* = h_{\mu_1}(T, \mathcal{P})$, where $0 < \epsilon^* < h^*$ is given. For each n define the number $m = m(n) = \lceil \frac{(h+2\epsilon^*)n}{h^*-\epsilon^*} \rceil$. As in the proof of Theorem 2.1 we can show that for n big enough, we have for almost all $x \in I$ that $\mathcal{P}_m(x) \subseteq \alpha_n(x)$. This means that then $K_n(x,y) \leq W_m(x,y)$ and therefore by Theorem 1.2

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log K_n(x,y)}{n} &\leq \limsup_{n\to\infty} \frac{\log W_m(x,y)}{m} \cdot \frac{m}{n} \\ &= (h + 2\epsilon^*)\tfrac{h^*}{h^*-\epsilon^*} \qquad (\mu_1 \times \mu_1) \text{ a.e.} \end{aligned}$$

Since this holds for all $0 < \epsilon^* < h^*$, we have the result. □

In the previous theorems, the condition that $h(T, \alpha) > 0$ was used in the proof. We will now show that this condition is indeed necessary by showing that for certain irrational rotations $\lim\limits_{n\to\infty} K_n(x)/n$ does not exist. To this end, consider the probability space $(I, \mathcal{B}, \lambda)$ and let the transformation $T_\theta : I \to I$ be defined by $T_\theta x = x + \theta \pmod 1$ with $\theta \in (0, 1)$ an irrational number. It is well known that $h_\lambda(T, \mathcal{P}) = 0$ for any partition $\mathcal{P}$. Suppose α is a generating partition of S with $H(\alpha) < \infty$ and $h = h(S, \alpha) > 0$. Define for each $x \in I$,

$$K_n^\theta(x) = \min\{j \geq 1 : T_\theta^j x \in \alpha_n(x)\}.$$

We have the following theorem.

Theorem 2.3. *Let $\theta \in (0, 1)$ be an irrational number of type η. Then, for almost all $x \in I$*

$$\liminf_{n\to\infty} \frac{\log K_n^\theta(x)}{n} = \frac{h}{\eta} \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\log K_n^\theta(x)}{n} = h \textit{ a.e.}$$

Proof. Let $0 < \epsilon < h$ be given. Then by the Shannon-McMillan-Breiman Theorem, we know that for almost all x, there exists an N such that for all $n \geq N$ we have

$$2^{-n(h+\epsilon)} < \lambda(\alpha_n(x)) < 2^{-n(h-\epsilon)}.$$

Therefore, for these x's we have that

$$\alpha_n(x) \subseteq B(x, 2^{-n(h-\epsilon)}),$$

so $K_n(x) \geq R_{B(x,2^{-n(h-\epsilon)})}(x)$. On the other hand, if we define for each $n \geq N$, the integer $m = m(n) = \lceil n(h + 2\epsilon) \rceil$ and the set

$$E_n(\epsilon) = \left\{ x \in I : \begin{array}{c} 2^{-n(h+\epsilon)} < \lambda(\alpha_n(x)) < 2^{-n(h-\epsilon)} \\ \\ B(x, 2^{-m}) \not\subseteq \alpha_n(x) \end{array} \right\},$$

by an argument similar to the proof of Theorem 2.1 we can see that $\sum_{n=1}^\infty \lambda(E_n(\epsilon)) < \infty$. Thus by the Borel Cantelli Lemma and the Shannon-McMillan-Breiman Theorem $K_n^\theta(x)(x) \leq R_{B(x,2^{-m})}(x)$ for all n sufficiently large. By Theorem 1.3

$$\liminf_{n\to\infty} \frac{\log K_n^\theta(x)}{n} = \frac{h}{\eta} \quad \lambda \text{ a.e.},$$

and

$$\limsup_{n\to\infty} \frac{\log K_n^\theta(x)}{n} = h \text{ a.e.}$$

□

The following corollary follows immediately from the above theorem.

Corollary 2.1. *Let $\theta \in (0,1)$ be an irrational number of type η. If $\eta = 1$, then $\lim_{n\to\infty} \frac{\log K_n^\theta(x)}{n} = h$ a.e., and if $\eta > 1$, then $\lim_{n\to\infty} \frac{\log K_n^\theta(x)}{n}$ does not exist.*

3. Higher Dimensions

Using the Theorem of Barreira and Saussol, some of the results of the previous section can be extended to dimension $d \in \mathbb{N}$ in case the partitions α_n are given by the d-dimensional dyadic hypercubes of order n. To this end, consider the probability space $(I^d, \mathcal{B}^d, \lambda^d)$, where I^d is the d-dimensional unit hypercube, $\mathcal{B}^d$ is the d-dimensional Borel σ-algebra and λ^d is the d-dimensional Lebesgue measure on $(I^d, \mathcal{B}^d)$. Let the partitions Q_n^d of I^d be given by

$$Q_n^d = \{\prod_{j=0}^{d-1} [\frac{i_j}{2^n}, \frac{i_j+1}{2^n}) : 0 \le i_0, \dots i_{d-1} \le 2^n - 1\}.$$

Then for each $\bar{x} \in I^d$,

$$(\text{diameter } Q_n^d(\bar{x})) = (\sum_{i=1}^{d} (2^{-n})^2)^{1/2} = \sqrt{d} \cdot 2^{-n}$$

and $\lambda^d(Q_n^d(\bar{x})) = 2^{-dn}$. Let $Q_n^{(i_1,\dots,i_d)}$ denote the partition element

$$Q_n^{(i_1,\dots,i_d)} = [\frac{i_1}{2^n}, \frac{i_1+1}{2^n}) \times \dots \times [\frac{i_d}{2^n}, \frac{i_d+1}{2^n}).$$

Using the result of Barreira and Saussol, we prove the following theorem.

Theorem 3.1. *Let T be an measure preserving transformation on the probability space $(I^d, \mathcal{B}^d, \mu)$. Assume that μ has long return time with respect to T, and is absolutely continuous with respect to λ^d with density g bounded away from zero and bounded from above. Define for each $\bar{x} \in I^d$,*

$$K_n^d(\bar{x}) = \min\{k \in \mathbb{N} : T^k \bar{x} \in Q_n^d(\bar{x})\},$$

then for μ-almost every $\bar{x} \in I^d$,

$$\lim_{n\to\infty} \frac{\log K_n^d(\bar{x})}{n} = d.$$

Proof. Notice first that $\lim_{r\to 0} \frac{\log \lambda^d(B(\bar{x}, r))}{\log r} = d$. Since $\mu \ll \lambda^d$ with density bounded away from zero and bounded from above, it follows that

$$\lim_{r\to 0} \frac{\log \mu(B(\bar{x}, r))}{\log r} = \lim_{r\to 0} \frac{\log \lambda^d(B(\bar{x}, r))}{\log r} = d.$$

Therefore all the conditions of the Theorem of Barreira and Saussol are satisfied. Since for each $\bar{x} \in I^d$, $Q_n^d(\bar{x}) \subseteq B(\bar{x}, \sqrt{d} \cdot 2^{-n})$, we have $K_n^d(\bar{x}) \geq R_{B(\bar{x},\sqrt{d}\cdot 2^{-n})}(\bar{x})$ and thus

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log K_n^d(\bar{x})}{n} &\geq \liminf_{n\to\infty} \frac{\log R_{B(\bar{x},\sqrt{d}\cdot 2^{-n})}(\bar{x})}{-\log(\sqrt{d}\cdot 2^{-n})} \cdot \frac{-\log(\sqrt{d}\cdot 2^{-n})}{n} \\ &\geq \liminf_{r\to 0} \frac{\log R_{B(\bar{x},r)}(\bar{x})}{-\log r}(1 - r\log\sqrt{d}).\end{aligned}$$

Using Theorem 1.4 then gives

$$\liminf_{n\to\infty} \frac{\log K_n^d(\bar{x})}{n} \geq \liminf_{r\to 0} \frac{\log \mu(B(\bar{x},r))}{\log r} = d$$

for μ-almost all $\bar{x} \in I^d$.

For the second part, using Kac's Lemma we get

$$\int_{I^d} K_n^d(\bar{x}) d\mu(\bar{x}) = \sum_{i_1,\dots,i_d=0}^{2^n-1} \int_{Q_n^{(i_1,\dots,i_d)}} K_n^d(\bar{x}) d\mu(\bar{x}) \leq 2^{dn}.$$

Let $\epsilon > 0$. For each $n \geq 1$, define the set $B_n^d = \{\bar{x} \in I^d : K_n^d(\bar{x}) > 2^{dn(1+\epsilon)}\}$. Then by Markov's Inequality

$$\mu(B_n^d) \leq \frac{1}{2^{dn(1+\epsilon)}} \int_{I^d} K_n^d(\bar{x}) d\mu(\bar{x}) \leq 2^{-dn\epsilon}.$$

By the Borel-Cantelli Lemma, then $\mu(B_n^d \text{ i.o.}) = 0$, so that

$$\limsup_{n\to\infty} \frac{\log K_n^d(\bar{x})}{n} \leq \limsup_{n\to\infty} \frac{\log 2^{dn(1+\epsilon)}}{n} = d + d\epsilon \quad \mu \text{ a.e.} \qquad \square$$

The above theorem can be generalized when the partitions Q_n^d are replaced by d-fold product partitions. $\alpha_n \times \cdots \times \alpha_n$, where α_n is some interval partition of $[0,1)$. For ease of notation, we restrict our attention to dimension 2. Given a rectangle R, we define the frame of R of width δ as the set

$$\mathcal{F}(R,\delta) = \{\bar{x} \in I^2 : \bar{x} \in R \text{ and } d(\bar{x}, \partial R) \leq \delta\},$$

where ∂R is the boundary of R and d indicates the usual Euclidian distance in the plane. Notice that the proportion of R taken up by its frame of width δ is bounded above by

$$\frac{4 \cdot (\text{length of } R) \cdot \delta}{\bar{\lambda}(R)}.$$

Now, consider the probability space $(I, \mathcal{B}, \nu)$. Suppose S is a measure preserving weakly mixing transformation on I, and let α be an interval partition on I with $H_\lambda(\alpha) < \infty$. Construct the sequence of partitions $\{\alpha_n\}$ by setting $\alpha_n = \bigvee_{i=0}^{n-1} S^{-i}\alpha$, and suppose that $h = h(S,\alpha) > 0$. Assume that for each $n \geq 1$ the partition α_n consists of intervals only. Now construct the two-dimensional dynamical system $(I^2, \bar{\mathcal{B}}, \bar{\nu}, S \times S)$ by setting $I^2 = I \times I$, $\bar{\mathcal{B}} = \mathcal{B} \times \mathcal{B}$ and $\bar{\nu} = (\nu \times \nu)$ Then $\bar{h} = h(S \times S, \alpha \times \alpha) = 2h$. We assume throughout that ν is equivalent to λ with density bounded away from 0 and ∞. This allows us to replace ν by λ in the Shannon-McMillan-Breiman theorem applied to the partition α_n. For this reason, we shall assume with no loss of generality that ν is λ.

Let T be a transformation on $(I^2, \bar{\mathcal{B}}, \mu)$, where μ is a T-invariant probability measure, absolutely continuous with respect to $\bar{\lambda}$ with density bounded away from zero and bounded from above. Assume furthermore that T has long return time with respect to μ. Define

$$\bar{K}_n(\bar{x}) = \min\{j \geq 1 : T^j \bar{x} \in (\alpha_n \times \alpha_n)(\bar{x})\}.$$

Then also in this case we have the following theorem.

Theorem 3.2. *For μ-almost every $\bar{x} \in I^2$,*

$$\lim_{n\to\infty} \frac{\log \bar{K}_n(\bar{x})}{n} = \bar{h}.$$

Proof. Again the proof is done by first checking that $\liminf_{n\to\infty} \frac{\log \bar{K}_n(\bar{x})}{n} \geq \bar{h}$ μ a.e. and then that $\limsup_{n\to\infty} \frac{\log \bar{K}_n(\bar{x})}{n} \leq \bar{h}$ for μ-almost all x. For the first part, let $\epsilon > 0$ be given and define for each $n \geq 1$

$$D_n(\epsilon) = \left\{ \bar{x} = (x,y) \in I^2 : \begin{array}{c} 2^{-n(h+\epsilon)} < \lambda(\alpha_n(x)) < 2^{-n(h-\epsilon)} \\ \\ 2^{-n(h+\epsilon)} < \lambda(\alpha_n(y)) < 2^{-n(h-\epsilon)} \end{array} \right\}.$$

Then for all $\bar{x} \in D_n(\epsilon)$, we have that $(\alpha_n(x) \times \alpha_n(y)) \subseteq B(\bar{x}, \sqrt{2}\cdot 2^{-n(h-\epsilon)})$, so $R_{B(\bar{x},\lfloor\sqrt{2}\cdot 2^{-n(h-\epsilon)}\rfloor)}(\bar{x}) \leq \bar{K}_n(\bar{x})$. The Shannon-McMillan-Breiman Theorem tells us that $\bar{\lambda}(D_n(\epsilon)^c) = 0$, so by absolute continuity also $\mu(D_n(\epsilon)^c) = 0$ and then by Theorem 1.4

$$\begin{aligned} \liminf_{n\to\infty} \frac{\log \bar{K}_n(\bar{x})}{n} &\geq \liminf_{n\to\infty} \frac{\log R_{B(\bar{x},\lfloor\sqrt{2}\cdot 2^{-n(h-\epsilon)}\rfloor)}(\bar{x})}{-\log(\lfloor\sqrt{2}\cdot 2^{-n(h-\epsilon)}\rfloor)} \cdot \frac{-\log(\lfloor\sqrt{2}\cdot 2^{-n(h-\epsilon)}\rfloor)}{n} \\ &\geq \bar{h} - 2\epsilon \quad \mu \text{ a.e.} \end{aligned}$$

For the other part, let $\epsilon^* > 0$ again be given and define for each $n \geq 1$ the number $m = m(n) = \lceil n(h + 4\epsilon^*)\rceil$ and the set

$$D_n(\epsilon^*) = \left\{ \bar{x} = (x,y) \in I^2 : \begin{array}{l} 2^{-n(h+\epsilon^*)} < \lambda(\alpha_n(x)) < 2^{-n(h-\epsilon^*)} \\ 2^{-n(h+\epsilon^*)} < \lambda(\alpha_n(y)) < 2^{-n(h-\epsilon^*)} \\ B(\bar{x}, 2^{-m}) \nsubseteq (\alpha_n(x) \times \alpha_n(y)) \end{array} \right\}.$$

If $\bar{x} \in D_n(\epsilon^*)$, then $B(\bar{x}, 2^{-m})$ overlaps with at least two elements of the partition $\alpha_n \times \alpha_n$, so that $\bar{x} = (x,y)$ must lie in the frame of $\alpha_n(x) \times \alpha_n(y)$ of width at most $2\cdot 2^{-m}$. Therefore,

$$\frac{\bar{\lambda}(\mathcal{F}(\alpha_n(x)\times\alpha_n(y), 2^{-m}))}{\bar{\lambda}(\alpha_n(x)\times\alpha_n(y))} \leq \frac{2^{-n(h-\epsilon^*)}\cdot 2\cdot 2^{-m}}{2^{-n(\bar{h}+2\epsilon^*)}} \leq 2\cdot 2^{-n\epsilon^*}.$$

Then $\bar{K}_n(\bar{x}) \leq R_{B(\bar{x},2^{-m})}(\bar{x})$, so by Theorem 1.4

$$\limsup_{n\to\infty} \frac{\log \bar{K}_n(\bar{x})}{n} \leq \limsup_{n\to\infty} \frac{\log R_{B(\bar{x},2^{-m})}(\bar{x})}{-\log 2^{-m}} \cdot \frac{m}{n} \leq 2(h + 4\epsilon^*) = \bar{h} + 8\epsilon^* \quad \mu \text{ a.e.}$$

And since this holds for all $\epsilon^* > 0$, this finishes the proof. □

A similar result can be obtained without using the Theorem of Barreira and Saussol, but a different set of assumptions instead. Let us consider a dynamical system $(I^2, \bar{\mathcal{B}}, \nu, S)$, where $\nu \ll \bar{\lambda}$ with density function f bounded away from zero by a constant $c_S > 0$ and from above by a constant $C_S < \infty$. Let S be an ergodic and ν-preserving transformation. Also consider a finite partition α of I^2 with $H_\nu(\alpha) < \infty$ and $h = h_\nu(S,\alpha) > 0$. Suppose furthermore that for each $n \geq 1$, the partition $\alpha_n = \bigvee_{i=0}^{n-1} S^{-i}\alpha$ consists of rectangles and that there exists a constant $\zeta_\alpha > 1$ independent of n such that for all $A \in \alpha_n$,

$$(\text{diameter } A)^2 \leq \zeta_\alpha \bar{\lambda}(A).$$

Let $(I^2, \bar{\mathcal{B}}, \mu, T)$ also be a dynamical system where $\mu \ll \bar{\lambda}$, with density g bounded away from zero by a constant $c_T > 0$ and bounded from above by a constant $C_T < \infty$. Suppose furtermore that T is an ergodic and μ-preserving transformation. We then have the following theorem.

Theorem 3.3. *Let $\mathcal{P}$ be a finite partition of I^2 with $H_\mu(\mathcal{P}) < \infty$. Suppose furthermore that $h^* = h_\mu(T, \mathcal{P}) > 0$. Define the sequence of partitions $\{\mathcal{P}_n\}$ by $\mathcal{P}_n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$ and suppose that for all $n \geq 1$ the partition $\mathcal{P}_n$ consists of rectangles only. Suppose furthermore that there exists a constant $\zeta_\mathcal{P}$, such that for all $n \geq 1$*

$$(\textit{diameter } P)^2 \leq \zeta_\mathcal{P} \bar{\lambda}(P)$$

for all $P \in \mathcal{P}_n$. Then

$$\lim_{n\to\infty} \frac{\log \bar{K}_n(\bar{x})}{n} = h \quad \mu \textit{ a.e.}$$

Notice that instead of the assuming a long return time, we now imposed the condition that the partition elements consist of rectangles, whose diameters can not become large with respect to their measures.

Proof. Let $\epsilon > 0$ be given. For each $n \geq 1$, define the number $m = m(n) = \lfloor \frac{(h-2\epsilon)n}{h^*+\epsilon} \rfloor$ and the set

$$D_n(\epsilon) = \left\{ \quad \bar{x} \in I^2 : \begin{array}{l} 2^{-m(h^*+\epsilon)} < \mu(\mathcal{P}_m(\bar{x})) < 2^{-m(h^*-\epsilon)} \\ 2^{-n(h+\epsilon)} < \nu(\alpha_n(\bar{x})) < 2^{-n(h-\epsilon)} \\ \alpha_n(\bar{x}) \not\subseteq \mathcal{P}_m(\bar{x}) \end{array} \right\}.$$

Using the same technique as in the second part of the proof of Theorem 3.2, we can show that

$$\frac{\bar{\lambda}(\mathcal{F}(\mathcal{P}_m(\bar{x}), \delta))}{\bar{\lambda}(\mathcal{P}_m(\bar{x}))} \leq 4\sqrt{\frac{\zeta_\mathcal{P}\zeta_\alpha C_T}{c_S}} 2^{-1/2n\epsilon}.$$

From this we can deduce that

$$\sum_{n=1}^{\infty} \bar{\lambda}(D_n(\epsilon)) < \infty.$$

So by using the Borel Cantelli Lemma, it can be shown that for n big enough $\alpha_n(\bar{x}) \subseteq \mathcal{P}_m(\bar{x})$, which means that

$$\liminf_{n\to\infty} \frac{\log K_n(\bar{x})}{n} \geq \liminf_{n\to\infty} \frac{\log R_{\mathcal{P}_m(\bar{x})}(\bar{x})}{m} \cdot \frac{m}{n} = (h - 2\epsilon)\frac{h^*}{h^* + \epsilon} \quad \mu \text{ a.e.}$$

The proof of the assertion that

$$\limsup_{n\to\infty} \frac{\log K_n(\bar{x})}{n} \leq h \quad \mu \text{ a.e.}$$

can be obtained in a similar way by taking $m = m(n) = \lceil \frac{(h+2\epsilon)n}{h^*-\epsilon} \rceil$ and

$$D_n(\epsilon) = \left\{ \quad \bar{x} \in I^2 : \begin{array}{l} 2^{-m(h^*+\epsilon)} < \mu(\mathcal{P}_m(\bar{x})) < 2^{-m(h^*-\epsilon)} \\ 2^{-n(h+\epsilon)} < \nu(\alpha_n(\bar{x})) < 2^{-n(h-\epsilon)} \\ \mathcal{P}_m(\bar{x}) \not\subseteq \alpha_n(\bar{x}) \end{array} \right\}.$$

□

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