Optimal Expansions in non-integer base

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For a given positive integer \( m \), let \( A = \{0, 1, \ldots, m\} \) and \( \beta \in (m, m + 1) \). A sequence \( (c_i) = c_1c_2 \ldots \) consisting of elements in \( A \) is called a \( \beta \)-expansion of \( x \) if \( \sum_{i=1}^{\infty} c_i \beta^{-i} = x \). It is well known that almost every \( x \in [0, m/(\beta - 1)] \) has uncountably many expansions. We call an expansion \( (d_i) \) of \( x \) optimal if for all \( n \geq 1 \), the inequality \( x - \sum_{i=1}^{n} d_i \beta^{-i} \leq x - \sum_{i=1}^{n} c_i \beta^{-i} \) holds for any other expansion \( (c_i) \) of \( x \). We show that optimal expansions almost always fail to exist except for a countable set \( P \) consisting of those bases \( \beta \in (m, m + 1) \) which satisfy one of the equalities

\[
1 = \frac{m}{\beta} + \cdots + \frac{m}{\beta^n} + \frac{p}{\beta^{n+1}}, \quad n \in \mathbb{N} \text{ and } p \in \{1, \ldots, m\}.
\]

More precisely, we have the following dichotomy:

**Theorem**

(i) If \( \beta \in P \), then each \( x \in [0, m/(\beta - 1)] \) has an optimal expansion.

(ii) If \( \beta \in (m, m + 1) \setminus P \), then the set of numbers \( x \in [0, m/(\beta - 1)] \) with an optimal expansion is nowhere dense and has Lebesgue measure zero.

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