Appendix: Order unit spaces and base norm spaces

This handout is a collection of results from the appendix of the book "Geometry of State Spaces of Operator Algebras" by Erik M. Alfsen and Frederic W. Shultz. They were introduced in the book "State Spaces of Operator Algebras. Basic Theory, Orientations, and C*-products" by the same authors, referenced as [AS] below. These results will be referenced throughout my seminar talk on bi- and preduals of JBW-algebras.

A 9. *An ordered normed vector space V with a generating cone V^+ is said to be a base norm space if V^+ has a base K located on a hyperplane H $(0 \notin H)$ such that the closed unit ball of V is $co(K \cup -K)$. The convex set K is called the distinguished base of V. [AS, Def. 1.10]*

A 11. *If V is a base norm space with distinguished base K, then the restriction map $f \mapsto f|_K$ is an order and norm preserving isomorphism of V^* onto the space $A_b(K)$ of all real valued bounded affine functions on K equipped with pointwise ordering and supremum norm. [AS, Prop. 1.11]*

A 19. *The dual of an order unit space A is a base norm space and the dual of a base norm space V is an order unit space. More specifically, the distinguished base of A^* is the state space of A, and the distinguished order unit of V^* is the unit functional on V. [AS, Thm. 1.19]*

A 26. *Each point $\omega \neq 0$ in a base norm space V can be decomposed as a difference of two orthogonal positive components, i.e., there exists $\rho, \sigma \in V^+$ such that $\omega = \rho - \sigma$ and $\|\omega\| = \|\rho\| + \|\sigma\|$. [AS, Prop. 1.26]*

A 31 (Bipolar theorem)**.** *Let X and Y be vector spaces over $\mathbb{R}$ or $\mathbb{C}$ in separating duality under a bilinear form $\langle\ ,\ \rangle$. If M is a weakly closed subspace of X with the annihilator M° in Y and if $M^{\circ\circ}$ is the annihilator of M° in X, then $M^{\circ\circ} = M$. [AS, Thm. 1.35]*

A 51. *Suppose A is a complete order unit space which is a power associative commutative algebra where the distinguished order unit 1 acts as an identity. Then A is an order unit algebra iff the following implication holds for $a \in A$:*

$$-1 \leq a \leq 1 \Rightarrow 0 \leq a^2 \leq 1.$$

[AS, Lemma 1.80]

A 52. *Suppose A is a real Banach space which is equipped with a power associative and commutative bilinear product with identity element* 1. *Then A is an order unit algebra with positive cone consisting of all squares, distinguished order unit* 1 *and the given norm, iff for $a, b \in A$,*

1. $\|ab\| \leq \|a\|\|b\|$,
2. $\|a^2\| = \|a\|^2$,
3. $\|a^2\| \leq \|a^2 + b^2\|$.

Moreover, the ordering of A is uniquely determined by the norm and the identity element 1. *[AS, Thm. 1.81]*

A 53. *Let A be a commutative order unit algebra that is the dual of a base norm space V such that multiplication in A is separately w*-continuous. Then for each $a \in A$ and each $\varepsilon > 0$ there are orthogonal projections $p_1, ..., p_n$ in the w*-closed sub algebra $W(a, 1)$ generated by a and* 1 *and scalars $\lambda_1, ..., \lambda_n$ such that*

$$\|a - \sum_{i=1}^{n} \lambda_i p_i\| < \varepsilon.$$

[AS, Thm. 1.84]