① $V \subseteq W$ operator spaces, H Hilbert, $\varphi: V \to B(H)$ complete contraction, then $\exists\, \psi: W \to B(H)$ extending φ and completely contractive.

② $H = \mathbb{C}^n$, $B(H) = M_n$

<u>Def</u> X, Y normed sp. $\varphi: X \to Y$ bdd linear then φ is a <u>quotient mapping</u> if
- $\tilde{\varphi}: X/\ker\varphi \to Y$ isometric
- or $\varphi(X_{\|\cdot\|<1}) = Y_{\|\cdot\|<1}$

<u>exact quotient mapping</u> if ... $Y_{\|\cdot\|\leq 1}$

(L1) φ isometry $\Leftrightarrow \varphi^*$ ... mapping

(2*) $\rho: CB(W, M_n) \to CB(V, ...$ ... $|_V$ is exact quotient mapping

$\|v\|_1 := \inf\{\|\alpha\|_2 ...$ ... $M_r(V), \alpha \in M_{n,r}, \beta \in M_{r,n}\}$

<u>Prop. 3.2.4</u> V complete then $\exists$ wk*-homeomorphic compl. isom. injection $\varphi: V^* \to B(H)$.

<u>4.2.2</u> V complete op. sp., $N \subseteq V^*$ wk*-closed lin. subspace, then $\exists$ wk* homeomorphic complete isometry $\pi^*: (V/N_\perp)^* \cong N$. $\quad N_\perp = \{v \in V : f(v) = 0, f \in N\}$

$(**) \Rightarrow \pi^*: (V/N_\perp)^* \cong (N_\perp)^\perp = N$

$(N_\perp)^\perp$ = wk*-closed convex balanced hull of N V1.9 Conway

Remains π^* compl. isometric & wk*-homeomorphic

4.1.9 equiv. π complete quotient mapping.

Saw in Pf 4.2.1 $\longrightarrow$ ☒

Pf (**) $\pi^* (V/N)^* \cong M^\perp$ as normed sp.

By 4.1.9 it suffices to show $\pi: V \to V/N$ is complete quotient.

$\ker \pi = N$

$\tilde{\pi}: V/N \to V/N \quad \tilde{\pi}(v+N) = \pi(v) = v+N$

$\Rightarrow \pi$ quotient

$\pi_n : M_n(V) \to M_n(V/N) := M_n(V)/M_n(N)$

$\pi_n(v) = (\pi(v_{ij})) = (v_{ij}+N) \cong v + M_n(N)$

$\Rightarrow \pi_n$ is a canonical map, hence quotient

$j: N \hookrightarrow V$ embedd. is compl. isometric $\xRightarrow{4.1.9}$ $j^*: V^* \to N^*$ complete quotient

and $j^*(f) = f|_N$ so $\ker j^* = N^\perp$

So $\overline{(j^*)}_n$ is isometric $\forall n$

$\overline{j^*}: V^*/N^\perp \to N^*$

Check $\overline{(j^*)}_n = (\overline{j^*})_n$ using $M_n(V^*/N^\perp) = M_n(V^*)/M_n(N^\perp)$

$\Rightarrow \overline{j^*}$ compl. isometric.

① $V \subseteq W$ operator spaces, H Hilbert, $\varphi : V \to B(H)$ complete contraction, then $\exists\, \tilde{\varphi} : W \to B(H)$ extending φ and completely contractive.

② $H = \mathbb{C}^n$, $B(H) = M_n$

<u>Def</u> X, Y normed sp. $\varphi : X \to Y$ bdd linear then φ is a <u>quotient mapping</u> if

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or

- $\varphi(X_{\text{ball}}) = Y_{\text{ball}}$

<u>exact quotient mapping</u> if $\cdot\ \varphi(X_{\text{ball}}) = Y_{\text{ball}}$

(L1) φ isometry $\Leftrightarrow$ φ^* (exact) quotient mapping

(2*) $\rho : CB(W, M_n) \to CB(V, M_n)$, $\rho(\phi) = \phi|_V$ is exact quotient mapping

$\|v\|_1 := \inf\{\|\alpha\|_2 \|\tilde{v}\| \|\beta\|_2 : v = \alpha \tilde{v} \beta,\ r \in \mathbb{N},\ \tilde{v} \in M_r(V),\ \alpha \in M_{n,r},\ \beta \in M_{r,n}\}$

Cor 4.1.9 V, W op. sp. $\varphi : V \to W$ linear, then φ complete quotient mapping $\Leftrightarrow$ φ^* complete isometry & φ^* is wk*-homeomorphic.

<u>Recall</u> if $M \subseteq V$, $M^\perp = \{f \in V^* : f(v) = 0,\ \forall v \in M\}$ and $\pi : V \to V/M$ canonical map, then

(**) $\pi^* : (V/M)^* \to M^\perp \subseteq V^*$ is an isometric isomorph.

<u>Prop. 4.2.1</u> N closed subsp. of complete op. sp. V then have <u>complete</u> isometries

$$(V/N)^* \cong N^\perp \quad \& \quad N^* \cong V^*/N^\perp$$

4.1.7 operator sp. V injective

iff $\forall W_0 \subseteq W$ op. sp., the restriction

$\rho: CB(W,V) \to CB(W_0,V)$ is

an exact complete quotient mapping

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means $\forall n$:

$$\begin{array}{ccc} \rho_n : M_n(CB(W,V)) & \longrightarrow & M_n(CB(W_0,V)) \\ \wr\wr & & \wr\wr ? \\ CB(W, M_n(V)) & \longrightarrow & CB(W_0, M_n(V)) \end{array}$$

are exact quotient,

i.e. $M_n(V)$ injective $\forall n$.

$\pi: V \hookrightarrow B(H)$ compl. isom. rep.

$\pi_n: M_n(V) \hookrightarrow M_n(B(H)) \cong B(H^n)$

V inj $\xRightarrow{(ii)} \exists\, \phi$ proj $B(H) \to B(H)$ onto V

$$\begin{array}{l} \phi_n : M_n(B(H)) \to M_n(B(H)) \text{ is compl. contr. proj onto } M_n(V) \\ \qquad\quad \| \\ \qquad B(H^n) \end{array}$$

$B(H^n)$ is injective $\xRightarrow{(i)}$ $M_n(V)$ injective.

Def operator sp V is *injective* if $\forall W_0 \subseteq W$ op. sp. and $\forall$ completely bdd linear map $\varphi_0 : W_0 \to V$ $\exists$ a linear extension $\varphi : W \to V$ with $\|\varphi\|_{cb} = \|\varphi_0\|_{cb}$.

Example : $B(H)$, H Hilbert.

4.1.6 (i) B injective, $\Phi : B \to B$ completely contractive projection, then $\Phi(B)$ is injective.

(ii) V injective, $V \subseteq B(H) \Rightarrow \exists$ compl. contr. projection $\Phi : B(H) \to B(H)$ s.t. $V = \Phi(B(H))$.

(i) Let $\varphi_0 : W_0 \to \Phi(B)$ compl. bdd

$$\begin{tikzcd} W \arrow[r, "\tilde{\varphi}"] & B \arrow[d, shift left, "\Phi"] \\ W_0 \arrow[u, phantom, "\cup"] \arrow[r, "\varphi_0"] & \Phi(B) \arrow[u, shift left, "j"] \end{tikzcd}$$

$\Rightarrow j \circ \varphi_0 : W_0 \to B$

B injective $\Rightarrow \exists\, \tilde{\varphi} : W \to B$ s.t. $\tilde{\varphi}|_{W_0} = j \circ \varphi_0$, $\|\tilde{\varphi}\|_{cb} = \|j \circ \varphi_0\|$.

Let $\varphi = \Phi \circ \tilde{\varphi} : W \to \Phi(B)$.

$$\underset{\in W_0}{\varphi(w_0)} = \Phi(\tilde{\varphi}(w_0)) = \Phi(j \circ \varphi_0(w_0)) = \Phi(\overset{\in \Phi(B)}{\varphi_0(w_0)}) = \varphi_0(w_0)$$

$\Rightarrow \varphi|_{W_0} = \varphi_0$.

(ii) $V \subseteq B(H)$, V injective

$I : V \to V$ identity then V inj. $\Rightarrow$

$\exists\, \Phi : B(H) \to V$ s.t. $\Phi|_V = I$, $\|\Phi\|_{cb} = \|I\|_{cb} = 1$

$$\Phi^2(\overset{\in B(H)}{w}) = \Phi(\overset{\in V}{\Phi(w)}) = I(\Phi(w)) = \Phi(w)$$

$\Rightarrow \Phi$ is a projection onto V

① $V \subseteq W$ operator spaces, H Hilbert, $\varphi : V \to B(H)$ complete contraction, then $\exists\, \psi : W \to B(H)$ extending φ and completely contractive.

② $H = \mathbb{C}^n$, $B(H) = M_n$

<u>Def</u> X, Y normed sp $\varphi : X \to Y$ bdd linear then φ is a <u>quotient mapping</u> if
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or
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<u>exact quotient mapping</u> if $\varphi(X_{\|\cdot\|\le 1}) = Y_{\|\cdot\|\le 1}$

(L1) φ isometry $\Leftrightarrow$ φ^* (exact) quotient mapping

(2*) $\rho : CB(W, M_n) \to CB(V, M_n)$, $\rho(\phi) = \phi|_V$ is exact quotient mapping

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$\|v\|_1 := \inf\{\|\alpha\|_2 \|\tilde{v}\| \|\beta\|_2 : v = \alpha \tilde{v} \beta,\ r \in \mathbb{N},\ \tilde{v} \in M_r(V),\ \alpha \in M_{n,r},\ \beta \in M_{r,n}\}$

$\Rightarrow \{\psi_F\}_{F \in \mathcal{F}}$ has cluster point ψ with $\|\psi\|_{cb} \le 1$.

We show $\psi|_V = \varphi$.

Since $\psi_F \xrightarrow{cl} \psi$, have $\forall$ open $U \ni \psi$ $\forall F_0 \in \mathcal{F}$ $\exists F \ge F_0$ st $\psi_F \in U$.

For $v \in V$, $\xi \in H$ let $F_0^{v,\xi}$ be the orthog proj. onto $\mathrm{sp}\{\xi, \varphi(v)\xi\}$, then for $F \ge F_0^{v,\xi}$, have $\operatorname{im} F \supseteq \operatorname{im} F_0^{v,\xi} \supseteq \{\xi, \varphi(v)\xi\}$, so

$$\psi_F(v)\xi = F\varphi(v)F\xi \underset{\substack{\xi \in \operatorname{im} F \\ F^2 = F}}{=} F\varphi(v)\xi \underset{\varphi(v)\xi \in \operatorname{im} F}{=} \varphi(v)\xi$$

For $\varepsilon > 0$ let $U = \{f : p_{v,\xi,\xi}(f - \psi) < \varepsilon\} \ni \psi$

Then by def of cl $\exists F \ge F_0^{v,\xi}$ s.t. $\psi_F \in U$

Cor 4.1.4 $\rho : CB(W, M_n) \to CB(V, M_n)$ is an exact quotient mapping

in fact $\|\phi\|_{cb} = \|\phi|_V\|_{cb}$

pf of ① For general $B(H)$ define

$\mathcal{F} = \{F \in B(H) : F \text{ orthogonal projection}, rk < \infty\}$

Then for $F \in \mathcal{F}$, if $n = rk(F)$, then $F(H) \cong \mathbb{C}^n$

So $B(F(H)) \cong M_n$.

Let $\varphi : V \to B(H)$ be a complete contraction.

Consider $F\varphi F : V \to B(F(H)) : v \mapsto F\varphi(v)F$

then $F\varphi F$ is also compl. contractive.

By 4.1.4 $\exists$ completely contractive extension $\tilde{\varphi}_F : W \to B(F(H))$ of $F\varphi F$.

For $\psi_F : \tilde{\varphi}_F F : W \to B(H)$, $w \mapsto \tilde{\varphi}_F(w) F$

then for $v \in V$ $\psi_F(v) = \tilde{\varphi}_F(v) F = F\varphi(v)F^2 = F\varphi(v)F$

ψ_F is also compl. contr.

So $\{\psi_F\}_{F \in \mathcal{F}}$ is a net in $CB(W, B(H))_{\|\cdot\|_{cb} \leq 1}$.

where we order $\mathcal{F}$ by $F_1 \leq F_2 \Leftrightarrow \operatorname{im} F_1 \subseteq \operatorname{im} F_2$.

Endow $CB(W, B(H))$ with "point - weak operator topology" (p-WOT)

i.e. $f_i \xrightarrow{p\text{-WOT}} f \Leftrightarrow \forall w \in W$ $f_i(w) \xrightarrow{WOT} f(w)$, equivalently top. generated by

$$p_{w,x,y}(f) = |\langle f(w)x, y\rangle|$$

Claim, $CB(W, B(H))_{\|\cdot\|_{cb} \leq 1}$ is compact wrt p-WOT

To show, $(M_n(V), \|\cdot\|_1)^* = M_n(V^*) \; (= CB(V, M_n))$

As sets via $M_n(V^*) \to M_n(V)^* : (f_{ij}) \mapsto \left((v_{ij}) \mapsto \sum_{i,j=1}^n f_{ij}(v_{ij})\right)$

$f \in M_n(V^*)$

$\|f\|_{T_n(V)^*} = \sup\{ |f(v)| : v \in M_n(V), \|v\|_1 \le 1 \}$

$= \sup\{ |\sum_{i,j=1}^n f_{ij}(v_{ij})| : v \in M_n(V), \|v\|_1 \le 1 \}$

$= \|f\|_{cb}$.

Also $M_n(V)^* = T_n(V^*)$.

Lemma 4.1.2. if $v \in M_n(V)$, then

$\|v\|_1 < 1 \iff v = \alpha \tilde{v} \beta$, with $\alpha, \beta \in M_n$, $\tilde{v} \in M_n(V)$, α and β invertible and $\|\tilde{v}\|, \|\alpha\|_2, \|\beta\|_2 < 1$.

4.1.3 $j : T_n(V) \hookrightarrow T_n(W)$ is isometric.

To show $\|v\|_{1, T_n(V)} = \|v\|_{1, T_n(W)}$

"$\geq$" immediate from def.

"$\leq$" If $\|v\|_{1, T_n(W)} < 1$ then by 4.1.2 take decomposition

$v = \alpha \tilde{w} \beta$ as in 4.1.2 $\tilde{w} \in M_n(W)$

$\Rightarrow \tilde{w} = \alpha^{-1} v \beta^{-1} \in M_n(V) \Rightarrow v = \alpha \tilde{w} \beta$ is an allowable decomp. in $\|\cdot\|_{1, T_n(V)}$

So $\|v\|_{1, T_n(V)} \le \|\alpha\|_2 \|\tilde{w}\| \|\beta\|_2 < 1$.

Consider $\frac{v}{\|v\|_{1, T_n(W)} + \delta}$, $\delta \downarrow 0$ establishes "$\leq$".

To show, $(M_n(V), \|\cdot\|_1)^* = M_n(V^*) \,(= CB(V, M_n))$

Lemma: $\forall v \in M_n(V)$: $\|v\| \le \|v\|_1 \le n\|v\|$

$\|v\|_1 < 1$ then $\exists \alpha, \beta, \tilde{v}$, $v = \alpha \tilde{v} \beta$ s.t. $\|\alpha\|_2 \|\tilde{v}\| \|\beta\|_2 < 1$

wlog: $\|\alpha\|_2, \|\tilde{v}\|, \|\beta\|_2 < 1$

So $\|v\| = \|\alpha \tilde{v} \beta\| = \|\alpha\| \cdot \|\tilde{v}\| \|\beta\|$

$\le \|\alpha\|_2 \|\tilde{v}\| \|\beta\|_2 < 1$

for general $v \in M_n(V)$:

$\left\| \frac{v}{\|v\|_1 + \delta} \right\|_1 < 1 \quad \forall \delta > 0$

By above: $\left\| \frac{v}{\|v\|_1 + \delta} \right\| < 1$

$\Rightarrow \|v\| \le \|v\|_1 + \delta$

$\delta \downarrow 0 \Rightarrow \|v\| \le \|v\|_1$.

Suppose $v \in M_n(V)$, then $v = I v I$, $I \in M_n$

$\Rightarrow \|v\|_1 \le \|I\|_2 \|v\| \|I\|_2 = n\|v\|$. $\boxtimes$

Cor. $T_n(V)^* = M_n(V)^* = M_n(V^*)$ as set,

Define for $v \in M_n(V)$: $\|v\|_1 := \inf\{\|\alpha\|_2 \|\tilde{v}\| \|\beta\|_2 : v = \alpha \tilde{v} \beta, r \in \mathbb{N}, \tilde{v} \in M_r(V), \alpha \in M_{n,r}, \beta \in M_{r,n}\}$

$\|f\|_{cb} = \sup\{ |\sum_{k,l=1}^{n} f_{k,l}(v_{k,l})| : \|v\|_1 \le 1, v \in M_n(V)\}$

Claim $\|\cdot\|_1$ defines a norm.

Goal: for $f \in M_n(V^*) = CB(V, M_n)$, $\quad f = (f_{k,l})$ $\in V^*$

want $\|f\|_{cb} = \|f\|_{T_n(V)^*}$.

$\|f\|_{cb} = \sup\{\|f_r\| : r \in \mathbb{N}\} = \sup\{\|f_r(\tilde{v})\| : r \in \mathbb{N}, \tilde{v} \in M_r(V), \|\tilde{v}\| \leq 1\}$

$f_r : M_r(V) \to M_r(M_n) = M_{r \cdot n} = B((\mathbb{C}^{r \cdot n}, \|\cdot\|_2))$

$f_r(\tilde{v}) \in B((\mathbb{C}^{r \cdot n}, \|\cdot\|_2))$ Hilb

$D_{r \cdot n} = \{x \in \mathbb{C}^{r \cdot n} : \|x\|_2 \leq 1\}$

and instead of index set $\{1, \dots, r \cdot n\}$ use $\{1, \dots, r\} \times \{1, \dots, n\}$

Then

$\|f_r(\tilde{v})\| = \sup\limits_{\xi, \eta \in D_{r \cdot n}} |\langle f_r(\tilde{v}) \eta, \xi \rangle| = \sup\limits_{\xi, \eta \in D_{r \cdot n}} \left| \sum_{i,j=1}^{r} \sum_{k,l=1}^{n} f_{k,l}(\tilde{v}_{i,j}) \eta_{(j,l)} \bar{\xi}_{(i,k)} \right|$

$= \sup\limits_{\xi, \eta \in D_{r \cdot n}} \left| \sum_{k,l=1}^{n} f_{k,l} \left(\sum_{i,j=1}^{r} \bar{\xi}_{(i,k)} \tilde{v}_{i,j} \eta_{(j,l)} \right) \right|$

For $\xi, \eta \in D_{r \cdot n}$ define $\alpha_{k,i} = \bar{\xi}_{(i,k)}$, $\beta_{j,l} = \eta_{(j,l)}$, $\alpha = (\alpha_{k,i}) \in M_{n,r}$, $\beta = (\beta_{j,l}) \in M_{r,n}$

define $\|\alpha\|_2 = \left(\sum_{k,i} |\alpha_{k,i}|^2 \right)^{1/2}$, then

$\Rightarrow \|f\|_{cb} = \sup\left\{ \left| \sum_{k,l=1}^{n} f_{k,l} \left(\sum_{i,j=1}^{r} \alpha_{k,i} \tilde{v}_{i,j} \beta_{j,l} \right) \right| : r \in \mathbb{N}, \tilde{v} \in M_r(V), \|\tilde{v}\| \leq 1, \alpha \in M_{n,r}, \beta \in M_{r,n}, \|\alpha\|_2, \|\beta\|_2 \leq 1 \right\}$

$= \sup\left\{ \left| \sum_{k,l=1}^{n} f_{k,l}((\alpha \tilde{v} \beta)_{k,l}) \right| : \dots \right\}$

$= \sup\left\{ \left| \sum_{k,l=1}^{n} f_{k,l}(v_{k,l}) \right| : v = \alpha \tilde{v} \beta, \dots \right\}$

Define for $v \in M_n(V)$: $\|v\|_1 := \inf\{\|\alpha\|_2 \|\tilde{v}\| \|\beta\|_2 : v = \alpha \tilde{v} \beta, r \in \mathbb{N}, \tilde{v} \in M_r(V), \alpha \in M_{n,r}, \beta \in M_{r,n}\}$

$\|f\|_{cb} = \sup\left\{ \left| \sum_{k,l=1}^{n} f_{k,l}(v_{k,l}) \right| : \|v\|_1 \leq 1, v \in M_n(V) \right\}$

Claim $\|\cdot\|_1$ defines a norm.

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<u>exact quotient mapping</u> if $\cdot\ \varphi(X_{\text{ball}}) = Y_{\text{ball}}$

(L1) φ isometry $\Leftrightarrow$ φ^* (exact) quotient mapping

(2*) $\rho : CB(W, M_n) \to CB(V, M_n)$, $\rho(\phi) = \phi|_V$ is exact quotient mapping

Recall: as sets $M_n(V^*) = CB(V, M_n) = M_n(V)^*$

ignore norms then

$\rho : M_n(W)^* \to M_n(V)^*$ is adjoint of embedding $j : M_n(V) \hookrightarrow M_n(W)$

$j^* : M_n(W)^* \to M_n(V)^*$ $\quad j^*(\phi)(v) = \phi(j(v)) = \phi(v) \Rightarrow j^*(\phi) = \phi|_{M_n(V)} = \phi|_V = \rho(\phi)$

$\overset{L1}{\Rightarrow}$ ρ exact quotient.

Problem: as normed sp $CB(V, M_n) \neq M_n(V)^*$

Solution:

- find $\|\cdot\|_1$ on $M_n(V)$ s.t. $(M_n(V), \|\cdot\|_1)^* = CB(V, M_n) = M_n(V^*)$
- Let $T_n(V) := (M_n(V), \|\cdot\|_1)$
- $j : T_n(V) \hookrightarrow T_n(W)$ is completely isometric
- the above works.

① $V \subseteq W$ operator spaces, H Hilbert, $\varphi : V \to B(H)$ complete contraction, then $\exists\, \psi : W \to B(H)$ extending φ and completely contractive.

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(L1) φ isometry $\Leftrightarrow \varphi^*$ (exact) quotient mapping

②* $\rho : CB(W, M_n) \to CB(V, M_n)$. $\rho(\phi) = \phi|_V$ is exact quotient mapping

Recall as sets $M_n(V^*) = CB(V, M_n) = M_n(V)^*$

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$\rho : M_n(W)^* \to M_n(V)^*$ is adjoint of embedding $j : M_n(V) \hookrightarrow M_n(W)$

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$\overset{L1}{\Rightarrow}$ ρ exact quotient

Problem: as normed sp $CB(V, M_n) \neq M_n(V)^*$

Solution:

- find $\|\cdot\|_2$ on $M_n(V)$ s.t. $(M_n(V), \|\cdot\|_2)^* = CB(V, M_n) = M_n(V^*)$
- Let $T_n(V) := (M_n(V), \|\cdot\|_2)$
- $j : T_n(V) \hookrightarrow T_n(W)$ is completely isometric
- the above works.