<u>Prop</u> Let H be a Hilbert.

(1) the OH-matrix norm on H is an operator space matrix norm.

(2) Consider the OH-norm on $\overline{H}$ and consider the matrix norm on H^* defined by $M_n(H^*) \overset{\text{(3.2.2)}}{\cong} CB(\underset{\text{OH-norm}}{H}, M_n)$

Then $\theta : \overline{H} \to H^*$ def by $\theta(u)(v) = \langle v, u \rangle$ is a complete isometry.

(3) If $|||\cdot|||$ is a matrix norm on H st $\theta : \overline{H} \to H^*$ is a complete isometry (with $|||\cdot|||$ on $\overline{H}$ and $CB(\underset{\text{с }|||\cdot|||}{H}, M_n)$-norm on H^*), then $|||\cdot|||$ equals the OH-norm.

<u>Proof</u> (1) (M1) Let $\xi, \eta \in M_n(H)$.

$$\Big\| \underbrace{\begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix}}_{\xi \oplus \eta} \Big\|_o^2 = \Big\| \Big\langle\Big\langle \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix}, \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \Big\rangle\Big\rangle \Big\|$$

$$= \left\| \begin{pmatrix} \langle\langle \xi, \xi \rangle\rangle & & & 0 \\ & \langle\langle \eta, \eta \rangle\rangle & & \\ & & \langle\langle \xi, \eta \rangle\rangle & \\ 0 & & & \langle\langle \eta, \xi \rangle\rangle \end{pmatrix} \right\|$$

$$= \max\{ \|\langle\langle \xi, \xi \rangle\rangle\|, \|\langle\langle \eta, \eta \rangle\rangle\|, \|\langle\langle \xi, \eta \rangle\rangle\|, \|\langle\langle \eta, \xi \rangle\rangle\| \}$$

$$\underset{\text{Haagerup ineq}}{\leq} \max\{ \|\xi\|_o^2, \|\eta\|_o^2 \}$$

§3.5 Pisier's self-dual Hilbert operator space

Hilbert space H

$\theta: \bar{H} \to H^*$ $\quad \theta(u)(v) = \langle v, u \rangle_H$

($\bar{H}$ with scalar multipl. $\bar{\alpha} x$)

Goal: there is a unique operator space structure on H s.t.

$\theta: \bar{H} \to H^*$ is complete isometry.

Matrix sesquilinear form, induced by $\langle \cdot, \cdot \rangle$ of H:

$$M_m(H) \otimes M_n(H) \to M_m \otimes M_n$$

$$\eta \otimes \xi \mapsto \langle\langle \eta, \xi \rangle\rangle =$$

m×n blocks

on block k,l:

$$\begin{pmatrix} \langle \eta_{kl}, \xi_{11} \rangle & \cdots & \langle \eta_{kl}, \xi_{1n} \rangle \\ \vdots & & \vdots \\ \langle \eta_{kl}, \xi_{n1} \rangle & \cdots & \langle \eta_{kl}, \xi_{nn} \rangle \end{pmatrix}$$

<u>Theorem</u> 3.5.1 (Haagerup)

$\forall n \in \mathbb{N}$, $\xi, \eta \in M_n(H)$

$$\| \langle\langle \eta, \xi \rangle\rangle \|_{M_m \otimes M_n} \le \| \langle\langle \eta, \eta \rangle\rangle \|^{1/2}_{M_n \otimes M_n} \, \| \langle\langle \xi, \xi \rangle\rangle \|^{1/2}_{M_n \otimes M_n}.$$

<u>Def</u> OH-matrix norm on H:

for $\xi \in M_n(H)$

$$\| \xi \|_o = \| \langle\langle \xi, \xi \rangle\rangle \|^{1/2}_{M_n \otimes M_n}.$$

Def $R: H \to B(\bar{H}, \mathbb{C})$

$$R(u)(v) = \theta(v)(u) \quad u \in H, \; v \in \bar{H}$$

lin isometry

$B(\bar{H}, \mathbb{C})$ is an operator space, so by R we get operator space structure on H:

For $\xi \in M_n(H)$, $R_n(\xi): \bar{H}^n \to \mathbb{C}^n$

$$R_n(\xi)\begin{pmatrix} \eta \\ \vdots \\ \eta_n \end{pmatrix} = \begin{pmatrix} \theta(\xi_{11}) & \cdots & \theta(\xi_{1n}) \\ & & \\ \theta(\xi_{n1}) & \cdots & \theta(\xi_{nn}) \end{pmatrix}\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

$$\|\xi\|_r = \|R_n(\xi)\|_{B(\bar{H}^n, \mathbb{C}^n)} \qquad \theta(\xi_{ij}) \in H^*$$

row Hilbert space structure.

<u>Theorem</u> 3.4.1 Let H, K be Hilbert spaces, then

$$B(H, K) \cong CB(H_c, K_c)$$

completely isometrically isomorphic.

Define matrix norm for $\xi \in M_{m,n}(H)$:

$$\|\xi\|_c = \| C_{m,n}(\xi) \|_{B(\mathbb{C}^n, H^m)}$$

This is indeed a matrix norm.

<u>Claim</u> $\varphi : M_{m,n}(H_c) \to B(\mathbb{C}^n, H^m)$

$\xi \mapsto (a \mapsto \xi a)$ is a complete isometry.

<u>"Pf"</u>. <u>For $p \in \mathbb{N}$:</u>

$$\begin{tikzcd}
M_p(M_{m,n}(H_c)) \arrow[r, equal] \arrow[d, "\varphi_p^{m,n}" ', "\text{apply } \varphi^{m,n} \text{ entrywise}"] & M_{pm,pn}(H_c) \arrow[d, "\varphi^{m,n}" ', "\text{isometry}"] \\
M_p(B(\mathbb{C}^n, H^m)) \arrow[r, equal] & B(\mathbb{C}^{pn}, H^{pm})
\end{tikzcd}$$

Hence $\varphi_p^{m,n}$ is also isometry, that is, $\varphi^{m,n}$ is complete isometry. □

<u>"Row Hilbert space operator structure:"</u>

Denote: $\overline{H} = H$ with same addition

scalar multiplication on $\overline{H}$

$$\alpha v = \underbrace{\bar{\alpha} v}_{\text{scalar multipl. in } H.}$$

inner product on $\overline{H}$:

$$\langle u, v \rangle_{\overline{H}} = \langle v, u \rangle_H .$$

$\theta : \overline{H} \to H^*$ is defined by

$$\theta(u)(v) = \langle v, u \rangle \quad v \in H, \; u \in \overline{H}.$$

Then $\theta : \overline{H} \to H^*$ is linear!

$$B(\overline{H}, \mathbb{C}) = B(H^*, \mathbb{C}).$$

$\min V \subseteq \underbrace{\min Z}_{=Z}$ so $\min V = V$. $\square$

Prop. 3.3.2. A complete operator space V is maximal if and only if there is a set S and a complete quotient space map $\varphi : \max \ell_1(S) \to V$ (onto).

§3.4 Column and row Hilbert spaces.

Let H be a Hilbert space.

Wants operator structure on H.

First way: Column Hilbert space.

For $\xi \in M_{m,n}(H)$, define

$$C_{m,n}(\xi) : \mathbb{C}^n \to H^m \text{ by}$$

$$C_{m,n}(\xi)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{m1} & \cdots & \xi_{mn} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

i-th entry equals $\sum_{k=1}^{n} \underset{H}{\underset{\uparrow}{\xi_{ik}}}\, \underset{\mathbb{C}}{\underset{\uparrow}{a_k}} \in H$.

$$\|g\|_{\min} = \sup\left\{ \left\| \begin{pmatrix} f(g_{11}) & \cdots & f(g_{1n}) \\ & \vdots & \\ f(g_{n1}) & \cdots & f(g_{nn}) \end{pmatrix} \right\| : \|f\|_{B(Z,\mathbb{C})} \le 1 \right\}$$

enough to look at pt evaluation for f

$$= \sup\left\{ \left\| \begin{pmatrix} g_{11}(x) & \cdots & g_{1n}(x) \\ \vdots & & \vdots \\ g_{n1}(x) & \cdots & g_{nn}(x) \end{pmatrix} \right\|_{M_n} : x \in \Omega \right\}$$

$$= \|g\|_{C_0(\Omega, M_n)} = \|g\|_{M_n(Z)} \quad \square.$$

<u>Claim</u>: For a Banach space E,

- $(\max E)^* = \min(E^*)$ isometrically
- $\max(E^*) = (\min E)^*$ completely isometrically.

<u>Prop 3.3.1</u> If V is operator space, then

V is minimal (i.e. $\min V = V$)

if and only if

V is completely isometric to a subspace of a commutative C^*-algebra.

"<u>Pf</u>" $\Rightarrow$) $V \le \ell^\infty(S)$ for some set S

then $V = \min V \subseteq \min \ell^\infty(S) = \ell^\infty(S)$ isometrically

completely isometrically.

$\Leftarrow$) If Z commutative C^*-algebra, $V \le Z$ complete isometrically, then $\min V \le \min Z$ completely isom.

Forgetful functor: $N : \mathcal{D} \to \mathcal{N}$

an operator space $\mapsto$ its underlying normed space.

Def. A functor $Q : \mathcal{N} \to \mathcal{D}$ is a **strict quantization** if

- $(N \circ Q)(E) = E$ for every normed space E.

and

- $\forall$ bdd lin $\varphi : E \to F$, $Q(\varphi) : Q(E) \to Q(F)$ satisfies $\|Q(\varphi)\|_{cb} = \|\varphi\|$

Thm: min is a strict quantization.

Claim: Let E normed space, W operator space, $\forall \varphi : E \to W$ linear then $\|\varphi\|_{CB(\max E, W)} = \|\varphi\|_{B(E,W)}$

Claim: If $\varphi : E \to F$ contraction then $\varphi : \max E \to \max F$ is a complete contraction.

Conclusion: max is a strict quantization.

Exa. Ω loc cpt Hsd, $Z = C_0(\Omega)$, $\|\cdot\|_\infty$. Then Z is minimal, i.e. $Z = \min Z$.

For $n \geq 1$: $M_n(Z) \cong C_0(\Omega, M_n)$

$$\begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \cong \omega \mapsto \begin{pmatrix} g_{11}(\omega) & \cdots & g_{1n}(\omega) \\ \vdots & & \vdots \\ g_{n1}(\omega) & \cdots & g_{nn}(\omega) \end{pmatrix}$$

So $\|f_n \circ \varphi_n\|_{B(V,M_n)} = \|(f\circ\varphi)_n\| \leq \|f\circ\varphi\|_{CB(V,\mathbb{C})}$

$$\leq \|\varphi\|_{B(V,E)}$$

So

$$\|\varphi_n(v)\|_{\min} = \sup\{\|f_n(\varphi_n(v))\| : \|f\| \leq 1\}$$

$$\leq \|\varphi\|_{B(V,E)} \qquad \square$$

Claim. If E, F are normed spaces, $\varphi: E \to F$ lin contraction, then $\varphi: \min E \to \min F$ is a complete contraction.

Pf By previous:

$$\|\varphi\|_{CB(\min E, \min F)} = \|\varphi\|_{B(\underbrace{\min E}_{=E \text{ as normed space}}, F)} = \|\varphi\|_{B(E,F)} \qquad \square$$

Conclusion For every normed space E $\min E$ is an operator space and contractions between E and F become complete contractions between $\min E$ and $\min F$.

$\mathcal{N}$ category of normed space
morphism: bdd lin maps

$\mathcal{D}$ category of operator spaces
morphisms: completely bdd maps.

Recall $b_r(V) = \{ f \in B(V, M_r), \|f\| \leq 1 \}$

$b(V) = \bigcup_{r \in \mathbb{N}} b_r(V)$.

Then clearly $\varphi \in b_n(V) \subseteq b(V)$, so

$\|v\|_{max} = \sup \{ \|f_n(v)\| \ f \in b(v) \} \geq \|\varphi_n(v)\|$
$= \|v\|$.

Further:

$f \in b_1(V) = B(V, \mathbb{C})_{\|\cdot\| \leq 1} = CB(V, \mathbb{C})_{\|\cdot\| \leq 1}$

$\Rightarrow \|f_n\| \leq 1 \Rightarrow$

$\|v\|_{min} = \sup \{ \underbrace{\|f_n(v)\|}_{\leq \|f_n\| \|v\|} \cdot f \in b_1(v) \} \leq \|v\|$.

$\square$.

E normed space, V operator space

<u>Claim</u> $\forall \varphi : V \to E$ linear:

$\|\varphi\|_{CB(V, \min E)} = \|\varphi\|_{B(V,E)}$.

So $CB(V, \min E) = B(V,E)$.

<u>Proof</u>. Clearly,

$\|\varphi\|_{CB(V, \min E)} \geq \|\varphi\|_{B(V, \min E)}$

$= \|\varphi\|_{B(V,E)}$

For $\leq$: Let $v \in M_n(V)$ with $\|v\| \leq 1$

$\|\varphi_n(v)\|_{min} = \sup \{ \|f_n(\varphi_n(v))\| \ \ \|f\|_{B(E,\mathbb{C})} \leq 1 \}$

$\|f\|_{B(E,\mathbb{C})} \leq 1 \Rightarrow \|f \circ \varphi\|_{CB(V,\mathbb{C})} = \|f \circ \varphi\|_{B(V,\mathbb{C})}$

$\leq \underbrace{\|f\|}_{\leq 1} \|\varphi\| \leq \|\varphi\|_{B(V,E)}$.

Now $n \in \mathbb{N}$, $x \in M_n(E)$.

$$\|x_n\|_{max} = \sup\{\|f_n(x)\| : f \in \bigcup_{r \in \mathbb{N}} b_r(E)\}$$

$$= \sup_{r \in \mathbb{N}} \sup_{f \in b_r(E)} \left\| \begin{pmatrix} f(x_{11}) & & f(x_{1n}) \\ \vdots & & \\ f(x_{n1}) & \cdots & f(x_{nn}) \end{pmatrix} \right\|$$

$$= \sup_{r \in \mathbb{N}} \sup_{f \in b_r(E)} \left\| \begin{pmatrix} j(x_{11})(r)(f) & & j(x_{1n})(r)(f) \\ \vdots & & \vdots \\ j(x_{n1})(r)(f) & \cdots & j(x_{nn})(r)(f) \end{pmatrix} \right\|$$

$$= \|j_n(x)\|_{Z,n}.$$

Since Z is an operator space, we obtain that $\|\cdot\|_{max}$ is a matrix norm on E, $(E, \|\ \|_{max})$

<u>Def</u>. The operator space E with $\|\ \|_{min}$ is denoted by $\min E$, called the minimal quantization of E.

Similarly $\|\cdot\|_{max}$: $\max E$ maximal ......

<u>Claim</u>: If V operator space, $n \in \mathbb{N}$, $v \in M_n(V)$, then

$$\|v\|_{min} \leq \|v\| \leq \|v\|_{max}$$

<u>Proof</u>: Let $v \in V$. Recall L2 3.4: $\exists n \in \mathbb{N}$, complete contraction $\varphi : V \to M_n$ s.t.

$$\|v\| = \|\varphi_n(v)\|_{M_{n^2}}.$$

$$\|x\|_{\min,n} = \sup_{f \in b_1(E)} \left\| \begin{pmatrix} f(x_{11}) & f(x_{1n}) \\ f(x_{n1}) & f(x_{nn}) \end{pmatrix} \right\|$$

$$= \sup_{f \in b_1(E)} \left\| \begin{pmatrix} j(x_{11})(f) & j(x_{1n})(f) \\ & \\ j(x_{n1})(f) & j(x_{nn})(f) \end{pmatrix} \right\|$$

$$= \| j_n(x) \|_{\ell^\infty(S,\mathbb{C}),n}$$

So $\| \ \|_{\min}$ corresponds to the matrix norm of $\ell^\infty(S,\mathbb{C})$. Hence $\| \ \|_{\min}$ is a matrix norm (as the restriction of matrix norm).

• $\| \ \|_{\max}$: We embed E into

$$\left\{ z \in \prod_{r \in \mathbb{N}} \ell^\infty(b_r(E), M_r) : z \text{ bounded} \right\} =: Z$$

with $\| \ \|_\infty$.

For $x \in E$, $r \in \mathbb{N}$, $f \in b_r(E)$

$$j(x)(r)(f) = f(x) \in M_r$$

Then $j(x)(r) : b_r(E) \to M_r$ and

$$\| j(x)(r) \|_\infty = \sup_{f \in b_r(E)} \| f(x) \|_{M_r} \le \|x\| \text{ for all } r$$

So indeed, $j(x) \in Z$.

For $x \in E$:

$$\|x\|_{\max} = \sup \left\{ \|f(x)\| : f \in \bigcup_{r \in \mathbb{N}} b_r(E) \right\}$$

$$= \sup_{r \in \mathbb{N}} \| j(x)(r) \|_\infty = \| j(x) \|_Z .$$

So $j : (E, \| \ \|_{\max}) \to Z$ isometry.

Define

$$b_r(E) = \{ f \in B(E, M_r), \|f\| \le 1 \}, r \in \mathbb{N}$$

$$b(E) = \bigcup_{r \in \mathbb{N}} b_r(E)$$

For $x \in M_n(E)$:

$$\|x\|_{min} := \sup\{ \|f_n(x)\| : f \in b_1(E) \}$$

$$\|x\|_{max} := \sup\{ \|f_n(x)\| : f \in b(E) \}$$

recall $x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$ $x_{ij} \in E$,

then $f_n(x) = \begin{pmatrix} f(x_{11}) & \cdots & f(x_{1n}) \\ \vdots & & \\ f(x_{n1}) & & f(x_{nn}) \end{pmatrix}$

if $f \in b_1(E)$: $f(x_{ij}) \in \mathbb{C}$

$f \in b_r(E)$: $f(x_{ij}) \in M_r$

<u>Claim</u> $\|\cdot\|_{min}$ and $\|\cdot\|_{max}$ are matrix norms.

<u>Proof</u>. • $\|\cdot\|_{min}$: take $S = b_1(E)$

For $x \in E$, $f \in S$:

$$j(x)(f) = f(x) \in \mathbb{C}.$$

Then

$$\sup_{f \in S = b_1(E)} |j(x)(f)| = \sup_{f \in b_1(E)} |f(x)| = \|x\|$$

So $j : E \to \ell^\infty(S, \mathbb{C})$ is an isometric embedding.

For $n \ge 1$ and $x \in M_n(E)$, $x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$

§3.3 The Min and Max Quantizations

Given a normed space $(E, \|.\|_E)$

Does there exist an operator space structure on E?

i.e. matrix norms $\|.\|_n$ $(n \in \mathbb{N})$ (satisfying (M1) and (M2)) s.t.

$$\|\,\|_1 = \|\,\|_E.$$

Such an operator space is then called a quantization of the normed space

First: $S \neq \phi$ set, $r \in \mathbb{N}$ fixed.

$\ell^\infty(S, M_r)$ is an operator space with for $n \in \mathbb{N}$, $x \in M_n(\ell^\infty(S, M_r))$,

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \text{ with } x_{ij} \in \ell^\infty(S, M_r)$$

$$\|x\|_n = \sup_{s \in S} \left\| \begin{pmatrix} x_{11}(s) & \cdots & x_{1n}(s) \\ \vdots & & \\ x_{n1}(s) & & x_{nn}(s) \end{pmatrix} \right\|_{M_{nr}}$$

Let E be a normed space.