$V \subseteq V'$ V linear space

V' operator space

Suppose we have a pairing

$\langle \, , \, \rangle \; V \times V' \to \mathbb{C}$

then we can define an o.s. structure

$\varphi_v \quad V' \longrightarrow \mathbb{C}$ bounded

$w \longmapsto \langle v, w \rangle$ $\forall v \in V$

$V \hookrightarrow (V')^*$

↑ op. space structure

"dual" op. space structure on V

let $\varphi = [\varphi_{ij}] \in M_m(CB(V, W))$

define $\bar{\varphi} : V \longrightarrow M_m(W)$

$v \longmapsto \bar{\varphi}(v) = [\varphi_{ij}(v)]$

$M_m(CB(V,W)) \simeq CB(V, M_m(W))$

as lin spaces.

use the norm on W define matrix norms

$\Rightarrow$ op space structure on $CB(V, W)$

__Prop__ if W is complete $\Rightarrow CB(V,W)$ is complete too.

Example W a wk $*$ closed subspace of $B(\mathcal{H})$ $(\simeq S_1(H)^*)$. there is a wk $*$ homeom. isometry

$$W \simeq (S_1(\mathcal{H})/W_\perp)^*$$

$$W_\perp = \{ x \in S_1(H) \mid \varphi^*(x) = 0 \ \forall \varphi^* \in W \}$$

$\uparrow$ pre-annihilator (notion from Banach spaces).

Def Given an op space W that is the dual of a complete op s V, we say that $\varphi\ W \to B(\mathcal{H})$ is a dual realisation if it is a wk $*$ homeomorphic completely isom injection

Prop V a complete operator space, then V^* has a dual realisation for some HS $\mathcal{H}$.

If $W \subseteq V^*$ is a wk $*$-closed subset of the dual op space then $M_n(W) \subseteq M_n(V^*)$ is automatically wk $*$-closed

Let $F = [F_{ij}] : V^* \to M_n$

F is continuous in the wk $*$-top if each F_{ij} is wk $*$-cont

Denote by $CB^{\sigma}(V^*, W^*)$ wk $*$-continuous completely bounded maps $\varphi : V^* \to W^*$

Thm: let H be a Hilbert space, $B(H) \supseteq K(H)$ the comp. op's, $S_1(H)$ trace class operators. then

$$B(H) \cong S_1(H)^*$$
$$S_1(H) \cong K(H)^*$$

PROP: let V, W operator spaces, $\varphi: V \to W$ completely bounded mapping

① $\|(\varphi^*)_n\| = \|\varphi_n\| \quad \forall n$

$\Downarrow$

② $\|\varphi^*\|_{cb} = \|\varphi\|_{cb}$

Pf

$$\|(\varphi^*)_n\| = \sup\left\{ \|\langle\langle (\varphi^*)_n(g), v \rangle\rangle\| \;\middle|\; \begin{array}{l} g \in M_n(W^*) \\ v \in M_n(V) \\ \|g\| \le 1 \; \|v\| \le 1 \end{array} \right\}$$

$$= \sup\{ \|\langle\langle g, \varphi_n(v) \rangle\rangle\| \mid \ldots \} = \|\varphi_n\|$$

$\square$

the wk * topology

let V be an op space, a net

$$f_\lambda = [f^\lambda_{kl}] \in M_n(V^*) = CB(V, M_n)$$

converges in the wk*-top to

$$f = [f_{kl}] \in M_n(V^*)$$

if for every $v \in M_n(V)$, n arbitrary

$$\langle\langle f_\lambda, v \rangle\rangle = [f^\lambda_{kl}(v_{ij})]$$

$$\downarrow \text{ in norm}$$

$$[f_{kl}(v_{ij})]$$

Prop the canonical inclusion $\iota_V : V \hookrightarrow V^{**}$ is completely isometric

Proof. $v \in M_n(V)$, $f \in M_n(V^*) = CB(V, M_n)$

$((\iota_V)_n(v))_n(f) = [\iota_V(v_{ij})(f_{kl})] = [f_{kl}(v_{ij})]$

$= \langle\langle f, v \rangle\rangle$

$\|(\iota_V)_n(v)\|_{cb} = \sup\{ \|((\iota_V)_n(v))_n f\| \mid f \in CB(V, M_n), \|f\| \leq 1 \}$

$= \sup\{ \|\langle\langle f, v\rangle\rangle\| \; f \in CB, \|f\| \leq 1 \} = \|v\|$ $\square$

Suppose $\varphi : V \to W$ completely bounded,

$\varphi^* : W^* \to V^*$ $\quad f \in CB(W, \mathbb{C})$

$\varphi^*(f) = f(\varphi(v))$

Let now $v \in M_n(V)$, $g \in M_m(W^*)$

$\underset{\in M_m(W^*)}{\langle\langle g}, \underset{M_n(W)}{\varphi_n(v)} \rangle\rangle = [g_{kl}\, \varphi(v_{ij})] = [\varphi^*(g_{kl})(v_{ij})]$

$= \langle\langle \underset{\in M_m(V^*)}{\varphi^*_m(g)}, \underset{\in M_n(V)}{v} \rangle\rangle$

Conversely the norms of $M_n(V^*)$ determines the norms on V

$$\|v\| = \sup\{\|f_n(v)\|,\ f \in CB(V, M_n),\ \|f\|_{cb} \le 1\}$$
$$= \sup\{\|\langle\langle f, v\rangle\rangle\| \mid f \in CB(V, M_n),\ \|f\|_{cb} \le 1\}$$

- therefore we have matrix norms on V
- Show that they det. an operator space structure

$f \in M_m(V^*)$, $\alpha \in M_{n,m}$, $\beta \in M_{m,n}$

$$\|(\alpha f \beta)_p\| = \|(\alpha \otimes 1_p) f_p (\beta \otimes 1_p)\|$$
$$\le \|\alpha\| \|f\|_{cb} \|\beta\| \quad R_2$$

- R_1 works similarly

Moreover V^* is complete $\Rightarrow$ all $M_n(V^*)$ are complete

$\square$

$\Rightarrow$ the dual of an OS is an OS

$\iota_V : V \longrightarrow V^{**}$ canonical inclusion

$\iota_V(v)(f) = f(v)$

$\iota_V(v) \in CB(V^*, \mathbb{C})$

Dual spaces & mapping spaces

Define the dual space of an OS V

$$V^* = CB(V, \mathbb{C})$$

to show that V^* is an OS we need matrix norms

$M_n(V^*)$

$f = [f_{ij}] \in M_n(V^*)$

f determines a lin map $V \to M_n(\mathbb{C})$

$v \mapsto [f_{ij}(v)]$

isomorphism

$$M_n(V^*) = CB(V, M_n)$$

can be used to construct norms on $M_n(V^*)$

$$\Rightarrow M_n(V^*) \neq (M_n(V))^*$$

Fact. the norm on $M_n(V)$ determines that on $M_n(V^*)$

$\forall f \in M_n(V^*)$

$$\|f\| = \sup\{\|f_n(v)\| \mid v \in M_n, \|v\| \leq 1\}$$

$$= \sup\{\|\langle\langle f, v\rangle\rangle\| \mid v \in M_n(V), \|v\| \leq 1\}$$

matrix pairing

Conjugate linear spaces

let V a lin space, (W, J) is a conj space for V if $J : V \to W$ is a conj lin isomorphism.

(W, J) and (W', J') both conj to V

$\Rightarrow \exists$ isomorphism btw W, W'

$$\begin{tikzcd} & V \arrow[dl, "J"'] \arrow[dr, "J'"] & \\ W \arrow[rr, "\cong"] & & W' \end{tikzcd}$$

we will talk about THE conj lin space

denote $\overline{V} = J(V)$

let V be an op space, $\overline{V}$ the conj lin space

$$\|[\overline{v_{ij}}]\| = \|\overline{[v_{ij}]}\| = \|[v_{ij}]\|$$

this gives an o.s. structure on $\overline{V}$

$V \subset B(\mathcal{H})$, take $\overline{\mathcal{H}}$ conj H.S.

$\forall\, T \in B(\mathcal{H})$, define $\overline{T} \in B(\overline{\mathcal{H}})$

$$\overline{T}(\overline{v}) = \overline{T(v)}$$

$$\overline{\pi} : \overline{V} \to B(\overline{\mathcal{H}})$$

$$\overline{\pi}(\overline{v}) = \overline{\pi(v)}$$

If V is an op space with closed subspaces

$V_1 \subseteq V_2 \subseteq \cdots$

st $\overline{\bigcup V_n}^{\text{norm}}$ closure in V

$\Rightarrow V$ is a direct limit.

eg $V_n = M_n(\mathbb{C})$

$M_n(\mathbb{C}) \longrightarrow M_{n+1}(\mathbb{C})$

$v \longmapsto v \oplus 0$

$\underrightarrow{\lim}\, M_n(\mathbb{C}) = K(\mathcal{H})$

We will use the asymptotic product os.

V_n $\quad \varphi_{n,\infty} : V_n \longrightarrow \prod_{n \in \mathbb{N}} V_n$

$$\downarrow \pi_\infty$$

$$\prod_{n\in\mathbb{N}} V_n / \sum_{n\in\mathbb{N}} V_n$$

$v \longmapsto \pi_\infty(\underbrace{0, \ldots, 0}_{n-1}, v, \varphi_n(v), \varphi_{n,n+2}(v) \ldots)$

$\overline{\bigcup_{n\in\mathbb{N}} \varphi_{n,\infty}(V_n)}^{\|\,\|}$ with $\varphi_{n,\infty}$ defined as above,

$\cap$

$\prod V_n / \sum V_n$

is a direct limit.

let $v = (v_n) \in M_m\left(\prod_{n\in\mathbb{N}} V_n\right)$

$$\| (\pi_\infty)_m (v_m) \| = \limsup_{n\to\infty} \| v_n \|$$

From now on assume all op spaces are norm complete

$$V_1 \xhookrightarrow{\varphi_1} V_2 \xhookrightarrow{\varphi_2} V_3 \hookrightarrow \cdots \quad 4.$$

$\varphi_i : V_i \to V_{i+1}$ is a compl isometric embedding

A direct limit op space $\varinjlim V$ is an op. space V together with

$\varphi_{n,\infty} : V_n \hookrightarrow V$ comp iso injection $\forall n \in \mathbb{N}$

st

$$\begin{tikzcd} V_n \arrow[r, hook, "\varphi_{n,\infty}"] \arrow[dr, "\varphi_n"'] & V \\ & V_{n+1} \arrow[u, "\varphi_{n+1,\infty}"'] \end{tikzcd}$$

① $\varphi_{n,\infty} = \varphi_{n+1,\infty} \circ \varphi_n \ \forall n$

② $\bigcup \varphi_{n,\infty}(V_n)$ norm dense inside V

$\forall n < m$

$\varphi_{n,m} = \varphi_{m-1} \circ \cdots \circ \varphi_{n+1} \circ \varphi_n : V_n \to V_m$

$\Rightarrow \forall n,m \ n<m \quad \varphi_{n,\infty} = \varphi_{m,\infty} \circ \varphi_{n,m}$

$S = \mathbb{N}$

Let $V = (V_n)_{n \in \mathbb{N}}$ a sequence of op. spaces. $\prod_{n \in \mathbb{N}} V_n$ is well-defined.

$$\sum V_n = \{ (v_n) \in \prod V_n \mid \lim_{n \to \infty} \|v_n\| = 0 \}$$

"Co-ordinate sum"

$\sum V_n$ closed subspace of $\prod_{n \in \mathbb{N}} V_n$

Use the quotient construction to define the asymptotic product OS.

$$\prod_{n \in \mathbb{N}} V_n \Big/ \sum_{n \in \mathbb{N}} V_n \xleftarrow{\pi_\infty} \prod_{n \in \mathbb{N}} V_n$$

PROP (Appendix) $(V_n)_{n \in \mathbb{N}}$ seq of Banach spaces

$$\pi_\infty : \prod_{n \in \mathbb{N}} V_n \to \prod V_n / \sum V_n$$

$$\| \pi_\infty(v) \| = \limsup_{n \to \infty} \| v_n \| \quad \forall v \in \prod_{n \in \mathbb{N}} V_n$$

Products of operator spaces

Family $(V_s)_{s\in S}$ of op. spaces indexed by S

$$\prod_{s\in S} V_s = \ell_\infty(S, V_s)$$

$$= \{(x_s) \mid x_s \in V_s \ \forall s, \ \sup \|x_s\| < \infty\}.$$

Matrix norms come from

$$M_n\Big(\prod_{s\in S} V_s\Big) = \prod_{s\in S} M_n(V_s)$$

$S = \{1 \ldots n\}$

$V_1 \oplus \ldots \oplus V_n$

If each $V_i \underset{\text{comp. isom. embedding}}{\hookrightarrow} B(\mathcal{H}_i)$

$$\Rightarrow V \hookrightarrow B(\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n)$$

Also true for general family

$\forall s \ V_s \hookrightarrow B(\mathcal{H}_s)$

$$\prod_{s\in S} V_s \hookrightarrow B\Big(\bigoplus_{s\in S} \mathcal{H}_s\Big)$$

③ $N \subseteq V$ is a closed subspace of an O.S. V

then $M_n(N) \subseteq M_n(V)$ closed.

$$M_n(V)/M_n(N) = M_n(V/N)$$

$$(*) \quad \|\tilde{v}\| = \inf\{\|v\| \mid v \in M_n(V),\ \pi(v) = \tilde{v}\}$$

$$\pi : M_n(V) \to M_n(V/N) = M_n(V)/M_n(N).$$

<u>Prop</u> If $N \subseteq V$ closed subspace of an operator space, then $(*)$ defines an operator space str on V/N.

<u>Pf</u> Use (R_1'), (R_2) and ε's

$\|\tilde{v}\|$ is defined as inf.

$\exists v$ st $\pi(v) = \tilde{v}$

$\|v\| < \|\tilde{v}\| + \varepsilon$

$\pi_n(\alpha v \beta) = \alpha \tilde{v} \beta$

$\|\alpha v \beta\| \leq \|\alpha\| (\|\tilde{v}\| + \varepsilon) \|\beta\|$ ✓

& similarly $\|v \oplus w\| \leq \max\{\|v\|, \|w\|\}$

SUBSPACES & QUOTIENTS

let V be a fixed op space

$W \subseteq V$ is a subspace

$M_n(W) \subseteq M_n(V)$

restrict the matrix norm on V to a matrix norm on W

$\Rightarrow$ subspaces of op. spaces are op. spaces

$\alpha \in M_{n,n'}$, $\beta \in M_{m,m'}$

$\alpha \otimes \beta \in M_n$

② If V is an operator space, then $M_p(V)$ is an op. space $\forall p \in \mathbb{N}$

Use $\underline{M_n(M_p(V))} \underset{\uparrow}{\simeq} \underline{M_{pn}(V)}$

as linear spaces

norms on $M_{pn}(V)$'s $\rightarrow$ norms on $M_n(M_p(V))$.

(R_1') $(\leq \max)$ ✓

as $mp \times mp$ matrix with entries in V

$\|\alpha v \beta\|_n = \|(\alpha \otimes 1_p) v (\beta \otimes 1_p)\| \leq \|\alpha \otimes 1_p\| \, \|v\| \, \|\beta \otimes 1_p\|$

R_2 in V

$\leq \|\alpha\| \, \|v\| \, \|\beta\|$

$v \in M_m(M_p(V))$

$\alpha \in M_{n,m}$, $\beta \in M_{m,n}$

$$t(\pi^{-k}) = (\pi^{-k})^{tr} = (\pi^{-k})^* = \pi^k$$

$$t(\alpha) = \sum_{k=0}^{n-1} \pi^k D_n(\alpha \pi^{-k})$$

$$\|t\|_{cb} \le \sum \|\alpha \mapsto \pi^k D_n(\alpha \pi^{-k})\| \le n$$

this bound is attained

(use column matrices that $\varepsilon_{i1}^{[n,1]}$)

$$\begin{tikzcd} M_n \arrow[r, "t"] \arrow[d] & M_n \arrow[d] \\ K_\infty \arrow[r, "t"] & K_\infty \end{tikzcd}$$

completely isom embeddings

$\square$

the transpose mapping

let $v \in M_n(V)$ define

$$t: M_n(V) \to M_n(V)$$

$$(t(v))_{ij} = v_{ji}$$

Prop for $n < \infty$ the transpose matrix

$t: M_n \to M_n$ is an isometry

$\|t\|_{cb} = n$

$t: K_\infty \to K_\infty$ is isometric

but not completely bounded

- $t : M_n(V) \longrightarrow M_n(V)$ is isometric.

$\xi = (\xi_i)_{i=1}^n$ and $\eta = (\eta_i)_{i=1}^n$ unit vectors in $\mathbb{C}^n$.

$$\langle \beta \xi, \eta \rangle = \langle \beta^t \bar{\xi}, \bar{\eta} \rangle$$

$\bar{\xi}, \bar{\eta}$ conjugates of ξ & η.

$\alpha \in M_n(V)$

$$\alpha = \begin{bmatrix} \alpha_{11} & & 0 \\ & \alpha_{22} & \\ 0 & & \alpha_{nn} \end{bmatrix} + \begin{bmatrix} 0 & \alpha_{12} & & \\ & 0 & \alpha_{23} & \\ & & & \ddots \\ \alpha_{n1} & & & \end{bmatrix}$$

$$+ \cdots + \begin{bmatrix} 0 & & & \alpha_{1n} \\ \alpha_{21} & 0 & & \\ & & \ddots & \\ & & \alpha_{n,n-1} & 0 \end{bmatrix}$$

$$\pi := \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & \end{pmatrix}$$

$$\alpha = \sum_{k=0}^{n-1} D_n(\alpha \pi^k) \pi^{-k}$$

$$D_n : M_n(V) \to M_n(V) \quad \begin{bmatrix} v_{11} & v_{12} & \\ & & \end{bmatrix} \mapsto \begin{bmatrix} v_{11} & & 0 \\ & v_{22} & \\ 0 & & \ddots \end{bmatrix}$$

fact. axiom R_1 can be replaced

by (R_1') $\|v \oplus w\|_{m+n} \leq \max\{\|v\|_m, \|w\|_n\}$

<u>PROP</u>. Suppose that V linear space with

mappings

$\|\cdot\|_n : M_n(V) \to [0,\infty)$

that satisfy (R_1') and (R_2)

then the $\|\cdot\|_m$ are SEMINORMS on V

if in addition $\|\cdot\|_1$ is a <u>norm</u> then there

def an operator space structure.

<u>the transpose mapping</u>

let $v \in M_n(V)$ define

$t : M_n(V) \to M_n(V)$

$(t(v))_{ij} = v_{ji}$

<u>Prop</u>. for $n < \infty$ the transpose matrix

$t : M_n \to M_n$ is an isometry

$\|t\|_{cb} = n$

$t : K_\infty \to K_\infty$ is isometric

but <u>not</u> completely bounded

Recall.

Def. An abstract operator space

V linear space +

matrix norm $(\|\cdot\|_m \ \forall M_n(V))$

st

$R_1)$ $\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$

$R_2)$ $\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_m \|\beta\|$

$\forall v \in M_m(V),\ w \in M_n(V),$

$\alpha \in M_{nm},\ \beta \in M_{mn}$

$(V, \|\cdot\|_1)$ is a Banach space

$B(\mathcal{H})$ is an operator space

Representation thm Every abstract operator space V can be represented on a Hilbert space, i.e. $\exists\ \mathcal{H}_V$ Hilbert space

$\varphi: V \to B(\mathcal{H}_V)$ complete isometry

Rem. right 'morphisms' are completely bounded maps ($\Rightarrow$ bndd)

$\|\varphi\|_{cb} = \sup \|\varphi\|_n < \infty$

φ_n is an isometry $\forall n$ $\quad \varphi: V \to M_n$