The representation theorem for operator spaces

Let S be a set, the $\ell^\infty(S) = \{ f : S \to \mathbb{C} \mid \sup_{s \in S} |f(s)| \}$ is a commutative C^*-algebra.

Prop. Let V be a normed space. Then there is a set S and a linear isometry $V \hookrightarrow \ell^\infty(S)$.
In particular, if V is a Banach space then V is isometrically isomorphic to a closed subspace of the commutative C^*-algebra $\ell^\infty(S)$.

Proof. $S = V^*_{\leq 1} = \{ f : V \to \mathbb{C} \mid \|f\| \leq 1 \}$

$\Phi : V \to \ell^\infty(S)$

$\Phi(v)(f) = f(v)$

$$\|\Phi(v)\| = \sup_{\|f\| \leq 1} |\Phi(v) f| = \sup_{\|f\| \leq 1} |f(v)| = \|v\| \quad \forall v \in V$$

(By Hahn-Banach $\exists f : V \to \mathbb{C}$ $\|f\| = 1$ with $|f(v)| = \|v\|$. □

Observation: for a Banach V, there is no canonical norm on V^n.

Examples of norms on V^n:

$$\|(v_j)_{j=1}^n\|_\infty = \max\{\|v_j\| \mid 1 \le j \le n\}$$

$$\|(v_j)\|_p = \left(\sum \|v_j\|^p\right)^{1/p} \qquad p \ge 1.$$

Suppose $V \subset B(H)$ (H Hilbert space) is closed subspace. Then since $B(H^n) \simeq M_n(B(H))$, there is a canonical norm on each $M_n(V) \subset M_n(B(H))$

Def. (Abstract operator space) Let V be a linear space. A matrix norm on V is a sequence of norms $\|\cdot\|_n$ ($n \in \mathbb{N}$) where $\|\cdot\|_n$ is a norm on $M_n(V)$, such that for all $v \in M_m(V)$ and $w \in M_n(V)$ we have

- $\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$, $v \oplus w = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$

- for $\alpha \in M_{n,m}(\mathbb{C})$, $\beta \in M_{m,n}(\mathbb{C})$ and $v \in M_m(V)$ then $\|\alpha \cdot v \cdot \beta\|_n \le \|\alpha\| \|v\|_m \|\beta\|$, $M_{n,m}(\mathbb{C}) \subset M_{\max\{n,m\}}(\mathbb{C})$

An operator space is a matrix normed space V that is a Banach space for $\|\cdot\|_1$. C^*-norm

The maps

$$M_n(V) \to M_{n+m}(V)$$
$$T \mapsto \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$$

are isometric.

Ex. If A is a C^*-algebra and $V \subset A$ a closed subspace then the inclusions $M_n(V) \subset M_n(A)$ determine an operator space structure on V via the C^*-norm on $M_n(A)$.

Lemma: If V is an operator space then $M_n(V)$ is a Banach space in the norm $\|\cdot\|_n$.

- $M_n(A)$ as operators on A^n where A^n has the norm $\|(a_j)\|_2 := \|\sum_{j=1}^{n} a_j^* a_j\|^{1/2}$
- Choose faithful $\pi: A \to B(H)$ and represent

$$M_n(A) \to B(H^n) \; (\simeq M_n(B(H)))$$
$$(a_{ij}) \mapsto (\pi(a_{ij}))$$

Ex. H, K are Hilbert spaces then $B(H,K)$ is an operator space via the embedding $B(H,K) \hookrightarrow B(H \oplus K)$ via $b \mapsto \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$

$$\|(a_{ij})\|_2 = \left\| \begin{pmatrix} a_1 & & \\ a_2 & & \\ a_3 & 0 & \\ \vdots & & \\ a_n & & \end{pmatrix} \right\|_{M_n(A)}$$

$$\left\| \begin{pmatrix} a_1 & & \\ a_2 & & \\ \vdots & 0 & \\ a_n & & \end{pmatrix} \right\|^2_{M_n(A)} = \left\| \begin{pmatrix} a_1 & \\ \vdots & 0 \\ a_n & \end{pmatrix}^* \begin{pmatrix} a_1 & \\ \vdots & 0 \\ a_n & \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} a_1^* & \cdots & a_n^* \\ & 0 & \end{pmatrix} \begin{pmatrix} a_1 & \\ \vdots & 0 \\ a_n & \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sum a_j^* a_j & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 0 & \\ 0 & & & \end{pmatrix} \right\|$$

$$(a_{ij}) \cdot \begin{pmatrix} x_1 & \\ \vdots & 0 \\ x_n & \end{pmatrix} = \begin{pmatrix} \sum a_{1j} x_j & \\ \vdots & 0 \\ \sum a_{nj} x_j & \end{pmatrix}$$

<u>Proof.</u> Let $E_k = (0 \cdots \underset{\substack{\uparrow \\ k\text{th entry}}}{1} 0 \cdots 0) \in M_{1,n}(\mathbb{C})$ (row vector)

Then $E_k^* = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ and for $v \in M_n(V)$ we have that

$v = (v_{ij})_{ij}$ and $v_{ij} = E_i \cdot v \cdot E_j^*$ hence

$$\|v_{ij}\| = \|E_i \cdot v \cdot E_j^*\| \le \|E_i\|\|v\|\|E_j^*\| = \|v\|_n$$

$$\|v\| = \Big\|\sum_{i,j} E_i^* v_{ij} E_j\Big\| \le \sum \|E_i^*\|\|v_{ij}\|\|E_j\| = \sum \|v_{ij}\|$$
$$\le n^2 \|v\|_n$$

Maps between operator spaces:

V, W operator spaces, $\varphi: V \to W$ linear.

Write $\varphi_n : M_n(V) \longrightarrow M_n(W)$

$$(v_{ij}) \longmapsto (\varphi(v_{ij}))$$

Then $\|\varphi\|_n := \|\varphi_n\|_{op(M_n(V) \to M_n(W))}$

and $\|\varphi\|_{cb} := \sup_n \|\varphi\|_n$.

Since $M_n(V) \hookrightarrow M_{n+1}(V)$, $M_n(W) \hookrightarrow M_{n+1}(W)$ isometrically it holds that $\|\varphi\|_n \leq \|\varphi\|_{n+1}$

Def. $\varphi: V \to W$ is completely bounded if $\|\varphi\|_{cb} < \infty$.

$CB(V, W) = \{\varphi: V \to W \mid \|\varphi\|_{cb} < \infty\}$

φ is a complete isometry if each φ_n is an isometry

φ is a complete isomorphism if φ^{-1} exists and $\|\varphi\|_{cb}, \|\varphi^{-1}\|_{cb} < \infty$

Goal: For an operator space V, construct a Hilbert space $H (= H_V)$ and a complete isometry $\varphi: V \to B(H)$.

Lemma: Let $m, n \in \mathbb{N}$, $m \geq n$ and $\eta \in \mathbb{C}^m \otimes \mathbb{C}^n \, (\cong \mathbb{C}^{mn})$

There exists an isometry $\beta : \mathbb{C}^n \to \mathbb{C}^m$ and $\tilde{\eta} \in \mathbb{C}^n \otimes \mathbb{C}^n$ such that

$$(\beta \otimes I_n)(\tilde{\eta}) = \eta.$$

"proof": $\eta = \sum_{j=1}^{n} \eta_j \otimes e_j$ (e_j basis for $\mathbb{C}^n$) $\eta_j \in \mathbb{C}^m$

then $E := \operatorname{span}\{\eta_j\} \subset \mathbb{C}^m$ has dimension <u>at most</u> $n \leq m$. So there is an isometry $E \cong \mathbb{C}^n \hookrightarrow \mathbb{C}^m$ and vectors $\tilde{\eta}_j \in \mathbb{C}^n$ $\beta(\tilde{\eta}_j) = \eta_j$

and then $\tilde{\eta} = \sum_{j=1}^{n} \tilde{\eta}_j \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^n$ satisfies

$$(\beta \otimes I_n)(\tilde{\eta}) = \eta. \quad \square$$

Prop. Let V be an operator space and $\varphi : V \to M_n(\mathbb{C})$ a linear map. Then

$$\|\varphi\|_{cb} = \|\varphi\|_n \quad \left(\|\varphi\|_n := \|\varphi_n\| \quad \varphi_n : M_n(V) \to M_n(M_n(\mathbb{C})) \cong M_{n^2}(\mathbb{C}) \right)$$

Proof. It suffices to show that $\|\varphi\|_m \leq \|\varphi\|_n$ for $m \geq n$. Let $\varepsilon > 0$ and choose $v \in M_m(V)$ with $\|\varphi\|_m - \varepsilon < \|\varphi_m(v)\|$, $\|v\| = 1$. Now $\varphi_m(v) \in M_{mn}(\mathbb{C}) = B(\mathbb{C}^m \otimes \mathbb{C}^n)$

Therefore

$$\|\varphi_m(v)\| = \sup_{\substack{\xi,\eta \in \mathbb{C}^m \otimes \mathbb{C}^n \\ \|\xi\|, \|\eta\| \le 1}} |\langle \varphi_m(v)\xi, \eta \rangle|$$

Thus $\exists\, \xi, \eta$, $\|\xi\| = \|\eta\| = 1$

with $\|\varphi_m\| - \varepsilon < |\langle \varphi_m(v)\xi, \eta \rangle|$

and isometries $\alpha, \beta : \mathbb{C}^n \to \mathbb{C}^m$ and $\tilde{\xi}, \tilde{\eta} \in \mathbb{C}^n$

with $\|\tilde{\xi}\| = \|\tilde{\eta}\| = 1$, such that $\xi = (\alpha \otimes I_n)(\tilde{\xi})$

(by previous lemma) $\eta = (\beta \otimes I_n)(\tilde{\eta})$

Hence $\|\varphi\|_m - \varepsilon < |\langle \varphi_m(v)(\alpha \otimes I_n)(\tilde{\xi}), (\beta \otimes I_n)(\tilde{\eta}) \rangle|$

$$= |\langle (\beta^* \varphi_m(v) \alpha)(\tilde{\xi}), \tilde{\eta} \rangle$$

$$= |\langle \varphi_n(\beta^* v \alpha)(\tilde{\xi}), \tilde{\eta} \rangle$$

$$\le \|\varphi_n(\beta^* v \alpha)\| \, \|\tilde{\xi}\| \, \|\tilde{\eta}\|$$

$$\le \|\varphi_n\| \, \|\beta^*\| \, \|v\| \, \|\alpha\|$$

$$= \|\varphi_n\|$$

Now, $\varepsilon > 0$ was arbitrary, and $\|\varphi\|_m \le \|\varphi\|_n$ $\square$

Cor. Let $f : V \to \mathbb{C}$ be a continuous linear functional.

Then $\|f\|_{cb} = \|f\|$.

Prop. Let V be an operator space and A a commutive C^*-algebra. Then any bounded linear map $\varphi: V \to A$ satisfies $\|\varphi\| = \|\varphi\|_{cb}$.

$(A = C_0(\Omega))$

Remarks. If $\mu \in M_n(\mathbb{C})$ is unitary and $v \in M_n(V)$ then $\|\mu \cdot v\| = \|v \cdot \mu\| = \|v\|$

$\|\mu \cdot v\| \leq \|\mu\| \|v\|$ $\quad \|v\| = \|\mu^* \mu v\| \leq \|\mu^*\| \|\mu v\| = \|\mu v\|$.

- If $v \in M_n(V)$ and $\alpha \in M_p(\mathbb{C})$ then $\|v \otimes \alpha\|_{pn} = \|\alpha \otimes v\|_{pn} = \|v\| \|\alpha\|$

(Use $\alpha = |\alpha| \mu$ polar decomposition)

Cor. Let $v \in V$, $\|v\| = 1$, then $\Theta_v : \mathbb{C} \to V$, $\lambda \mapsto \lambda v$ is a complete isometry.

Proof. $\|(\Theta_v)_n(\alpha)\| = \|\alpha \otimes v\| = \|\alpha\| \|v\| = \|\alpha\|$

Cor. Let V, W be operator spaces with either $\dim V = n$ or $\dim W = n$. Then any map $\varphi: V \to W$ satisfies $\|\varphi\|_{cb} \leq n \|\varphi\|$

Proof. Suppose $\dim W = n$, choose a basis $w_1, \ldots, w_n$ with $\|w_i\| = 1$ with dual basis $g_i \in W^*$, $\|g_i\| = 1$, $g_i(w_j) = \delta_{ij}$.

Then $Id_W = \sum_{i=1}^{n} \Theta_{w_i} \circ g_i$ and $\varphi = \sum_{i=1}^{n} \Theta_{w_i} \circ g_i \circ \varphi$

Hence

$$\|\varphi\|_{cb} = \Big\|\sum_{i=1}^{n} \theta_{w_i}\circ g_i \circ \varphi\Big\|_{cb}$$
$$\le \sum_{i=1}^{n} \|\theta_{w_i}\|_{cb}\,\|g_i\circ\varphi\|_{cb}$$
$$\le \sum_{i=1}^{n} \|g_i\circ\varphi\| \le n\|\varphi\|. \ \square$$

If $\dim V = n$, replace W by $\varphi(V)$ $\square$

The representation theorem.

Need "matrix version" of Hahn-Banach:

* "For every $v \in M_n(V)$ $\exists\, \varphi : V \to M_n(\mathbb{C})$ such that φ is a complete contraction and $\|v\| = \|\varphi_n(v)\|$."

We will achieve this by showing that every $F \in M_n(V)^*$ factors as

$F(v) = \langle \varphi_n(v)\xi, \eta\rangle$, where $\varphi : V \to M_n\mathbb{C}$ a complete contraction and $\xi, \eta \in \mathbb{C}^n \otimes \mathbb{C}^n$ are vectors.

Once we have * we can consider the spaces $S_n(V) = CB(V, M_n(\mathbb{C}))_{\le 1}$ and the Hilbert space

$$H = \bigoplus_{n\in\mathbb{N}} \bigoplus_{\varphi\in S_n(V)} \mathbb{C}^n$$

and the map

$\Phi : V \to B(H)$ given by $v \mapsto (\varphi(v))_{\varphi\in S(V)}$

where $S(V) = \bigcup_k S_k(V)$. Then

$\Phi_n : M_n(V) \to B(H^n)$ is given by

$$v \mapsto (\varphi_n(v))_{\varphi \in S(V)}$$ and

$\|\Phi_n(v)\| \leq \|v\|$ by construction for all v,

$\|\Phi_n(v)\| \geq \|v\|$ for $v \in M_n(V)$ by the matrix Hahn-Banach theorem.

$\delta : \mathcal{A} \to A$ $\quad \partial(ab) = \partial(a)b + a\partial(b)$ $\quad \|f\| + \|\partial f\|$

$\frac{\partial}{\partial x} : C^1(S^1) \to C(S^1)$ $\quad \begin{pmatrix} f & 0 \\ \partial f & f \end{pmatrix} \in M_2(C(S^1))$

$$\begin{pmatrix} f & 0 \\ \partial f & f \end{pmatrix} \cdot \begin{pmatrix} g & 0 \\ \partial g & g \end{pmatrix} = \begin{pmatrix} fg & 0 \\ (\partial f)g + f\partial g & fg \end{pmatrix}$$

Lemma: Let $F \in M_n(V)^*$, $\|F\| = 1$. Then there exist states p_0 and q_0 on $M_n(\mathbb{C})$ such that for all $\alpha, \beta \in M_n(\mathbb{C})$ and $v \in M_n(V)$ we have

$$|F(\alpha v \beta)| \leq p_0(\alpha\alpha^*)^{1/2} \|v\| \, q_0(\beta^*\beta)^{1/2}$$

Proof. Reduce to proving that $\exists\, p_0, q_0$ such that for $\|v\|=1$

$$\mathrm{Re}(\alpha v \beta) \leq \frac{1}{2}\left(p_0(\alpha\alpha^*) + q_0(\beta^*\beta)\right)$$

Let S_n be the state space of $M_n(\mathbb{C})$ and $K = S_n \times S_n \subset (M_n(\mathbb{C}) \oplus M_n(\mathbb{C}))^*$.

Then K is compact and convex. For $\alpha, \beta \in M_n(\mathbb{C})$, $v \in M_n(V)$ define $\ell_{\alpha,v,\beta} : K \to \mathbb{R}$, $\ell_{\alpha,v,\beta}(p,q) = p(\alpha\alpha^*) + q(\beta^*\beta) - 2\,\mathrm{Re}\,F(\alpha v \beta)$

$e_{\alpha,v,\beta}$ respects convex combinations of states

Moreover if p is such that $p_\alpha(\alpha\alpha^*) = \|\alpha\|^2$ and q is such that $q_\beta(\beta^*\beta) = \|\beta\|^2$ then

$-\ e_{\alpha,v,\beta}(p_\alpha, q_\beta) \geq 0$. Moreover, the

$-$ Set $\mathcal{E} = \{e_{\alpha,v,\beta}\}$ is cone

$t\, e_{\alpha,v,\beta} = e_{t^{1/2}\alpha, v, t^{1/2}\beta}$

$e_{\alpha,v,\beta} + e_{\alpha',v',\beta'} = e_{\alpha'',v'',\beta''}$

Then it follows that $\exists\ p_0, q_0$ Such that $e_{\alpha,v,\beta}(p_0, q_0) \geq 0$ for all $e_{\alpha,v,\beta}$.

Lemma. Let $F \in M_n(V)^*$, $\|F\| = 1$, then $\exists$ complete contraction $\varphi: V \to M_n(\mathbb{C})$ and unit vectors $\xi, \eta \in \mathbb{C}^n \otimes \mathbb{C}^n$ such that $F(v) = \langle \varphi_n(v)\xi, \eta\rangle$.

$S(V) = \bigcup_k S_k(V)$ $\quad S_k(V) = CB(V, M_k(\mathbb{C}))_{\leq 1}$

$\bigoplus_k \bigoplus_{\varphi \in S_k(V)} \mathbb{C}^k$