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Lecture 2 : The representation theorem for operator spaces
(B. Mesland)

Let S be a set, $\ell^\infty(S) = \{ f: S \to \mathbb{C} \mid \sup_{s \in S} |f(s)| \}$

is a commutative C^* algebra. (pointwise structure)

The theory of operator spaces is sometimes said to be a quantisation of the theory of Banach spaces.

Indeed, we have the Hahn-Banach theorem!

PROP 1 Let V be a normed space. Then there exist a set S and a linear isometry $V \to \ell^\infty(S)$.
In particular, if V is a Banach space, then V is isometrically isomorphic to a closed linear subspace of the commutative C^*-algebra $\ell^\infty(S)$.

NB: every closed subspace of a commutative C^*-algebra is trivially a Banach space.

WHAT ABOUT CLOSED SUBSPACES of a N.C. C^*-ALGEBRA?

Proof of PROP 1 Let $S = V^*_{\leq 1} = \{ f: V \to \mathbb{C} \mid \|f\| \leq 1 \}$

$\Phi: V \to \ell^\infty(S)$
$\phi(v)(f) = f(v)$

$\Rightarrow \|\phi(v)\| = \sup_{\|f\| \leq 1} |\phi(v) f| = \sup_{\|f\| \leq 1} |f(v)| = \|v\|$

By the Hahn-Banach thm, $\forall v \in V \; \exists f_v : V \to \mathbb{C}$ s.t
$\|f\| = 1$ and
$\|f(v)\| = \|v\|$ □

Observation: For a Banach space V there is no canonical norm on V^n. They could all be equivalent!

Examples of norms: ∞-norm $\|(v_j)\|_\infty = \max\{\|v_j\| \mid 1 \le j \le n\}$

p-norms $\|(v_j)\|_p = \left(\sum_j \|v_j\|^p\right)^{1/p}$ $p \ge 1$

Suppose, however, $V \subseteq B(H)$ for H a Hilbert space.
Then there is a canonical norm on V^n

$B(H^n) \simeq M_n(B(H))$, there is a canonical norm on each

$M_n(V) \subseteq M_n(B(H))$, hence V^n can be seen as column / row

Def 2 (Abstract operator space).
Let V be a linear space. A matrix norm on V is a sequence of norms $\|\cdot\|_n$ $(n \in \mathbb{N})$ where $\|\cdot\|_n$ is a norm on $M_n(V)$, such that, for all $v \in M_m(V)$ & $w \in M_n(V)$ we have

- $\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$, $v \oplus w = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$

- for $\alpha \in M_{n,m}(\mathbb{C})$, $\beta \in M_{m,n}(\mathbb{C})$ & $v \in M_m(V)$

  $\|\alpha v \beta\|_n \le \|\alpha\| \|v\|_m \|\beta\|$ (module structure)

[* where norms on $M_{n,m}(\mathbb{C}) \subset M_{\max(m,n)}(\mathbb{C})$ C^*-norm (op norm)]

An operator space is a matrix normed space V that is a Banach space for $\|\cdot\|_1$. (complete)

~ an operator space is a Banach space together with norms on each matrix space that satisfy the axioms above.

NB: column embedding: right module structure on V^n
row embedding: left module structure on V^n

Observations: The maps $M_n(V) \to M_{n+m}(V)$

$$(T) \mapsto \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$$ are isometric by the 1st axiom

$\Rightarrow$ all the metric spaces are complete once $(V, \|\cdot\|_1)$ is complete.

EXAMPLE • If A is a C^* algebra and $V \subset A$ is a closed subspace, then the inclusions $M_n(V) \subseteq M_n(A)$ determine an operator space structure via the C^* norm on A

• if H, K are Hilbert spaces, then $B(H,K)$ is an operator space via the embedding $B(H,K) \to B(H \oplus K)$

$$b \mapsto \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

Proof (sketch): let $E_k = (0, .., 0, 1, 0 ...)$ (row vector)
↑ k-th entry $\in M_{1,n}(\mathbb{C})$

and $E_k^* = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$ the corresponding column vector

For every $v \in M_n(V)$ we have $v = (v_{ij})_{ij}$ $\quad v_{ij} = E_i \cdot v \cdot E_j^*$

hence, the norm $\|v_{ij}\| = \|E_i v E_j^*\| \leq \|E_i\| \|v\| \|E_j^*\| = \|v\|$
↑ Axiom 2

the norm of each matrix entry is dominated by the norm of the matrix v, which can be recovered by "multiplying in the other order"

$$\|v\| = \|\sum_{ij} E_j^* v_{ij} E_i\| \leq \sum \|E_j^*\| \|v_{ij}\| \|E_i\| = \sum_{i,j} \|v_{ij}\|$$
$$\leq n^2 \|v\|_n$$

NB: many proofs in the theory of operator spaces boil down to matrix multiplication tricks

MAPS between operator spaces

As you may expect, bounded maps are not enough because we want compatibility with the matrix norm structure

let V, W operator spaces. $\varphi: V \to W$ linear. write

$$\varphi_n: M_n(V) \to M_n(W)$$
$$v_{ij} \to \varphi(v_{ij})$$

Then $\|\varphi\|_n = \|\varphi_n\|_{op: M_n(V) \to M_n(W)}$

and $\|\varphi\|_{cb} := \sup_{n \in \mathbb{N}} \|\varphi\|_n$

Since we have isometric embeddings, $M_n(V) \hookrightarrow M_{n+1}(V)$, $M_n(W) \hookrightarrow M_{n+1}(W)$

it holds that $\|\varphi\|_n \leq \|\varphi\|_{n+1}$

Def 3: $\varphi: V \to W$ is called completely bounded if $\|\varphi\|_{cb} < \infty$

completely bounded $\Rightarrow$ bounded

$CB(V, W) = \{ \varphi: V \to W \mid \|\varphi\|_{cb} < \infty \}$ set of completely bounded maps

1. φ is a complete isometry if each φ_n is an isometry.
2. φ is a complete isomorphism if $\|\varphi\|_{cb}$ & $\|\varphi^{-1}\|_{cb} < \infty$ (φ^{-1} exists and)
3. φ is completely contractive if $\|\varphi\|_{cb} \leq 1$

GOAL: for an operator space V, construct a Hilbert space H_V & a completely bounded isometry $\varphi: V \to B(H_V)$

LEMMA 4 let $m, n \in \mathbb{N}$ and $\eta \in \mathbb{C}^m \otimes \mathbb{C}^n$. there exists an isometry $\beta: \mathbb{C}^n \to \mathbb{C}^m$ & $\tilde{\eta} \in \mathbb{C}^n \otimes \mathbb{C}^n$ s.t

$$(\beta \otimes I_n)(\tilde{\eta}) = \eta$$

Proof: $\eta = \sum \eta_j \otimes e_j$ (e_j basis for $\mathbb{C}^n$)
$\eta_j \in \mathbb{C}^m$

then span $\{\eta_j\} \subset \mathbb{C}^m$ has dimension at most $n \leq m$.

So there is an isometry $E = \mathbb{C}^n \hookrightarrow \mathbb{C}^m$ and vectors $\tilde{\eta}_j \in \mathbb{C}^n$

$\beta(\tilde{\eta}_j) = \eta_j$. & then $\tilde{\eta} = \sum \tilde{\eta}_j \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^n$

satisfies $(\beta \otimes Id)(\tilde{\eta}) = \eta$. □

PROP 5 let V an operator space and $\varphi: V \to M_n(\mathbb{C})$ a linear map.
Then
$\|\varphi\|_{cb} = \|\varphi\|_n$ ⟹ the increasing sequence of norms stabilises at n!

Proof: it suffices to show that $\|\varphi\|_m \leq \|\varphi\|_n$ for $m \geq n$ (the other '≥' is automatic)

let $\varepsilon > 0$ and choose $v \in M_m(V)$ $\|\varphi\|_m - \varepsilon < \|\varphi_m(v)\|$, $\|v\| = 1$
⟶ because this is the sup

Now, $\varphi_m(v) \in M_{nm}(\mathbb{C}) = B(\mathbb{C}^n \otimes \mathbb{C}^m)$.

Therefore, $\|\varphi_m(v)\| = \sup_{\substack{\xi, \eta \in \mathbb{C}^n \otimes \mathbb{C}^m \\ \|\xi\|, \|\eta\| \leq 1}} |\langle \varphi_m(v)\xi, \eta \rangle|$

[↑* $\|T\| = \sup_{\substack{\|\xi\| \leq 1 \\ \|\eta\| \leq 1}} |\langle T\xi, \eta \rangle|$

⟹ there exist ξ, η, $\|\xi\| \leq 1$ $\|\eta\| \leq 1$ s.t

$$\|\varphi_m\| - \varepsilon \leq |\langle \varphi_m(v)\xi, \eta \rangle| \quad \&$$

isometries $\alpha, \beta: \mathbb{C}^n \to \mathbb{C}^m$, $\tilde{\xi}, \tilde{\eta} \in \mathbb{C}^n$ $\|\tilde{\xi}\| = \|\tilde{\eta}\| = 1$, such that $\xi = (\alpha \otimes I_n)(\tilde{\xi})$ & $\eta = (\beta \otimes I_n)(\tilde{\eta})$

by LEMMA 4.

Hence $\|\varphi\|_m - \varepsilon < |\langle \varphi_m(v)(\alpha \otimes I_n)(\tilde{\xi}), (\beta \otimes I_n)(\tilde{\eta})\rangle|$

$$= |\langle (\beta^* \varphi_m(v) \alpha)(\tilde{\xi}), \tilde{\eta} \rangle| =$$

$$= |\langle \varphi_n(\beta^* v \alpha)(\tilde{\xi}), \tilde{\eta} \rangle|$$

$$\leq \|\varphi_n(\beta^* v \alpha^*)\| \|\tilde{\xi}\| \|\tilde{\eta}\| =$$

$$= \|\varphi_n\| \|\beta^*\| \|v\| \|\alpha\| = \|\varphi_n\|$$

ε arbitrary $\Rightarrow$ $\|\varphi\|_m \leq \|\varphi_n\|$

Corollary 6 Let $f: V \to \mathbb{C}$ be a continuous linear functional.
Then $\|f\|_{cb} = \|f\|$

PROP 7 Let V be an operator space and A a commutative C^*-algebra.
Then any bounded linear map $\varphi: V \to A$ satisfies
$\|\varphi\| = \|\varphi\|_{cb}$

Proof sketch Use $A = C_0(\Omega)$ + PROP 5.

Remarks: ⓐ If $\mu \in M_n(\mathbb{C})$ is unitary and $v \in M_n(V)$, then
$\|\mu v\| = \|v \mu\| = \|v\|$

(this follows from the 2nd operator space axiom:
$\|\mu v\| \leq \|\mu\| \|v\|$

$\|v\| = \|\mu^* \mu v\| \leq \|\mu^*\| \|\mu v\| = \|\mu v\|$)

ⓑ If $v \in M_n(C)$ and $\alpha \in M_p(\mathbb{C})$
$\|v \otimes \alpha\|_{pn} = |\alpha \otimes v\|_{pn} = \|v\| \|\alpha\|$ (use $\alpha = |\alpha| \mu$ POLAR DECOMPOSITION)

Corollary 8 Let $v \in V$, $\|v\| = 1$. Then $\theta_v : \mathbb{C} \to V$, $\lambda \mapsto \lambda v$ is a complete isometry.

Proof: $\|(\theta_v)_n(\alpha)\| = \|\alpha \otimes v\| = \|\alpha\|\,\|v\| = \|\alpha\|$

Corollary 9 Let V, W be operator spaces with either $\dim V = n$ or $\dim W = n$. Then any map $\varphi : V \to W$ satisfies $\|\varphi\|_{cb} \leq n \|\varphi\|$.

Proof Suppose $\dim W = n \Rightarrow$ choose a basis of norm-1 vectors $w_1 \dots w_n$, $\|w_i\| = 1$, with dual basis $g_i \in W^*$, $\|g_i\| = 1$,
$g_i(w_j) = \delta_{ij}$

Then $Id_W = \sum_{j=1}^{n} \theta_{w_i} \cdot g_i$ & $\varphi = \sum \theta_{w_j} \cdot g_i \circ \varphi$

Hence,
$$\|\varphi\|_{cb} = \Big\|\sum_{i=1}^{n} \theta_{w_i} \cdot g_i \cdot \varphi\Big\|_{cb}$$
$$\leq \sum \|\theta_{w_i}\|_{cb} \|g_i \cdot \varphi\|_{cb}$$
$$\leq \sum_{i=1}^{n} \|g_i \cdot \varphi\|_{cb} \leq n \|\varphi\|$$

If $\dim V = n \Rightarrow$ replace W by $\varphi(V)$. □

The representation theorem

We need a matrix version of Hahn-Banach.

(*) For every $v \in M_n(V)$, there exists $\varphi : V \to M_n(\mathbb{C})$ such that φ is a complete contraction and $\|v\| = \|\varphi_n(v)\|$.

We will achieve this by showing that every $F \in M_n(V)^*$ factors as $F(v) = \langle \varphi_n(v)\xi, \eta \rangle$ where $\varphi : V \to M_n(\mathbb{C})$ is a complete contraction and $\xi, \eta \in \mathbb{C}^n \otimes \mathbb{C}^n$ vectors.

Once we have (*), we can consider the spaces
$$S_n(V) = CB(V, M_n(\mathbb{C}))_{\leq 1} \quad \& \quad H = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\varphi \in S_n(V)} \mathbb{C}^n$$
$$\Phi : V \to B(H) \qquad v \mapsto (\varphi(v))_{\varphi \in S(V)} \qquad S(V) := \bigcup S_n(V)$$

Then $\Phi_n : M_n(V) \to B(H^n)$ is given by

$v \longmapsto (\varphi_n(v))_{\varphi \in S(V)}$

and $\|\phi_n(v)\| \leq \|v\|$ for all v, by construction.

$\|\Phi_n(v)\| \geq \|v\|$ for $v \in M_n(V)$ by the matrix HB theorem.

LEMMA 10: let V an abstract op. space, let $F \in M_n(V)^*$. $\|F\| = 1$.
Then there exist states p_0 and q_0 on $M_n(\mathbb{C})$ s.t for all α & β in $M_n(\mathbb{C})$ and $v \in M_n(V)$ we have

$$\| F(\alpha v \beta) \| \leq p_0(\alpha\alpha^*)^{1/2} \|v\| q_0(\beta^*\beta)^{1/2}$$

$$\forall \|v\| = 1$$

Proof: reduces to showing that $\forall v$ $\exists$ p_0, q_0 s.t
$Re(\alpha v \beta) \leq \frac{1}{2}(p_0(\alpha\alpha^*) + q_0(\beta^*\beta))$. $(*)$

let S_n be the state space of $M_n(\mathbb{C})$ and $K = S_n \times S_n \subseteq (M_n(\mathbb{C}) \oplus M_n(\mathbb{C}))^*$

Then K is compact & convex.

For $\alpha, \beta \in M_n(\mathbb{C})$, $v \in M_n(V)$, define $e_{\alpha v \beta} : K \to \mathbb{R}$ by

$$e_{\alpha v \beta}(p, q) = p(\alpha\alpha^*) + q(\beta^*\beta) - 2 Re F(\alpha v \beta)$$

$e_{\alpha v \beta}$ respects convex combinations of states

moreover, if p_α is st $p_\alpha(\alpha\alpha^*) = \|\alpha\|^2$ &
q_β is st $q_\beta(\beta^*\beta) = \|\beta\|^2$ } then $e_{\alpha v \beta}(p_\alpha, q_\beta) \geq 0$

Moreover, the set $\mathcal{E} = \{ e_{\alpha, v, \beta} \}$ is a cone.

it follows that there exist p_0, q_0 s.t $e_{\alpha, v, \beta}(p_0, q_0) \geq 0$ $\forall e_{\alpha, v, \beta}$

(lemma about cones on compact convex spaces)

+ Reduction argument leading to the inequality (*)

Lemma 11 let $F \in M_n(V)^*$, $\|F\| = 1$ then there exist a complete contraction $\varphi : V \to M_n(\mathbb{C})$

and unit vectors $\xi, \eta \in \mathbb{C}^n \otimes \mathbb{C}^n$ st $F(v) = \langle \varphi_n(v) \xi, \eta \rangle$

Idea: use the GNS rep of the 2 states p_0, q_0
⇒ cyclic vectors ⇒ existence of $\varphi : V \to M_n(\mathbb{C})$

Question $S(V) = \bigcup S_k(V)$ $S_k(V) = CP(V, M_k(\mathbb{C})) \leq 1$

if V separable as B. space ⇒ $\bigoplus_{k \in \mathbb{N}} \bigoplus_{\varphi \in S_k(V)} \mathbb{C}^k$

$\simeq \ell^2$ space.

*

?= How to prove this ?