① Tensor products of linear spaces (over $\mathbb{C}$)
② Polar decomposition
③ Schatten p-classes
④ Matrices of operators

① Tensor products

Device for converting multilinear phenomena into linear ones

Def: Let $X_1, \dots, X_n, Y$ be linear spaces.
A map $f: X_1 \times \cdots \times X_n \longrightarrow Y$ is multilinear if

$$f(x_1, \dots, x_{k-1}, \lambda x_k' + \mu x_k'', x_{k+1}, \dots, x_n)$$
$$= \lambda f(x_1, \dots, x_k', \dots, x_n) + \mu f(x_1, \dots, x_k'', \dots, x_n)$$

for all $x_i \in X_i$, $x_k', x_k'' \in X_k$, $\lambda, \mu \in \mathbb{C}$

Def. An algebraic tensor product of $X_1, \dots, X_n$ is a pair (Z, p), where Z is a linear space, and $p: X_1 \times \cdots \times X_n \longrightarrow Z$ is multilinear, satisfying the following universal property:

For any linear space Y and multilinear map $f: X_1 \times \cdots \times X_n \longrightarrow Y$, there is a unique linear map $h: Z \longrightarrow Y$ such that $f = h \circ p$

$$\begin{tikzcd}[column sep=small]
 & X_1 \times \cdots \times X_n \arrow[dl, "p"', "\text{multilinear}"] \arrow[dr, "f", "\text{multilinear}"'] & \\
Z \arrow[rr, dashed, "h", "\text{linear}"'] & & Y
\end{tikzcd}$$

Notation. Denote $p(x_1, \ldots, x_n)$ by $x_1 \otimes \cdots \otimes x_n$

Such elements are called <u>elementary tensors</u>

$$\begin{tikzcd}
 & X_1 \times \cdots \times X_n \arrow[dl, "p"'] \arrow[d, "p'"] \arrow[dr, "p"] & \\
Z \arrow[r, dashed, "h"] & Z' \arrow[r, dashed, "h'"] & Z
\end{tikzcd}$$

Thm. Given linear spaces $X_1, \ldots, X_n$, an algebraic tensor product (Z, p) exists

Each $t \in Z$ can be written as $\sum_{j=1}^{m} x_{1j} \otimes \cdots \otimes x_{nj}$ with $x_{kj} \in X_k$

If (Z', p') also has the universal property, then there is an inverse pair of unique linear maps $h : Z \to Z'$, $h' : Z' \to Z$ such that $p' = h \circ p$, $p = h' \circ p'$

Pf:. Let Z' be the linear space of all formal linear combinations of elements of $X_1 \times \cdots \times X_n$

Let Z'' be the linear subspace of Z' spanned by elements of the form

$$(x_1, \ldots, x_{k-1}, x_k' + x_k'', x_{k+1}, \ldots, x_n) - (x_1, \ldots, x_k', \ldots, x_n) - (x_1, \ldots, x_k'', \ldots, x_n)$$

and $(x_1, \ldots, x_{k-1}, \lambda x_k, x_{k+1}, \ldots, x_n) - \lambda (x_1, \ldots, x_k, \ldots, x_n)$, $\lambda \in \mathbb{C}$

Let $Z = Z'/Z''$ be the quotient space

$$p : X_1 \times \cdots \times X_n \to Z, \quad p(x_1, \ldots, x_n) = (x_1, \ldots, x_n) + Z'' =: x_1 \otimes \cdots \otimes x_n$$

- Given multilinear $f : X_1 \times \cdots \times X_n \to Y$, define $h : Z \to Y$ by $h(x_1 \otimes \cdots \otimes x_n) = f(x_1, \ldots, x_n)$
- If (Z', p') also has the universal property, then there are unique linear maps $h : Z \to Z'$, $h' : Z' \to Z$ such that $p' = h \circ p$ and $p = h' \circ p'$

The maps h and h' are inverses of each other by uniqueness of the identity maps on Z and Z'

$\square$

Given a complex linear space V,
we have linear isomorphisms

$$V\oplus\cdots\oplus V := V^n \cong \mathbb{C}^n\otimes V \cong V\otimes\mathbb{C}^n$$

$$(v_i) \longmapsto \sum_{i=1}^{n} \varepsilon_i\otimes v_i \quad , \quad \varepsilon_i = (0,\dots,0,\overset{i\text{th coordinate}}{1},0,\dots,0)$$

Each linear map $\varphi: V\to W$ determines a linear map

$$\begin{array}{ccccc} \varphi^n : & V^n & \longrightarrow & W^n, & (v_i)\mapsto(\varphi(v_i)) \\ & \cong & & \cong & \\ id\otimes\varphi : & \mathbb{C}^n\otimes V & \longrightarrow & \mathbb{C}^n\otimes W, & \xi\otimes v\mapsto \xi\otimes\varphi(v) \end{array}$$

For $m,n\in\mathbb{N}$, $\quad M_{m,n}(V)\cong M_{m,n}\otimes V \cong V\otimes M_{m,n}$

$$[v_{ij}] \longmapsto \sum \varepsilon_{ij}\otimes v_{ij} \quad , \quad \varepsilon_{ij} \text{ matrix unit}$$

$$[\alpha_{ij}v_0] \longleftarrow \underset{[\alpha_{ij}]}{\underset{\shortparallel}{\alpha}}\otimes v_0$$

$$M_p\otimes M_q \cong M_{pq}$$

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② Polar decomposition (cf. Thm 1.2.1 for finite dim case)

Def: A bounded linear operator T on a Hilbert space H is <u>positive</u> if $\langle Tx, x\rangle \geq 0$ for all $x \in H$

Note: For any $T \in B(H)$, T^*T is positive.

Fact: Every positive operator on H is of this form.

Fact: Every positive operator $T \in B(H)$ has a unique positive square root, ie, $S \in B(H)$ that is positive and $S^2 = T$

Write $|T|$ for $(T^*T)^{\frac{1}{2}}$, $T \in B(H)$

Def: A bounded linear map $U: H_1 \to H_2$ between Hilbert spaces is a <u>partial isometry</u> if U is isometric on $\ker(U)^\perp$

Thm: Let $T \in B(H)$.
such that

Given a linear space V, write V' for the dual space

There is a linear map $V\otimes W' \longrightarrow \mathcal{L}(W,V) = \left\{\begin{matrix}\text{all linear maps}\\ W\to V\end{matrix}\right\}$

$$(v\otimes f)(w) = f(w)\,v$$

There is also a linear map $V\otimes W' \longrightarrow \mathcal{L}(V,W)'$

$$(v\otimes f)(T) = f(Tv)$$

Tensor product of Hilbert spaces

If H, K are Hilbert spaces, then the algebraic tensor-product $H\otimes K$ is a pre-Hilbert space with the inner product given on elementary tensors by

$$\langle \eta_1\otimes\eta_2, \xi_1\otimes\xi_2\rangle = \langle\eta_1,\xi_1\rangle\langle\eta_2,\xi_2\rangle$$

Write $H\otimes K$ also for the completion with respect to

$$\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$$

Given Hilbert spaces H_i, K_i and bounded linear operators $b_i : H_i \to K_i$, the linear map $b_1\otimes b_2 : H_1\otimes H_2 \to K_1\otimes K_2$ on the algebraic tensor product

$$h_1\otimes h_2 \mapsto b_1(h_1)\otimes b_2(h_2)$$

satisfies $\|b_1\otimes b_2\| = \|b_1\|\cdot\|b_2\|$

so it extends to a bounded linear operator on the completion

Thm: Let $T \in B(H)$. Then there is a unique partial isometry $U \in B(H)$ such that $T = U|T|$ and $\ker(U) = \ker(T)$.
Moreover, $U^*T = |T|$.

Pf: If $x \in H$, then $\| |T|x \|^2 = \langle |T|x, |T|x \rangle = \langle |T|^2 x, x \rangle$
$= \langle T^*Tx, x \rangle = \langle Tx, Tx \rangle$
$= \|Tx\|^2$

so the map $U_0 : |T|(H) \longrightarrow H$, $|T|x \mapsto Tx$,
is well-defined and isometric. It is also linear.
Thus it has a unique linear isometric extension U_0 to $\overline{|T|(H)}$

Define $U = \begin{cases} U_0 & \text{on } \overline{|T|(H)} \\ 0 & \text{on } \overline{|T|(H)}^{\perp} \end{cases}$ Then $U|T| = T$

U is isometric on $\ker(U)^{\perp}$ because $\ker(U) = \overline{|T|(H)}^{\perp}$
Thus U is a partial isometry, and $\ker(U) = \ker(|T|)$

Now $\langle U^*Tx, |T|y \rangle = \langle Tx, Ty \rangle = \langle T^*Tx, y \rangle$
$= \langle |T|x, |T|y \rangle$ for all $x, y \in H$.
so $\langle U^*Tx, z \rangle = \langle |T|x, z \rangle$ for all $z \in \overline{|T|(H)}$
and thus for all $z \in H$

Hence $U^*T = |T|$.
and $\ker(|T|) = \ker(T)$
$\ker(|T|) = \ker(U)$

If W is another partial isometry such that $T = W|T|$ and $\ker(W) = \ker(T)$, then W equals U on $\overline{|T|(H)}$
and also on $\overline{|T|(H)}^{\perp} = \ker(T) = \ker(W) = \ker(U)$. $\square$

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③ Schatten p-classes

Recall: H separable Hilbert space

$T \in B(H)$ is compact if it maps the unit ball of H to a relatively compact set.

Denote by $K(H)$ the set of all compact operators on H

Fact: T is compact $\Leftrightarrow$ $\exists$ sequence (T_n) of finite rank operators on H such that $\|T_n - T\| \to 0$.

Fact: $K(H)$ is the only proper closed 2-sided ideal in $B(H)$.

Fact: $T \in K(H) \Leftrightarrow T^*T \in K(H) \Leftrightarrow |T| \in K(H)$

For self-adjoint compact operator T, spectral decomposition gives

$$Tx = \sum_{n=1}^{\infty} \mu_n \langle x, e_n \rangle e_n ,$$

(μ_n) sequence of nonzero eigenvalues of T (with multiplicity)

(e_n) orthonormal

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Define

For arbitrary compact operator T,
note that $|T|$ is a self-adjoint compact operator,
so $|T|x = \sum_{n=1}^{\infty} \mu_n \langle x, e_n \rangle e_n$, μ_n nonzero eigenvalues of $|T|$

$Tx = U|T|x = \sum_{n=1}^{\infty} \mu_n \langle x, e_n \rangle \sigma_n$, $\sigma_n = Ue_n$

(μ_n) seq of positive numbers (since $|T|$ is a positive operator)
$(e_n), (\sigma_n)$ orthonormal

This will be called the singular value decomposition of T

Def: For $1 \le p < \infty$, define the Schatten p-class, denoted $S_p(H)$, to be the set of all $T \in K(H)$ with $(\mu_n) \in \ell_p$

Convention: $S_\infty(H) = B(H)$
[Note: different convention in book]

$S_1(H)$ called trace class
$S_2(H)$ called Hilbert-Schmidt class

Observe ① Finite rank operators are in S_p
② $T \in S_p \Leftrightarrow T^* \in S_p$

Prop: $T \in B(H)$, $1 \le p < \infty$
Then $T \in S_p \Leftrightarrow$ There is a sequence (F_n) of finite rank operators with $\operatorname{rank}(F_n) \le n$, and $\sum_{n=1}^{\infty} \|T - F_n\|^p < \infty$

Idea: Enough to consider self-adjoint T
because $T = \frac{1}{2}(T + T^*) + i \frac{1}{2i}(T - T^*)$
In this case, if $T \in S_p$, $Tx = \sum_{m=1}^{\infty} \lambda_m \langle x, e_m \rangle e_m$
Define $F_n x = \sum_{m=1}^{n} \lambda_m \langle x, e_m \rangle e_m$

Prop: Suppose $1 \le p \le \infty$, $T \in S_p$, $S \in B(H)$. Then $TS, ST \in S_p$

Pf: Use previous Prop.

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Trace on $\mathcal{S}_1$

Lem: If $T \in \mathcal{S}_1$, then the series $\sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$ converges absolutely for any orthonormal basis (e_k), and the sum is independent of the choice of orthonormal basis

Def: For $T \in \mathcal{S}_1$ and any orthonormal basis (e_k) of H, define the trace of T to be $tr(T) = \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$

Prop: For $S, T \in \mathcal{S}_1$, $R \in B(H)$, $\alpha, \beta \in \mathbb{C}$.

1. $tr(\alpha S + \beta T) = \alpha\, tr(S) + \beta\, tr(T)$
2. $tr(T^*) = \overline{tr(T)}$
3. If $T \geq 0$, then $tr(T) \geq 0$ (with equality $\Leftrightarrow T = 0$)
4. $tr(RT) = tr(TR)$

Pf of 4: Recall that each $R \in B(H)$ is a linear combination of ≤ 4 unitary operators

So we may assume R is unitary

In this case, (Re_k) is also an orthonormal basis

and $tr(TR) = \sum \langle TRe_k, e_k \rangle = \sum \langle TRe_k, R^*Re_k \rangle$

$= \sum \langle RTRe_k, Re_k \rangle = tr(RT)$ $\square$

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$\mathcal{S}_p$ as a Banach space

Def: For $T \in \mathcal{S}_p$ $(1 \le p < \infty)$, define $\|T\|_p = \left(\sum_n \mu_n^p\right)^{\frac{1}{p}}$, where (μ_n) is the singular value sequence of T (with multiplicity) (eigenvalues of $|T|$)

Note: $\|T\|_p = \| \, |T| \, \|_p$ for $T \in \mathcal{S}_p$

$\|T\| \le \|T\|_p$

Lem: Suppose $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $T \in B(H)$

Then $T \in \mathcal{S}_p \iff \sup\left\{ |tr(FT)| : \begin{array}{l} F \text{ finite rank}, \\ \|F\|_q \le 1 \end{array} \right\} < \infty$

In this case, this sup equals $\|T\|_p$

Thm: For $1 \le p < \infty$, $(\mathcal{S}_p, \|\cdot\|_p)$ is a Banach space, and the finite rank operators are dense in $(\mathcal{S}_p, \|\cdot\|_p)$

Lem: Suppose $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $T \in \mathcal{S}_p$, $S \in \mathcal{S}_q$, then $TS, ST \in \mathcal{S}_1$, and $|tr(ST)| \le \|ST\|_1 \le \|S\|_q \|T\|_p$

Thm: Suppose $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ Then for ea

Thm: Suppose $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ Then for each $T \in \mathcal{S}_q$, the equation $f_T(S) = tr(ST)$ defines a bounded linear functional on $\mathcal{S}_p$

The map $T \mapsto f_T$ is an isometric isomorphism from $\mathcal{S}_q$ onto $\mathcal{S}_p^*$.

Also, $K(H)^* \cong \mathcal{S}_1$

Idea: Note $\|f_T\| \leq \|T\|_q$. Use Lem ① to show $\|f_T\| \geq \|T\|_q$

Let $f \in \mathcal{S}_p^*$. Let $L(x,y) = f(x \otimes y)$ for $x, y \in H$, $(x \otimes y)z = \langle z, y \rangle x$ for $z \in H$.

L is linear in x, conjugate linear in y,

$$|L(x,y)| \leq \|f\| \|x\| \|y\|.$$

So $\exists T \in B(H)$ such that $L(x,y) = \langle Tx, y \rangle$

Rest of proof: Check $T \in \mathcal{S}_q$. and $f = f_T$

④ Matrices of operators (section 1.3 of book)

Given a Hilbert space H, and $n \in \mathbb{N}$, $M_n(B(H)) \cong B(\overbrace{H \oplus \cdots \oplus H}^{= H^n})$

Equivalently, $M_n \otimes B(H) \cong B(\mathbb{C}^n \otimes H)$

Given $b \in M_m(B(H))$, $c \in M_n(B(H))$,
define $b \oplus c$ to be the matrix $\begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \in M_{m+n}(B(H))$

Prop 1.3.1: ① $\|b \oplus c\| = \max\{\|b\|, \|c\|\}$

② For $\alpha \in M_{n,m}$, $\beta \in M_{m,n}$,

$$\|\alpha b \beta\| \leq \|\alpha\| \, \|b\| \, \|\beta\|$$

Pf: ① Identifying $M_n(B(H))$ with $B(H^n)$, then the result follows from:
if $b_s : H_s \to K_s$ with $\sup_s \|b_s\| < \infty$,
then $\|\bigoplus_s b_s\| = \sup_s \|b_s\|$

② Identifying $M_n(B(H))$ with $B(\mathbb{C}^n \otimes H)$,

$$\|\alpha b \beta\| \leq \|(\alpha \otimes I) b (\beta \otimes I)\| \leq \|\alpha\| \, \|b\| \, \|b\|$$

$\square$

$\forall b \in M_m(B(H)),\ c \in M_n(B(H)),\ \alpha \in M_{m,n}$,

① If $b \geq 0$, $c \geq 0$, then $b \oplus c \geq 0$

② If $b \geq 0$, then $\alpha^* b \alpha \geq 0$

Prop 1.3.2: $\forall\, b \in M_n(B(H))$, $\|b\| \leq 1 \Longleftrightarrow \begin{bmatrix} I_n & b \\ b^* & I_n \end{bmatrix} \geq 0$

Fact: If A is a unital C^*-algebra (eg. $B(H)$), then for self-adjoint $a \in A$,

$$\|a\| = \min\{\alpha : -\alpha \cdot 1 \leq a \leq \alpha \cdot 1\}$$

Idea: If $b \in M_n(B(H))$ with $\|b\| \leq 1$, then $\tilde{b} = \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} \in M_{2n}(B(H))$

is self-adjoint and $\|\tilde{b}\| = \max\{\|b\|, \|b^*\|\} = \|b\| \leq 1$ so $\tilde{b} \geq 0$

Using fact, $\begin{bmatrix} I_n & b \\ b^* & I_n \end{bmatrix} = I_{2n} + \tilde{b} \geq 0$

Conversely, if $I_{2n} + \tilde{b} \geq 0$, then

$$I_{2n} - \tilde{b} = \begin{bmatrix} I_n & -b \\ -b^* & I_n \end{bmatrix} = \begin{bmatrix} I_n & \\ & -I_n \end{bmatrix} \begin{bmatrix} I_n & b \\ b^* & I_n \end{bmatrix} \begin{bmatrix} I_n & \\ & -I_n \end{bmatrix} \geq 0$$

so $-I_{2n} \leq \tilde{b} \leq I_{2n}$

Using fact, $\|b\| = \|\tilde{b}\| \leq 1$. □

Cor 1.3.3: For $b \in B(H)$, $\|b\| = \inf\left\{\alpha > 0 : \begin{bmatrix} \alpha I & b \\ b^* & \alpha I \end{bmatrix} \geq 0\right\}$

Cor 1.3.4: If $b \in B(H)$, $\begin{bmatrix} I & b \\ b^* & 0 \end{bmatrix} \geq 0$, then $b = 0$

Pf. Given $\varepsilon > 0$, show $\begin{bmatrix} \varepsilon I & b \\ b^* & \varepsilon I \end{bmatrix} \geq 0$, get $\|b\| \leq \varepsilon$ □

Prop 1.3.5 (used in §5): For self-adjoint $a, b \in B(H)$,

(1) if $-a \leq b \leq a$, then $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \geq 0$

(2) if a_1, a_2 are positive operators with $\begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix} \geq 0$, then $-\tilde{a} \leq b \leq \tilde{a}$, where $\tilde{a} = \frac{1}{2}(a_1 + a_2)$

If b is an arbitrary operator, and $b = (b_1 - b_2) + i(b_3 - b_4)$ with $b_j \geq 0$, then $\begin{bmatrix} \sum b_j & b \\ b^* & \sum b_j \end{bmatrix} \geq 0$

Conversely, if a_1, a_2 are positive with $D = \begin{bmatrix} a_1 & b \\ b^* & a_2 \end{bmatrix} \geq 0$, then $b = \frac{1}{4} \sum_{k=0}^{3} i^k [1\ \ i^k] D [1\ \ i^k]^*$

Pf uses property ② about ordering