## §2.5 Convolutions

- $G$ locally compact group
- $M(G)$ complex Radon measures

$$I(\varphi) = \iint \varphi(xy)\, d\mu(x)\, d\lambda(y)$$

$x, y \in G$, $\varphi \in C_c(G)$

is a linear functional

$$|I(\varphi)| \le \|\varphi\|_\infty \|\mu\| \|\lambda\|$$

$I$ is given by a measure, denoted by $\mu * \lambda$

$$\int \varphi \, d(\mu * \lambda) = \iint \varphi(xy)\, d\mu(x)\, d\lambda(y)$$

$$\|\mu * \nu\| \le \|\mu\| \|\nu\|.$$

$(M(G), *)$ is a Banach algebra.

- $G$ is abelian if and only if $*$ is commutative
- multiplicative unit $\delta = \delta_1$

$$\int \varphi \, d(\mu * \delta) = \iint \varphi(xy)\, d\delta(x)\, d\mu(y) = \int \varphi(y)\, d\mu(y)$$

- $\mu^*(E) = \overline{\mu(E^{-1})}$ or $\int \varphi \, d\mu^* = \int \varphi(x^{-1})\, d\bar{\mu}(x)$

involution on $M(G)$

How does this look on $L^1(G)$?

- $f, g \in L^1(G)$: $(f * g)(x) = \int f(y) g(y^{-1}x)\, dy$

$$\int \varphi(x) (f * g)(x)\, dx = \iint \varphi(x) f(y) g(y^{-1}x)\, dx\, dy$$

$$x \to yx \qquad = \iint \varphi(yx) f(y) g(x)\, dx\, dy$$

- By Fubini the integral is absolutely convergent a.e. $x$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$
- Involution on $L^1(G)$: $f^*(x) = \Delta(x^{-1}) \overline{f(x^{-1})}$

$\quad \hookrightarrow$ modular function

$L^1(G)$ is a Banach $*$-algebra.

**Proposition** Suppose $1 \leq p \leq \infty$, $f \in L^1(G)$, $g \in L^p(G)$

(i) $f * g(x) = \int f(y) g(y^{-1}x)\, dy$ converges a.e. $x$ in $L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$

(ii) $g * f$ holds the same if $G$ is unimodular

(iii) $G$ not unimodular, then $g * f \in L^p(G)$ if $f$ has compact support.

**Proof:** (i) $\|f*g\|_p = \| \int f(y) g(y^{-1}x)\,dy \|_p$

$$\le \int |f(y)| \, \|L_y g\|_p \, dy$$

$$= \|f\|_1 \|g\|_p .$$

(ii) $\|g*f\|_p = \| \int R_{y^{-1}} g(x) f(y) \Delta(y^{-1})\,dy \|_p$

$$\le \|R_{y^{-1}} g\|_p \|f\|_1$$

$$= \|g\|_p \|f\|_1$$

(iii) $\|g*f\|_p \le \int \|R_{y^{-1}} g\|_p |f(y)| \Delta(y^{-1})\,dy$

$$\le \|g\|_p \|f\|_1 \cdot C$$

where $C = \sup_{\mathrm{supp}\, f} \Delta(y)^{\frac{1}{p}-1}$ □

**Proposition (2.41):** Suppose $G$ is unimodular. If $f \in L^p(G)$, $g \in L^q(G)$ with $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ then $f * g \in C_0(G)$ and

$$\|f * g\|_\infty \le \|f\|_p \|g\|_q$$

**Proof:** Hölder's inequality

$$|(f*g)(x)| \le \|f\|_p \|g\|_q$$

If $f, g \in C_c(G)$ then $f*g \in C_c(G)$. Let $f_n \to f$ in $L^p$ and $g_n \to g$ in $L^q$ with $f_n, g_n \in C_c(G)$. $C_c(G)$ is dense in $L^p$ and $L^q$.

$f_n$

$f_n * g_n \longrightarrow f * g$ uniformly

$f * g \in C_0(G)$ $\square$

<u>Proposition</u> If $f \in L^1(G)$, $g \in L^\infty(G)$ then $f * g$ ($g * f$) is left (right) uniformly continuous.

Whenever $G$ is discrete then $\delta(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{elsewhere} \end{cases}$ is a multiplicative unit in $L^1(G)$

<u>Proposition</u> Let $\mathcal{U}$ be a neighborhood base of $1$ in $G$. For each $U \in \mathcal{U}$ let $\varphi_U$

(i) $\operatorname{supp} \varphi_U \subseteq U$ compact

(ii) $\varphi_U \geq 0$ and $\int \varphi_U = 1$.

Then $\| \varphi_U * f - f \|_p \to 0$ as $U \to \{1\}$ for $f \in L^p(G)$ whenever $1 \leq p < \infty$ or if $p = \infty$ and $f$ is left uniformly continuous.

If in addition

(iii) $\varphi_U(x^{-1}) = \varphi_U(x)$ $\forall x \in G$.

$\| f * \varphi_U - f \|_p \to 0$ as $U \to \{1\}$, $f \in L^p$

A family $\{\varphi_u\}_{u \in \mathcal{U}}$ with (i)–(iii) is an <u>approximate unit</u>.

Approximate units are in great abundance.

Take $U$ compact and symmetric and apply Urysohn's Lemma.

<u>Theorem (4.25)</u> Let $J$ be a closed subspace of $L^1(G)$. $J$ is a left (right) ideal if and only if it is closed under left (right) translations.

<u>Proof</u>: $J$ be a left ideal, $x \in G$ and $\{\psi_u\}$ an approximate unit. Then

$$L_x(\psi_u * f) = (L_x \psi_u) * f \in J$$

$$L_x f = \lim L_x(\psi_u * f) \in J.$$

Conversely, if $f \in J$, $g \in L^1(G)$ we have

$$g * f = \int g(y) L_y f \, dy \qquad L_y f \in J, \; g * f \in J \quad \square$$

$f_n * g_n \to$

$f * g \in C_0$

<u>Proposition</u>

$(g * f)$ is

. Whenever $G$

is a mult

## §26 Homogeneous spaces

For $g_0 \in G$ | $H$ closed subgroup

then $[\pi(g_0)f](g) = f(g_0^{-1}g)$, $g \in H$, $f \in L^2(G/H)$.

For a suitable measure ($G$-invariant)

$\pi$ becomes a unitary representation

- $G$ locally cpt group
- $S$ locally cpt Hdf space

A (left) action of $G$ on $S$ is a continuous

map $(x,s) \mapsto xs$ $(G \times S \to S)$

(i) fixed $x \in G$: $s \mapsto xs$ is a homeomorp[hism]

(ii) $x(ys) = (xy)s$ $\forall x,y \in G$ and $s \in S$

A space $S$ with an action of $G$ is call[ed]

a $G$-space.

A $G$-space $S$ is <u>transitive</u> if for $s,t \in S$

$\exists x \in G$ st $xs = t$.

nous

omorphism
$s \in S$.
called

$\in S$

<u>Remark:</u> $H$ is closed subgroup of $G$
Then $G$ acts on $G/H$ by left multiplication.
Let $S$ be a transitive $G$-space. transitively
Pick $s_0 \in S$ and define $\varphi : G \to S$. $\varphi(x) = x s_0$
Then $H = \{x \in G : x s_0 = s_0\}$ is a closed subgroup.
$\varphi$ is a continuous surjection (transitive) and
$\varphi$ is constant on left cosets of $H$.
Therefore, $\varphi$ induce a continuous bijection
$\Phi : G/H \to S$ s.t. $\Phi \circ q = \varphi$

Generally $\Phi^{-1}$ is not continuous.
<u>Proposition</u> If $G$ is $\sigma$-compact then $\Phi$ is a homeomorphism.
<u>Proof:</u> Suffices to show $\varphi$ is open.
Let $U \subseteq G$ open and $x_0 \in U$. Pick a compact symmetric neighborhood $V$ of $1$ s.t. $x_0 V V \subseteq U$
Let $\{y_n\}$ in $G$ countable s.t. $y_n V$ covers $G$
Then $S = \bigcup_{n=1}^{\infty} \varphi(y_n V)$. By (i) $\varphi(y_n V)$ are
to $\varphi(V)$. Also $\varphi(y_n V)$ are closed
homeomorphic
Baire's category theory $\varphi(V)$ has interior point, $x_1 \in V$

Then $\varphi(x_0)$ is an interior point of $\varphi(x_0 x_1^{-1} V)$ and

$x_0 x_1^{-1} V \subseteq x_0 V V \subseteq U$. $\Box$

<u>Def</u> A transitive $G$-space $S$ that is isomorphic to a quotient space $G/H$ is called a <u>homogeneous space</u>.

Identification of $S$ depends on $s_0 \in S$ upto homeomorphism.

General examples:

- $S^{n-1} \cong O(n)/O(n-1)$
- A "suitable" differentiable manifold with group diffeomorphisms form a homogeneous space.

$G$ with left Haar measure $dx$

$H$ subgroup " " " $d\xi$

$\Delta_G, \Delta_H$ modular functions

$G$-invariant measure $\mu(E) = \mu(xE)$ $E \subseteq G/H$, $x \in G$.

$P: C_c(G) \longrightarrow C_c(G/H)$

$$Pf(xH) = \int_H f(x\xi)\,d\xi.$$

$\left( \int_G f(x)\,dx = \int_{G/H} Pf\, d\mu \right)$

<u>Lemma ①</u>: If $E \subseteq G/H$ is compact then there exists a $K \subseteq G$ compact $q(K) = E$.

<u>Lemma ②</u>: If $F \subseteq G/H$ is compact then there is $f \in C_c^+(G)$ such that $Pf = 1$ on $F$

<u>Prop</u>: For all $\varphi \in C_c(G/H)$ there is a $f \in C_c(G)$ with $Pf = \varphi$ and $q(\operatorname{supp} f) = \operatorname{supp} \varphi$ and $f \geq 0$ whenever $\varphi \geq 0$.

<u>Proof</u>: Let $\varphi \in C_c(G/H)$. By ② let
let $g \in C_c^+(G)$ with $Pg = 1$ on $\operatorname{supp} \varphi$
Define $f = (\varphi \circ q)g$. We get

$$Pf = P((\varphi \circ q)g) = \varphi \cdot Pg = \varphi.$$

<u>Theorem</u>: $G$ loc cpt group, $H \leq G$ closed subgroup Then th
k∘ $G$-invariant measure $\mu$ on $G/H$ if and only if $\Delta_G|_H = \Delta_H$. In this case, $\mu$ is unique upto scalar and it suitable chosen

$$\int_G f(x)\,dx = \int_{G\backslash H} Pf\, d\mu = \int_{G/H} \int_H f(x\xi)\,d\xi\, d\mu(xH).$$

Proof: Suppose $\mu$ is a $G$-invariant measure. Then $f \mapsto \int Pf \, d\mu$ is nonzero left invariant positive linear functional on $C_c(G)$

$$\int Pf \, d\mu = c \int_G f(x)\, dx \qquad c > 0$$

$\mu$ is completely determined.

Replace $\mu$ by $c^{-1}\mu$.

For $\eta \in H$ and $f \in C_c(G)$ we have

$$\begin{aligned}
\Delta_G(\eta) \int_G f(x)\, dx &= \int_G f(x\eta^{-1})\, dx \\
&= \int_{G/H} \int_H f(x\xi\eta^{-1})\, d\xi \, d\mu(xH) \\
&= \Delta_H(\eta) \int_{G/H} \int_H f(x\xi)\, d\xi \, d\mu(xH) \\
&= \Delta_H(\eta) \int_G f(x)\, dx.
\end{aligned}$$

Indeed $\Delta_G|_H = \Delta_H$.

Conversely, let $f \in C_c(G)$ with $Pf = 0$

We show $\int f(x)\, dx = 0$.

By ② $\varphi \in C_c(G)$ with $P\varphi = 1$ on $q(\mathrm{supp}\, f)$

We get:

$$0 = Pf(xH) = \int f(x\xi)\,d\xi = \int f(x\xi^{-1})\,\Delta_H(\xi^{-1})\,d\xi$$

$$\left( \underline{d\lambda(x^{-1}) = \Delta(x^{-1})\,d\lambda(x)} \right) \qquad = \int f(x\xi^{-1})\,\Delta_G(\xi^{-1})\,d\xi$$

$$0 = \int_G \varphi(x) \int_H f(x\xi^{-1})\,\Delta_G(\xi^{-1})\,d\xi\,dx$$

$$= \int_H \int_G \varphi(x) f(x\xi^{-1})\,\Delta_G(\xi^{-1})\,dx\,d\xi$$

$$= \int_H \int_G \varphi(x\xi) f(x)\,dx\,d\xi$$

$$= \int_G P\varphi(xH) f(x)\,dx = \int f(x)\,dx$$

If $Pf = Pg$ then $\int_G f = \int_G g$.

So $Pf \longmapsto \int_G f$ $G$-invariant position linear functional. Associated Radon measure $\mu$ is desired one. □

Quasi-invariant measure

$\mu$ Radon measure on $G/H$. For $x \in G$ define $\mu_x(E) = \mu(xE)$

$\mu$ is quasi-invariant if all $\mu_x$ are equivalent (mutually absolutely continuous) with $\mu$.

$\mu$ is strongly quasi-invariant if there is a cont $\lambda : G \times (G/H) \to (0,\infty)$ $d\mu_x(p) = \lambda(x,p)\,d\mu(p)$

**Lemma** There exists a function $f : G \to (0,\infty)$ s.t.

(i) $\{y : f(y) > 0\} \cap xH \neq \phi \quad \forall x \in G$

(ii) $(\operatorname{supp} f) \cap KH$, $K \subseteq G$ compact is compact

**Proof.** Zorn's Lemma.

A rho-function for the pair $(G, H)$ is a continuous $\rho : G \to (0,\infty)$ s.t.

$$\rho(x\xi) = \frac{\Delta_H(\xi)}{\Delta_G(\xi)} \rho(x) \qquad \begin{matrix} x \in G \\ \xi \in H \end{matrix}$$

**Aim:** $\int_{G/H} \rho f \, d\mu = \int_G f(x)\rho(x)\, dx$

**Prop.** $(G, H)$ admits a rho-function

**Proof:** Let $f$ be as in the Lemma.

$$\rho(x) = \int_H \frac{\Delta_G(\eta)}{\Delta_H(\eta)} f(x\eta)\, d\eta$$

Property (ii) of $f$ guarantees $\rho(x)$ is continuous

$$\rho(x\xi) = \int \frac{\Delta_G(\eta)}{\Delta_H(\eta)} f(x\xi\eta)\, d\eta = \int \frac{\Delta_G(\xi^{-1}\eta)}{\Delta_H(\xi^{-1}\eta)} f(x\eta)\, d\eta$$

$$= \frac{\Delta_H(\xi)}{\Delta_G(\xi)} \rho(x). \qquad \square$$

$\rho(x)\,dx$.

nction

mma.

$d\eta$

$x)$ is continuous

$f(x\eta)\,d\eta$

$x)$.

$\square$

<u>Theorem</u> Suppose $G$ loc. cpt group, $H \leq G$ closed subgroup. Given any rho-function on $(G,H)$ there is a strongly quasi-invariant measure on $G/H$, such that

$$\int_{G/H} Pf\,d\mu = \int_G f(x)\rho(x)\,dx \qquad \circledast$$

and

$$\frac{d\mu_x}{d\mu}(yH) = \frac{\rho(xy)}{\rho(y)}$$

<u>Proof.</u> The map $Pf \mapsto \int f\rho$ is positive linear functional so it defines a Radon measure $\mu$ on $G/H$ s.t. $\circledast$ holds.

We aim to show that $\mu$ is strongly quasi-inv

$\lambda : G \times (G/H) \to (0,\infty)$

$$\lambda(x, q(y)) = \rho(xy)/\rho(y)$$

well-defined as $\frac{\rho(xy)}{\rho(y)}$ only depends on left cosets.

$$\begin{aligned}
\int_{G/H} Pf(p)\,d\mu_x(p) &= \int_{G/H} Pf(x^{-1}p)\,d\mu(p) \\
&= \int_G f(x^{-1}y)\rho(y)\,dy \\
&= \int_G f(y)\rho(xy)\,dy \\
&= \int_G f(y)\lambda(x, q(y))\rho(y)\,dy \\
&= \int_{G/H} Pf(p)\lambda(x,p)\,d\mu(p)
\end{aligned}$$

$\square$

**Theorem**: Every strongly quasi-invariant measure on $G/H$ arises from a rho-function as in ⊗ and all such measures are strongly equiv

$S$ is a homogeneous $G$-space and $\mu$ a strongly quasi-invariant measure $\lambda \in C^{+}(G \times S)$ with $d\mu_x(s) = \lambda(x, s)\, d\mu(s)$

$$[\tilde{\pi}(x) f](s) = \lambda(x, x^{-1}s)^{-\frac{1}{2}} f(x^{-1}s) \qquad x \in G,\ s \in S,\ f \in L^2(\mu)$$

$$\begin{aligned} \| \tilde{\pi}(x) f \|_2 &= \int \lambda(x, x^{-1}s)^{-1} |f(x^{-1}s)|^2 \, d\mu(s) \\ &= \int \lambda(x, s)^{-1} |f(s)|^2 \, d\mu_x(s) \\ &= \int |f(s)|^2 \, d\mu(s) = \| f \|_2 \end{aligned}$$