EUCLID'S ALGORITHM
IN LARGE DEDEKIND DOMAINS

H.W. Lenstra, Jr.

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Abstract. It is proved that any Dedekind domain with many more elements than prime ideals is Euclidean.

Key words: Euclidean ring, Dedekind domain.


Let $A$ be a Dedekind domain, and denote by $Z$ the set of its non-zero prime ideals. It is well known that $A$ is a principal ideal domain if $Z$ is finite. An infinite analogue of this result was obtained by Claborn [1, 2, chapter III, section 13]. He proved that $A$ is a principal ideal domain if

$$\#A > (\#Z)^2,$$

where $\text{a}$ is the least infinite cardinal and $\#S$ denotes the cardinality of $S$.

If $Z$ is finite then $A$ is not only a principal ideal domain but even a Euclidean domain [4, Proposition 5]. The latter statement means that there exists a map $\phi$ from $A - \{0\}$ to a well-ordered set $W$ such that for all $a, b \in A$ with $b \neq 0, a \notin Ab$, there exists $r \in a + Ab$ with $\phi(r) < \phi(b)$.

For finite $Z$ one can take for $W$ the set of non-negative integers.

It is a natural question whether Claborn’s result can be extended in a similar way, i.e. whether $A$ is Euclidean if (1) holds. In the present paper we show that this is indeed the case. For $W$ we take a well-ordered set of order type $\omega^2$, where $\omega$ is the least infinite ordinal. The elements of $W$ can be written in a unique way as $\omega a + b$, where $a, b$ are non-negative integers; and $\omega a + b < \omega a' + b'$ if and only if either $a < a'$ or $a = a'$, $b < b'$.

We shall see that the other results that Claborn obtained in [1] can be extended in an analogous way.

We let $K$ denote the field of fractions of $A$, and $v_p$, for $p \in Z$, the normalized exponential valuation of $K$ corresponding to $p$. The group of units of $A$ is denoted by $A^*$. Claborn’s first result [1, Proposition; 2, Proposition 13.7] states that $A$ is a principal ideal domain if $A$ contains a field $k$ satisfying $\#A = \#k > \#Z$. A sharper result is as follows.

(2) Proposition. Let $A$ be a Dedekind domain, and suppose that $A$ contains a subset $k$ with the properties

$$\#k > \#Z,$$

(2)

$$\lambda - \mu \in A^* \cup \{0\} \text{ for all } \lambda, \mu \in k.$$

Then $A$ is Euclidean.

Proof. For $x \in A - \{0\}$, let $\phi(x) = \sum_{p \in Z} v_p(x)$. We prove that $A$ is Euclidean with respect to $\phi$.

Let $a, b \in A$, $b \neq 0, a \notin Ab$. First suppose that for some $\lambda \in k$ we have $A(a + \lambda b) = Aa + Ab$. Then
\[ v_\wp(a + \lambda b) = \min\{v_\wp(a), v_\wp(b)\} \leq v_\wp(b) \]
for all \( \wp \in \mathcal{Z} \), with strict inequality for at least one \( \wp \). Hence the element \( r = a + \lambda b \) of \( A + \mathcal{B} \) satisfies \( \phi(r) < \phi(b) \), as required.

Next suppose that no such \( \lambda \) exists. Then for every \( \lambda \in k \) there exists \( \wp_\lambda \in \mathcal{Z} \) such that \( a + \lambda b \in \wp_\lambda (Aa + Ab) \). The map \( k \to \mathcal{Z} \) sending \( \lambda \) to \( \wp_\lambda \) is not injective, by (3), so there are \( \lambda, \mu \in k \), \( \lambda \neq \mu \), with \( \wp_\lambda = \wp_\mu \). Then \( (\lambda - \mu)b = (a + \lambda b) - (a + \mu b) \in \wp_\lambda (Aa + Ab) \), so \( b \in \wp_\lambda (Aa + Ab) \), by (4). We conclude that \( Aa + Ab = A(a + \lambda b) + Ab \) is contained in \( \wp_\lambda (Aa + Ab) \), which is a contradiction. This proves (2).

If \( A \) is the ring of integers in an algebraic number field then condition (3) can be substantially weakened, see [2, Theorem (1.4)].

For a subset \( \wp \subseteq \mathcal{Z} \), we define the subring \( A_\wp \subseteq K \) by
\[ A_\wp = \{ x \in K : v_\wp(x) \geq 0 \text{ for all } \wp \in \wp \}. \]
Notice that \( A_\wp = A \). Claborn [2, Theorem 13.8] proved that every ideal of \( A_\wp \) is generated by an element of \( A \) if the inequality \( \#A > (\# \wp)^2 \) is satisfied. To formulate our stronger result we need a definition. Let the pair \( (A, \wp) \) be called \textit{Euclidean} if there exist a well-ordered set \( W \) and a map \( \phi: A - \{0\} \to W \) such that for all \( a, b \in A \), \( b \neq 0 \), \( a \in A_\wp b \), there exists \( r \in a + Ab \) with \( \phi(r) < \phi(b) \). We have \( A_\wp = A \), and \( (A, \wp) \) is Euclidean if and only if \( A \) is.

Let \( (A, \wp) \) be Euclidean and \( b \) a non-zero \( A_\wp \)-ideal. Then \( b \) is generated by \( b \cap A \), and if \( b \cap b \cap A \) has minimal \( \wp \)-value then it follows easily that \( A_\wp b = b \). Hence, if \( (A, \wp) \) is a Euclidean pair, then every ideal of \( A_\wp \) is generated by an element of \( A \). This shows that the following theorem is indeed sharper than Claborn's result.

\textbf{(5) Theorem.} Let \( A \) be a Dedekind domain, and \( \wp \) a set of non-zero prime ideals of \( A \) such that \( \#A > (\# \wp)^2 \), where \( \wp \) denotes the least infinite cardinal. Then \( (A, \wp) \) is a Euclidean pair.

The proof uses the following lemma. Let \( W \) be the well-ordered set of order type \( \omega^2 \) defined above.

\textbf{(6) Lemma.} Let \( A \) be Dedekind, \( \wp \subseteq \mathcal{Z} \) a subset, and suppose that there exists a finite subset \( X \subseteq \wp \setminus A \) with the property that for every \( x \in A \setminus A \wp \) there exists \( q \in A \) such that \( (x + q)^{-1} \in A \wp \). Then \( (A, \wp) \) is a Euclidean pair with respect to the map \( \phi: A - \{0\} \to W \) defined by
\[ \phi(x) = \omega \sum_{\wp \in X} v_\wp(x) + \sum_{\wp \in \wp \setminus X} v_\wp(x). \]

\textit{Proof of (6).} Let \( a, b \in A \), \( b \neq 0 \), \( a \in A_\wp b \). We have to find \( r \in a + Ab \) such that \( \phi(r) < \phi(b) \).

First suppose that \( v_\wp(a) \geq v_\wp(b) \) for all \( \wp \in X \). Then \( x = a/b \) belongs to \( A_\wp \), but not to \( A \wp \), so by the hypothesis of the lemma there exists \( q \in A \) such that \( (x + q)^{-1} = b/(a + qb) \) belongs to \( A_\wp \). Then \( b \in A_\wp (a + qb) \), and therefore \( A_\wp (a + qb) = A_\wp a + A_\wp b \). Hence \( r = a + qb \in a + Ab \) satisfies
\[ v_\wp(a + qb) = \min\{v_\wp(a), v_\wp(b)\} \leq v_\wp(b) \]
for all \( \wp \in \wp \), with strict inequality for at least one \( \wp \) because \( a \in A_\wp b \). It follows that \( \phi(r) < \phi(b) \).

Secondly, suppose that \( v_\wp(a) < v_\wp(b) \) for at least one \( \wp \in X \). Since \( X \) is finite, the approximation theorem for Dedekind domains implies that there exists \( r \in A \) with the following properties:
\[ v_\wp(r - a) \geq v_\wp(b) \text{ for all } \wp \in \wp \text{ with } v_\wp(a) < v_\wp(b), \]
\[ [1, \text{ Section 2.4, Proposition 2}] \]
\[ v_p(r) = v_p(b) \text{ for all } p \in X \text{ with } v_p(a) \geq v_p(b), \]
\[ v_p(r) = v_p(b) \text{ for all } p \in Z - X \text{ with } v_p(a) \geq v_p(b) > 0. \]

Then we have \( v_p(r - a) \geq v_p(b) \) for all \( p \in Z \), so \( r \in a + Ab \). Also, \( v_p(r) \leq v_p(b) \) for all \( p \in X \), with strict inequality if \( v_p(a) < v_p(b) \), which occurs for at least one \( p \in X \). Hence \( \sum_{p \in X} v_p(r) < \sum_{p \in X} v_p(b) \), and it follows that \( \varphi(r) < \varphi(b) \), as required. This proves (6).

Notice that the lemma implies that \((A, Y)\) is a Euclidean pair if \( Y \) is finite.

**Proof of the theorem.** It suffices to show that some for finite subset \( X \subseteq Y \) the condition of the lemma is satisfied. By the remark just made we may assume that \( Y \) is infinite. Let \( \mathfrak{p} \in Z \), and let \( \hat{A}_\mathfrak{p} \) be the \( \mathfrak{p} \)-adic completion of \( A \). Then from

\[ (\# Y)^0 < (\# A / \# Y) \]

we see that \( \# Y < (\# A / \mathfrak{p}) \). So \( A / \mathfrak{p} \) is infinite for every \( \mathfrak{p} \in Z \).

Suppose that there does not exist a finite subset \( X \subseteq Y \) satisfying the condition of (6), i.e.,

\[ \text{for every finite } X \subseteq Y \text{ there exists } x \in A_X - A_Y \text{ such that } (x + q)^{-1} \not\in A_Y \text{ for all } q \in A. \]

We derive a contradiction.

Using (7) we construct a sequence \((x_m)_{m=0}^\infty\) of elements of \( K - A_Y \) with the following two properties:

\[ (x_n + q)^{-1} \not\in A_Y \text{ for all } n \geq 0 \text{ and all } q \in A, \]
\[ \text{if } X_n = \{ \mathfrak{p} \in Y : v_\mathfrak{p}(x_n) < 0 \} \text{ then } X_i \cap X_j = \emptyset \text{ for all } i, j \geq 0, i \neq j. \]

The construction is by induction on \( m \). Let \( m \geq 0 \), and let \( x_m \), for \( 0 \leq n < m \), be such that (8), (9) hold when restricted to \( i, j, n < m \). Applying (7) to \( X = \bigcup_{n < m} X_n \) we find \( x_m, y_m \in A_X - A_Y \) such that \((x_n + q)^{-1} \not\in A_Y \) for all \( q \in A \). For \( n < m \) we then have \( x_m, y_m \in A_X \cap A_X \), so \( X_n \cap X_m = \emptyset \). Hence (8) and (9) hold for \( i, j, n \leq m \). This concludes the induction step and the construction of the sequence \((x_m)_{m=0}^\infty\).

If \((a_m)_{m=0}^\infty\) is any sequence of elements of \( A \), then plainly also \((y_m)_{m=0}^\infty = (x_m + a_m)_{m=0}^\infty\) satisfies (8) and (9), with \( x \) replaced by \( y \). We claim that for a suitable choice of \((a_m)_{m=0}^\infty\) the sequence \((y_m)_{m=0}^\infty\) has the following additional property:

\[ \text{there is no } \mathfrak{p} \in Y \text{ such that there exist } i, j, k \text{ with } v_\mathfrak{p}(y_i - y_j) > 0, v_\mathfrak{p}(y_j - y_k) > 0, i < j < k. \]

The proof is again by induction. Let \( m \geq 0 \), and let \( a_m \in A \), for \( n < m \), be such that (10) holds when restricted to \( k < m \). The only \( \mathfrak{p} \in Y \) which can possibly violate (10), with \( k = m \), are those for which \( v_\mathfrak{p}(y_i - y_j) > 0 \) for certain \( i, j \) with \( i < j < m \). There are only finitely many such \( \mathfrak{p} \), since \( y_i = y_j \) would imply that \( X_i = X_j \), so \( X_i = \emptyset \) by (9), contradicting that \( x_i \not\in A_Y \). Notice that \( v_\mathfrak{p}(y_i - y_j) > 0 \), with \( i < j < m \), implies that \( \mathfrak{p} \not\in X_i \) and \( \mathfrak{p} \not\in X_j \). If \( \mathfrak{p} \in X_m \), then regardless of the choice of \( a_m \) we have \( v_\mathfrak{p}(y_j - y_m) < 0 \). If \( \mathfrak{p} \not\in X_m \), then we have \( v_\mathfrak{p}(y_j - y_m) = 0 \) provided that

\[ a_m \equiv y_j - x_m \text{ mod } \mathfrak{p} \]

(in the local ring at \( \mathfrak{p} \)). Hence, for (10) to be valid with \( k = m \), it suffices that \( a_m \) avoids a finite set of residue classes modulo each of a finite number of prime ideals of \( A \). Since \( A / \mathfrak{p} \) is infinite for all \( \mathfrak{p} \in Z \), the approximation theorem guarantees the existence of an element \( a_m \in A \) satisfying these conditions. This completes our inductive proof of (10).
From (8), (9) (with $y_i$ for $x_i$) and (10) we derive a contradiction. Fix $q \in A$. Then for each $n \geq 0$ there exists $p_n \in Y$ with $v_{p_n}(y_n + q) > 0$, by (8). If $v_i = v_j = v_k$ for $i < j < k$, then with $p = v_i$ we obtain a contradiction to (10). Hence each $p \in Y$ occurs at most twice as $v_{p_n}$, and the map $f_q : \{0,1,2,\ldots\} \to Y$ defined by $f_q(n) = v_n$ has infinite image.

The number of maps $\{0,1,2,\ldots\} \to Y$ is $(\# Y)^\omega$, so from $\# A > (\# Y)^\omega$ it follows that there exist $q \neq r$ in $A$ with $f_q = f_r$. For $p = f_q(n)$ we then have $v_p(y_n + q) > 0$, $v_p(y_n + r) > 0$, and therefore

$$v_p(q - r) > 0$$

for all $p$ in the image of $f_q$.

But $f_q$ has infinite image, so it follows that $q - r = 0$, a contradiction.

This proves the theorem.

(11) **Corollary.** Let $A$ be a Dedekind domain, and suppose that the set $Z$ of non-zero prime ideals of $A$ satisfies $\# A > (\# Z)^\omega$. Then $A$ is Euclidean.

This follows from (5), with $Y = Z$.

**References.**


