TWO THEOREMS ON PERFECT CODES

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Abstract. Two theorems are proved on perfect codes. The first one states that Lloyd's theorem is true without the assumption that the number of symbols in the alphabet is a prime power. The second theorem asserts the impossibility of perfect group codes over non-prime-power-alphabets.

§0. Introduction

Let $V$ be a finite set, $|V| = q \geq 2$, and let $1 \leq e \leq n$ be rational integers. We put $N = \{1, 2, ..., n\}$. For $v = (v_i)_{i=1}^n \in V^n$, $v' = (v'_i)_{i=1}^n \in V^n$ we define $d(v, v') = |\{i \in N \mid v_i \neq v'_i\}|$. A perfect $e$-error-correcting code of block length $n$ over $V$ is a subset $C \subset V^n$ such that for every $v \in V^n$ there exists exactly one $c \in C$ satisfying $d(v, c) \leq e$.

If $q$ is a prime power, a necessary condition for the existence of such a code is given by Lloyd's theorem [6]. This theorem has recently been used to determine all $n$, $e$ for which a perfect code over an alphabet $V$ of $q$ symbols, $q$ a prime power, exists [5; 6].

In §1 I show that Lloyd's theorem holds for all $q$. The proof, which is modelled after [6, 5.4], makes use of some elementary notions from commutative algebra. A different proof has been obtained by P. Delsarte [2]. It seems hard to use Lloyd's theorem to prove non-existence theorems for perfect codes over non-prime-power-alphabets.

In §2 I prove the following theorem: if $G_i$ ($1 \leq i \leq n$) is a group with underlying set $V$, and $C \subset \prod_{i=1}^n G_i$ is a subgroup which as a subset of $V^n$ is a perfect $e$-error-correcting code, $e < n$, then $q$ is a prime power and each $G_i$ is abelian of type $(p, p, ..., p)$. A special case of this theorem was proved in [4].
§ 1. Lloyd’s theorem

Theorem 1. If a perfect $e$-error-correcting code of block length $n$ over $V$ exists then the polynomial

$$P(X) = \sum_{i=0}^{e} \left(-1\right)^i \binom{n-X}{e-i} \binom{X-1}{i} (q-1)^{e-i},$$

where

$$\binom{a}{i} = \Pi_{j=1}^{i} \frac{a-j+1}{j}.$$

has $e$ distinct integral zeros among $1, 2, \ldots, n$.

Proof. Let $K$ be a field of characteristic zero, and let $M$ be a $K$-vector space of dimension $q^n$ with the elements of $V^n$ as basis vectors:

$$M = \{ \sum_{v \in V^n} k_v \cdot v \mid k_v \in K \text{ for } v \in V^n \}.$$

If $D \subset V^n$ is a subset, we denote $\sum_{v \in D} v \in M$ by $\Sigma D$. Define the $K$-endomorphisms $\phi_i$ ($1 \leq i \leq n$) of $M$ by

$$\phi_i(v) = \sum \{ v' = (v'_j)_{j=1}^n \in V^n \mid v'_j = v_j \text{ for all } j \neq i \}, \quad v = (v_j)_{j=1}^n \in V^n.$$

One easily checks:

(1) \quad $\phi_i \phi_j = \phi_j \phi_i \quad (1 \leq i \leq j \leq n)$.

(2) \quad $\phi_i^2 = q \cdot \phi_i \quad (1 \leq i \leq n)$.

Let $K[X_1, \ldots, X_n]$ be the commutative polynomial ring in $n$ symbols over $K$. The ideal generated by $\{X_i^2 - qX_i \mid 1 \leq i \leq n\}$ is denoted by $B$, and $R$ is the factor ring $K[X_1, \ldots, X_n] / B$. By (1) there exists a $K$-linear ring homomorphism $K[X_1, \ldots, X_n] \to \text{End}_K(M)$ (the ring of $K$-endomorphisms of $M$) mapping $1$ to the identity and $X_i$ to $\phi_i$ ($1 \leq i \leq n$). The kernel of this ring homomorphism contains $B$, by (2), so we obtain a ring homomorphism $f: R \to \text{End}_K(M)$, mapping $x_i = (X_i \text{ mod } B) \in R$.
to $\phi_i$. Therefore we can make $M$ into an $R$-module by defining $r \cdot m = f(r)(m)$ ($r \in R$, $m \in M$) \cite[11.1; 3, III.1]{1}.

Put $y_I = \prod_{i \in I}(x_i - 1) \in R$ for $I \subset N$. Then

$$y_I \cdot v = \sum \{v' \in V^n \mid j \in N, \text{ then: } v_j = v_j' \iff j \notin I\},$$

$I \subset N, v \in V^n$. Therefore, $\{y_I \cdot v \mid I \subset N\} \subset M$ is linearly independent over $K$, for $v \in V^n$. Then certainly $\{y_I \mid I \subset N\} \subset R$ is linearly independent over $K$. Moreover, it is easily shown that $\{y_I \mid I \subset N\}$ generates $R$ as a $K$-vector space. This proves: $\{y_I \mid I \subset N\}$ is a $K$-basis for $R$, and

$$\dim_K(R) = 2^n \quad \text{(by } \dim_K \text{ we mean dimension over } K).$$

The permutation group $S_n$ on $n$ symbols acts as a group of $K$-linear ring automorphisms on $R$ by permuting $\{x_i \mid i \in N\}$. The set of invariants

$$A = \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in S_n\}$$

is a subring of $R$. Put

$$z_j = \sum_{I \subset N, |I| = j} y_I, \quad 0 \leq j \leq n.$$ Then it is easy to see that $\{z_j \mid 0 \leq j \leq n\}$ is a $K$-basis for $A$, and

$$(3) \quad z_j \cdot v = \sum \{v' \in V^n \mid d(v, v') = j\}, \quad 0 \leq j \leq n, v \in V^n.$$ Since $A$ is a subring of $R$, $M$ is also an $A$-module.

Choose $u \in V^n$ arbitrary but fixed, and define $w(v) = d(v, u)$ for $v \in V^n$. Let $S_{V^n}$ be the full permutation group of $V^n$, and let $G$ be the subgroup $G = \{\sigma \in S_{V^n} \mid \sigma(u) = u, \text{ and } d(v, v') = d(\sigma(v), \sigma(v')) \text{ for all } v, v' \in V^n\}$. By permuting the basis vectors, $G$ acts $K$-linearly on $M$. This action is even $A$-linear, since for $\sigma \in G, 0 \leq j \leq n, v \in V^n$ we have:

$$\sigma(z_j \cdot v) = \sigma(\sum \{v' \mid d(v, v') = j\}) = \sum \{\sigma(v') \mid d(v, v') = j\}$$

$$= \sum \{v' \mid d(v, \sigma^{-1}(v')) = j\} = \sum \{v' \mid d(\sigma(v), v') = j\}$$

$$= z_j \cdot \sigma(v).$$
Therefore, \( M^G = \{ m \in M \mid \sigma(m) = m \text{ for all } \sigma \in G \} \) is an \( A \)-submodule of \( M \), and the map \( T: M \to M^G \), defined by

\[
T(m) = \sum_{\sigma \in G} \sigma(m),
\]

is an \( A \)-homomorphism. We wish to determine the structure of \( M^G \) as an \( A \)-module.

It is not hard to see that the orbits of the \( G \)-action on \( V^n \) are \( \{ \{ v \in V^n : w(v) = j \} \mid 0 \leq j \leq n \} \). Put

\[
m_j = \sum \{ v \in V^n : w(v) = j \} \in M, \ 0 \leq j \leq n,
\]

then it follows that \( \{ m_j \mid 0 \leq j \leq n \} \) is a \( K \)-basis for \( M^G \). Define the \( A \)-homomorphism

\[
A \xrightarrow{\psi} M^G \text{ by } \psi(a) = a \cdot u
\]

(we consider \( A \) as an \( A \)-module by left multiplication, \([1; 3]\)). Then

\[
\psi(z_j) = z_j \cdot u = \sum \{ v \in V^n : \langle v, u \rangle = j \} = m_j.
\]

So \( \psi \) maps a \( K \)-basis for \( A \) one to one onto a \( K \)-basis for \( M^G \). This implies that \( \psi \) is bijective. We have shown:

\[
A \cong M^G \text{ as } A \text{-modules}.
\]

Now suppose that a perfect \( e \)-error-correcting code \( C \subset V^n \) exists. Then one easily constructs \( e + 1 \) perfect \( e \)-error-correcting codes \( C_0, \ldots, C_e \subset V^n \) such that \( i \in w[C_i] \) \((0 \leq i \leq e)\). We first prove:

\[
\{ T(\Sigma C_i) \mid 0 \leq i \leq e \} \subset M^G \text{ is linearly independent over } K.
\]

**Proof of (5).** Let \( T(\Sigma C_i) = \sum_{j=0}^n k_{ij} m_j \) \((k_{ij} \in K)\); since \( C_i \) is \( e \)-error-correcting, we have \( w[C_i] \cap \{0, 1, \ldots, e\} = \{i\} \); therefore, if \( 0 \leq i \leq e, 0 \leq j \leq e \), the coefficient \( k_{ij} \) is nonzero if and only if \( i = j \), and (5) follows.
Put
\[ s = \sum_{i=0}^{e} z_i \in A . \]

By (3), the perfectness of \( C_i \) implies
\[ s \cdot \Sigma C_i = \Sigma V^n, \quad 0 \leq i \leq e . \]

Applying the \( A \)-linear map \( T \) we find
\[ s \cdot T(\Sigma C_i) = T(\Sigma V^n), \quad 0 \leq i \leq e . \]

Using (5) we conclude \( \dim_K \{ m \in M^g \mid s \cdot m = 0 \} \geq e \), and by (4) this is the same as
\[ (6) \quad \dim_K \{ a \in A \mid s \cdot a = 0 \} \geq e . \]

Therefore it seems useful to study the structure of \( A \).

For \( I \subset N \) we define the ring homomorphism \( \chi_I : R \to K \) by
\[ \chi_I(k) = k, \quad k \in K , \]
\[ \chi_I(x_i) = 0 \quad \text{if} \quad i \in I , \]
\[ \chi_I(x_i) = q \quad \text{if} \quad i \notin I . \]

The maximal ideals \( \ker(\chi_I) \) of \( R \) are mutually different, so \( \ker(\chi_I) + \ker(\chi_J) = R \) for \( I \neq J \). By the Chinese remainder theorem [3, II.2; 1, I.8.11] it follows that the \( K \)-linear ring homomorphism
\[ \chi = \prod_{I \subset N} \chi_I : R \to \prod_{I \subset N} K \]
is surjective (in \( \prod_{I \subset N} K \) addition and multiplication are defined componentwise); comparison of \( K \)-dimension shows that \( \chi \) is injective, so \( \chi \) is a ring isomorphism. For \( \sigma \in S_n, I \subset N, r \in K \) we have \( \chi_{\sigma[I]}(\sigma(r)) = \chi_I(r) \).

This implies: if \( I, J \subset N \) satisfy \( |I| = |J| \) then \( \chi_I \) and \( \chi_J \) have the same restriction to \( A \). Therefore
\[ \chi[A] \subset \{(k_J)_{J \subseteq N} \in \Pi_{J \subseteq N} K \mid k_J = k_{J'}, \text{ if } |J| = |J'|\}, \]

and counting dimension over \( K \) shows that this inclusion is in fact an equality. Putting

\[ I_x = \{1, 2, \ldots, x\}, \quad \chi_x = \chi_{I_x} | A \quad (0 \leq x \leq n), \]

we conclude that

\[ \cdot' = \Pi_{x=0}^{n} \chi_x : A \to \Pi_{x=0}^{n} K \]

is a \( K \)-linear ring isomorphism.

For \( k = (k_x)_{x=0}^{n} \in \Pi_{x=0}^{n} K \) we have obviously

\[ \dim_K \{k' \in \Pi_{x=0}^{n} K \mid k \cdot k' = 0\} = \{x \mid 0 \leq x \leq n, k_x = 0\} \}

Putting \( k = \chi'(s) \) and using (6) we find:

(7) \[ \{x \mid 0 \leq x \leq n, \chi_x(s) = 0\} \geq e. \]

From the definitions we compute

\[ \chi_x(z_j) = \Sigma_{I \subseteq N, \cup I_j = j} \chi_{I_x}(y_j) \]

\[ = \Sigma_{I \subseteq N, \cup I_j = j} (-1)^{\cup I_x \cap I_j} (q-1)^{\cup I_x \setminus I_j} \]

\[ = \Sigma_{i=0}^{l} \binom{x}{i} \binom{n-x}{j-i} (-1)^i (q-1)^{-i} \]

(8) \[ \chi_x(s) = \Sigma_{j=0}^{e} \chi_x(z_j) \]

\[ = \Sigma_{i=0}^{e} (-1)^i \binom{n-x}{j-i} \binom{x-1}{i} (q-1)^{-i} \]

\[ = P(x). \]

Since \( P(0) = \Sigma_{i=0}^{e} \binom{n}{e-i} (q-1)^{-i} \neq 0 \), Lloyd's theorem now follows from (7) and (8).
§2. Perfect group codes

Theorem 2. Let \( G_i, 1 \leq i \leq n, \) be a group with underlying set \( V. \) Suppose there exists a subgroup \( C \subset \Pi_{i=1}^n G_i \) such that the underlying set of \( C \) is a perfect \( e \)-error-correcting code of block length \( n \) over \( V, \) with \( e < n. \) Then \( q \) is a power of a prime \( p \) and each \( G_i \) is abelian of type \((p, p, \ldots, p).\)

Proof. Without loss of generality we may assume that the groups \( G_i \) have the same unit element \( 1 \in V (1 \leq i \leq n). \) Put \( u = (1)_{i=1}^n, \) and let \( w(g) = d(g, u) \) for \( g \in \Pi_{i=1}^n G_i, \) as in §1.

Let \( C \subset \Pi_{i=1}^n G_i \) be as in the statement of Theorem 2. Then \( u \in C \) since \( u \) is the unit element of \( \Pi_{i=1}^n G_i. \) If

\[
g = (g_i)_{i=1}^n \in \Pi_{i=1}^n G_i
\]

satisfies \( w(g) = e + 1, \) then the unique element \( c = (c_i)_{i=1}^n \in C \) for which \( d(g, c) \leq e \) cannot equal \( u, \) and therefore \( w(c) \geq 2e + 1. \) This is only compatible with \( w(c) = e + 1 \) and \( d(g, c) \leq e \) if \( w(c) = 2e + 1 \) and \( c_i = g_i \) for all \( i \) such that \( g_i \neq 1. \) We shall use this remark two times below.

Choose \( a_2 \in G_2 \) such that the order of \( a_2 \) in \( G_2 \) is a prime number \( p, \) and choose \( a_i \in G_i, a_i \neq 1, \) for \( 3 \leq i \leq e + 1. \) It is sufficient to prove

(i) every \( a \in G_1, a \neq 1, \) has order \( p \) in \( G_1; \)
(ii) \( a\beta = \beta a \) for all \( a, \beta \in G_1.\)

(i) Let \( \alpha \in G_1, \alpha \neq 1. \) Put

\[
g = (\alpha, \alpha_2, ..., \alpha_{e+1}, 1, ..., 1) \in \Pi_{i=1}^n G_i.
\]

Then \( w(g) = e + 1. \) By the above remark, some \( c \subset C \) has the following shape:

\[
c = (\alpha, \alpha_2, ..., \alpha_{e-1}, (\text{exactly } e \text{ of the remaining components } \neq 1)).
\]

Since \( C \) is a subgroup, \( c^p \in C, \) and

\[
c^p = (\alpha^p, 1, (\text{at most } e-1 \text{ of the remaining components } \neq 1)).
\]

Therefore \( w(c^p) \leq 2e \) which implies \( c^p = u \) and \( \alpha^p = 1. \)

(ii) Let \( \alpha, \beta \in G_1, \alpha \neq 1 \neq \beta. \) Put
The above remark yields $c, c' \in C$ which look like:

$$c = (\alpha, \alpha_2, \ldots, \alpha_{e+1}, 1, \ldots, 1),$$
$$c' = (\beta, \alpha_2, \ldots, \alpha_{e+1}, 1, \ldots, 1).$$

Then $d(cc', c'c') \leq e + 1$, and since $cc', c'c \in C$ it follows that $cc' = c$ and $\alpha \beta = \alpha x$. This completes the proof of Theorem 2.

References