Lecture 4: Adjunction, Yoneda and (co)limits

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Abstract

1 Introduction

2 Yoneda's lemma

References: for example, should be on wikipedia. But the lemma is rather trivial to state and prove, and the interesting thing is how you apply it, which is harder to give a reference for.

Let C be a category. Let $\mathrm{Fun}(C^{op}, \mathbf{Set})$ be the category whose objects are functors from C^{op} to $\mathbf{Set}$, and whose morphisms are natural transformations of functors. There is an obvious functor from C to $Fun(C^{op}, \mathbf{Set})$, sending an object $c \in \mathrm{ob}\, C$ to the functor sending $- \in \mathrm{ob}\, C$ to $\mathrm{Hom}_C(-, c)$, and sending a morphism $f \colon c \to d$ to the natural transformation

$$\mathrm{Hom}_C(d, -) \to \mathrm{Hom}_C(c, -); g \mapsto g \circ f.$$

Remark 2.1. Note that one can replace C by C^{op} at the start to obtain an equivalent but slightly different-looking setup. This latter version is slightly more commonly written (e.g.. on Wikipedia last time I checked), but the version I present here is the one I use every day.

Anyway, what is the lemma? We have written down a functor $C \to \mathrm{Fun}(C^{op}, \mathbf{Set})$, let us denote it '$h$' ('the 'Yoneda functor'). Then

Lemma 2.2. *(Yoneda's lemma) The functor* $h \colon C \to \mathrm{Fun}(C^{op}, \mathbf{Set})$ *is fully faithful.*

Proof. Proof omitted. You can look it up online easily if you want, but you should really try to prove it as an exercise. □

Recall that a functor being fully faithful means that it induces a bijection on the hom sets. Equivalently, we can say that 'h realises C as a full subcategory of $\mathrm{Fun}(C^{op}, \mathbf{Set})$'. Recall that 'full faithful' is in some sense the categorical version of being injective:

- A functor is an equivalence iff it is fully faithful and essentially surjective;
- A fully faithful functor realises the first category as a full subcategory of the second, just as an injective map of sets realises one set as a subset of the other.

Note that the Yoneda functor is in general *not* essentially surjective. A stupid example: if C is a category with one object and one morphism, then $\mathrm{Fun}(C^{op}, \mathbf{Set})$ is equivalent to $\mathbf{Set}$, and the Yoneda embedding maps the object of C to the set $\{\mathrm{id}_c\}$ - clearly not an equivalence! The Yoneda lemma thus has perhaps two kinds of applications:

1. Use it to understand the morphisms in C better;
2. Find some clever way to cut out the full subcategory C inside $\mathrm{Fun}(C^{op}, \mathbf{Set})$, then use this as another way to view C. Maybe a slightly larger full subcategory is better behaved — this is in some sense the idea behind generalising schemes to algebraic spaces — analogous to generalising differential forms to currents ('Dirac delta' etc).

A less trivial example: fix an integer $n \geq 0$, and let $\mathbf{Mfd}_n$ be the category of all topological manifolds (with morphisms the continuous maps), and let $\mathbf{D}$ be the full subcategory whose objects are isomorphic to the open unit disc in $\mathbb{R}^n$. There are obvious functors

$$\mathbf{D} \to \mathbf{Mfd}_n \to \mathrm{Fun}(\mathbf{Mfd}_n^{op}, \mathbf{Set}) \to \mathrm{Fun}(\mathbf{D}^{op}, \mathbf{Set}),$$

where the last arrow just restricts the functor to $\mathbf{D}$. The first arrow is by definition fully faithful, and Yoneda tells us the second is too. On the other hand, the third arrow is certainly not fully faithful: let F_0 be the functor sending each manifold to the empty set (and each morphism to the identity on it), and let $F_1 = \mathrm{Hom}(-, \emptyset)$. Then $F_0(T) = F_1(T)$ unless T is empty, in which case they differ. In particular, the restrictions of F_0 and F_1 to $\mathbf{D}^{op}$ are isomorphic, but F_0 and F_1 are not isomorphic, so $\mathrm{Fun}(\mathbf{Mfd}_n^{op}, \mathbf{Set}) \to \mathrm{Fun}(\mathbf{D}^{op}, \mathbf{Set})$ cannot be fully faithful (check this for yourself if you are not so familiar with full faithfulness).

Given this, the following lemma is perhaps surprising:

Lemma 2.3. *The composite functor*

$$H \colon \mathbf{Mfd}_n \to \mathrm{Fun}(\mathbf{D}^{op}, \mathbf{Set})$$

is *fully faithful.*

Proof. Let M, N be two n-dimensional manifolds. We first prove injectivity on the hom-sets. Let f, $g\colon M \to N$ be two morphisms (continuous maps), and suppose $f \neq g$. Then there exists a point $m \in M$ such that $f(m) \neq g(m)$. We have two natural transformations, $f \circ -$ and $g \circ -$ from $H(M)$ to $H(N)$, and we want to prove they are not equal. Writing D for the unit disc in $\mathbb{R}^n$, let $\varphi\colon D \to M$ be the constant map sending D to m. Then $f \circ \varphi$ is the constant map sending D to $f(m)$, and $g \circ \varphi$ is the constant map sending D to $g(m)$, which are not equal. Hence $H(M) \neq H(N)$.

Injectivity was pretty trivial, the content is in surjectivity, which we now prove. Let M, N be as above, and suppose we have a natural transformation $F\colon H(M) \to H(N)$. We want to build a continuous map from M to N. Choose an open cover $\mathcal{U} = \{u_i\colon U_i \to M\}_{i \in I}$ of M by copies of D (so each U_i is isomorphic to D — but we give them different names for convenience). Note that this is possible because M is a manifold. For each pair i, $j \in I$, choose an open cover $\mathcal{U}_{i,j} = \{u_k^{i,j}\colon D \to M\}_{i \in K_{i,j}}$ of the intersection $U_i \cap U_j$ (intersection as subsets of M).

Now to define the map $f\colon M \to N$. Given a point $m \in M$, let $i \in I$ be such that m lies in the image of $u_i\colon U_i \to M$. Then $F(u_i)$ is a map $U_i \to N$, and we define

$$f(m) = F(u_i)(u_i^{-1}(m)).$$

A-priori this depends on the choice of i (there could be many different u_i having m in their images). But if $u_j\colon U_j \to M$ contains m, then let $\mathcal{U}_{i,j} = \{u_k^{i,j}\colon D \to M\}_{i \in K_{i,j}}$ be a cover of the intersection of U_i and U_j as above, and suppose that $u_k^{i,j}\colon D \to M$ hits m. Write $g_i\colon D \to U_i$ for the induced inclusion, and similarly $g_j\colon D \to U_j$, so that $u_k^{i,j} = u_i \circ g_i = u_j \circ g_j$ as maps $D \to M$. By definition of a natural transformation we have that

$$F(u_k^{ij}) = F(u_i) \circ g_i = F(u_j) \circ g_j\colon D \to N.$$

Evaluating these maps at the point $g_i^{-1}(u_i^{-1}(m)) = g_j^{-1}(u_j^{-1}(m))$, we see that

$$F(u_i)(u_i^{-1}(m)) = F(u_j)(u_j^{-1}(m)),$$

and so $f(m)$ is well defined independent of the choice of neighbourhood.

It still remains to check that the map f so defined is continuous. But locally it is given by pulling back along a homeomorphism from D to a patch of M, then going forward along another homeomorphism from D to a patch of N, so it is clearly continuous. □

Remark 2.4.

1. Intuitively, we have just shown that you can recover an n-manifold if you know all the maps from the open unit disc in $\mathbb{R}^n$ to it. Note that you can absolutely *not* recover a manifold just knowing the maps from a point to it (unless $n = 0$) — this only tells you the cardinality, which is probably $2^{\aleph_0}$ (the cardinality of the real numbers).

2. It is really important that we worked with manifolds, in order to get the surjectivity on the hom-sets. If for example you try to run the above argument replacing M by say the Cantor set, then the surjectivity will really fail — intuitively, D can't 'see' the topology on the Cantor set.

3. This construction may seem a little strange, but it is actually a prototype for the idea that a representable functor should be a sheaf in every (subcanonical) Grothendieck topology, and a sheaf is determined by what it does to a base of the topology. This was for example a key step in my recent paper [BH16].

2.1 Defining schemes

To emphasise the power of the ideas we have just introduced, we can use them to very quickly give the definition of a scheme (of course, if you want to do anything with it, you have a lot more work to do!). Firstly, the category **Aff** of affine schemes is the opposite to the category of commutative rings. Let $\mathbf{F} = \mathrm{Fun}(\mathbf{Aff}^{op}, \mathbf{Set})$. Then the category of schemes is the smallest full subcategory of $\mathbf{F}$ which contains the affine schemes (via the Yoneda embedding) and which is closed under fibred coproducts along open immersions, where an open immersion is a formally étale (see [Sta13, Tag 00UQ]) monomorphism which is locally of finite presentation (i.e. preserves all limits and colimits, see later).

3 Limits and colimits

We start with limits — definition and examples. Then define colimits dually, and look at a bunch more examples. Could give both definitions at once, but it makes one more likely to get the directions of the arrows mixed up. First we need to recall initial and terminal objects.

3.1 Initial and terminal objects

Brief recap on terminal objects: $d \in \mathrm{ob}\, D$ is *terminal* if for every $e \in \mathrm{ob}\, D$ there exists a unique map $e \to d$. Exercise: a terminal object is unique up to unique isomorphism if it exists. What is the terminal object in the category of sets (if there is one)? What about (abelian) groups? What about topological spaces? What about commutative rings? What about non-empty sets?

An initial object in C is a terrain object in C^{op}. Same exercises as above, but for initial!

Terminology: terminal = final, initial = cofinal.

3.2 Limits

Let C be a category and I another category (maybe small/finite). A *diagram in C of shape I* is a functor from I to C. If $F\colon I \to C$ is a diagram of shape I, then a *cone to F* is an object N of C together with morphisms $\psi_i\colon N \to F(i)$ for every object $i \in \operatorname{ob} I$, such that all the obvious triangles commute — more precisely, for all $g\colon i \to j$ in I, we have

$$\psi_j = g \circ \psi_i.$$

If $(N, \{\psi_i\}_I)$ and $(M, \{\varphi_i\}_I)$ are two cones to F, a morphism from $(N, \{\psi_i\}_I)$ to $(M, \{\varphi_i\}_I)$ is a map $h\colon N \to M$ in C such that again the obvious triangles commute; for all $i \in \operatorname{ob} I$,

$$\psi_i = \varphi_i \circ h.$$

In this way we have defined a category of cones to F. A *limit of F* is a terminal object in the category of cones to F, if one exists.

This definition may seem a bit involved, but it is actually very useful and nice to work with. Key is to look at a bunch of examples.

3.2.1 Very simple example

Take a cat with two objects and a single morphisms between them. Then the limit always exists, and is the image of the source morphism (together with the identity map to itself, and the given map to the other object — it has to be a cone, so we should specify this data, though often people do not bother).

3.2.2 Products

Let C be the category of sets or topological spaces or.... Let I be a discrete category, i.e. only morphisms are identities. Then to give a *diagram of shape I* is just to give a collection of objects of C of cardinality $\#\operatorname{ob} I$. If F is such a diagram then the limit over F is just the product of that collection of objects. Let's check this for sets in the simple case where I has only two objects, call them 1 and 2. Say the diagram sends 1 to a set S_1 and 2 to a set S_2. A cone over F is just a set together with maps to S_1 and S_2 (no commutativity to check). You can check easily that if (S, f_1, f_2) is a set together with maps to S_1 and S_2, then there is a unique map from the product $S_1 \times S_2$ to S such that the obvious diagrams commute. Hmm, I feel I am saying a very simple thing at such length that is becomes confusing — but if you work out the details for yourself it should be very easy.

3.2.3 Equalisers and kernels

Let I be a category with objects 1 and 2, and with two arrows a and b from 1 to 2 (and the identities on 1 and 2). If F is a diagram to C of shape I then the limit of this diagram is called the 'equaliser' of $F(a)$ and $F(b)$. For example

- If $C = \mathbf{Set}$, then the equaliser is exactly the set of elements $x \in F(1)$ such that $F(a)(x) = F(b)(x)$;
- if $C = \mathbf{Gp}$ the category of groups (or abelian groups or rings or ...) and $F(a)$ is the zero map, then the equaliser of $F(a)$ and $F(b)$ is exactly the kernel of $F(b)$.

3.3 Colimits

Colimits are just defined dually to limits. Given a diagram to C of shape I, a *cone from* F is an object of C together with compatible maps from $F(i)$ for each $i \in \operatorname{ob} I$, and a morphism of cones from F is analogously defined. Then the colimit of F is an initial object of the category of cones from F.

Again, colimits are unique up to unique isomorphism of they exist. A diagram F can have both a limit and a colimit (or either one, or neither), though often one is more interesting than the other.

3.3.1 Coproducts

Coproducts are colimits of diagrams with discrete shape category I (just as products are limits of such diagrams). For example, in **Set**, a colimit is exactly a disjoint union.

3.3.2 Coequalisers and Cokernels

Let $C = \mathbf{AbGp}$, the category of abelian groups. Let I be the same shape category as we used to define equalisers. Then the colimit of this diagram is the coequaliser. If $F(a)$ is the zero map, then the coequaliser is called a cokernel, and it is (in classical language) the quotient of $F(2)$ by the image of $F(b)$.

3.4 Fibred product and coproducts

Let I be a diagram with three objects 1,2,3 and with maps $a\colon 1 \to 2$, $b\colon 1 \to 3$ (and the identities). Let $F\colon I \to C$ be a diagram of shape I (recall that this just means a functor!). Then the limit over this diagram is just $F(1)$, so not very interesting. The colimit is called the 'fibred coproduct of 2 and 3 over 1'. For example, if $C = \mathbf{Top}$ and $F(a)$, $F(b)$ are inclusions of open subsets, then the fibred coproduct exists, and is just given by glueing 2 to 3 along the open subset 1. If $C = \mathbf{Rng}$ (commutative rings) then the fibres coproduct is just the tensor product of algebras.

Let I be a diagram with three objects 1,2,3 and with maps $a\colon 2 \to 1$, $b\colon 3 \to 1$. Let $F\colon I \to C$ be a diagram of shape I. The colimit of the diagram is $F(1)$, so not interesting. The limit is called the 'fibred product of 2 and 3 over 1'. For example, if $C = \mathbf{Set}$ then the fibred product is the subset of 2×3 consisting of pairs (x_2, x_3) such that $F(a)(x_2) = F(b)(x_3)$. Fibred products of affine schemes are just given by taking tensor product of the rings (=fibred coproduct, see above). Fibred products of schemes always exist, and are very useful. Fibred coproducts of schemes do not always exist, but can still be useful (though are much less ubiquitous).

Some examples: if $C = \mathbf{Set}$ or $\mathbf{Top}$ and a, b are injective, then the fibred product is just the intersection of 2 and 3. If 1 is a terminal object in C then the fibred coproduct is just the product. A terminal object in **Top** is a singleton, so you can perhaps think of fibred products in **Top** as interpolating between products and intersections...

3.5 Terminology and aide-memoire

Copied from Wikipedia: Older terminology referred to limits as "inverse limits" or "projective limits," and to colimits as "direct limits" or "inductive limits." This has been the source of a lot of confusion.

There are several ways to remember the modern terminology. First of all,

- cokernels,
- coproducts,
- coequalizers
- codomains

are types of colimits, whereas

- kernels,
- products
- equalizers, and
- domains

are types of limits.

4 Adjoint functors

By this point I am probably very short of time, and I do not have all that much to add on this topic over what can be found in standard sources such as Wikipedia. Since I do not expect to have time to cover it, this section of the notes is rather hastily-written, and you may prefer to read these things elsewhere, but if you just want a quick overview it should suffice.

Let C and D be two categories, let $f : C \to D$ and $g : D \to C$ be functors. We say f and g are adjoint (f is right adjoint to g, and g is left adjoint to f) if there exists a natural isomorphism between the two functors

$$\hom_C(g-, -) : D^{\mathrm{op}} \times C \to \mathrm{Set}$$

and

$$\hom_D(-, f-) : D^{\mathrm{op}} \times C \to \mathrm{Set}\,.$$

In particular, such a natural isomorphism specifies a bijection

$$\hom_C(g(d), c) \cong \hom_D(d, f(c))$$

for all $c \in C$ and $d \in D$. [Possible aid to memory: the functor which is right adjoint appears on the right inside the hom, and conversely].

4.1 Examples

4.1.1 Free abelian groups

Let $f : \mathrm{Ab} \to \mathrm{Set}$ be the forgetful functor. This has a *left* adjoint $g : \mathrm{Set} \to \mathrm{Ab}$; it is the functor taking a set to the free abelian group generated by that set.

4.1.2 Add more examples here

Get them from Wikipedia...

4.1.3 Schemes

Let $f : LRS \to \mathrm{Rng}^{\mathrm{op}}$ be the functor sending $(X, \mathcal{O})$ to $\mathcal{O}(X)$. Then f has a *right* adjoint $\mathrm{Spec} : \mathrm{Rng}^{\mathrm{op}} \to LRS$ - see Lecture 1. [Homs from a scheme to an affine scheme are the same as homs from the global sections to the ring. If you get confused, consider the case of $\mathbb{P}^1$ over spec of a field.]

Table 1: Easy to characterise

maps out of	maps into
colimits tensor products cokernels stalks	limits products kernels
left adjoint sheafification	right adjoint

5 Commuting of adjoints and (co)limits

Let $f : C \to D$ be a functor, and $g : D \to C$ be a right adjoint to f. Let S be a category, and $\varphi : S \to C$, $\psi : S \to D$ functors.

1. Suppose $\operatorname{colim}\varphi$ exists. Then $\operatorname{colim}(f \circ \varphi)$ also exists, and we have

$$f(\operatorname{colim}\varphi) = \operatorname{colim}(f \circ \varphi)$$

2. Suppose $\lim\psi$ exists. Then $\operatorname{colim}(g \circ \psi)$ also exists, and we have

$$g(\lim\psi) = \operatorname{colim}(g \circ \psi).$$

Proof. We do only the first one.

We will show that every cone from $f \circ \varphi$ admits a unique map from $f(\operatorname{colim}\varphi)$, in other words $f(\operatorname{colim}\varphi)$ is initial in the category of cones from $f \circ \varphi$, which is what we wanted.

Consider a cone N from $f \circ \varphi$, so for all $s \in S$ we get maps $f(\varphi(s)) \to N$ (such that a bunch of diagrams commute). Apply adjunction, this is equivalent to a bunch of maps $\varphi(s) \to g(N)$, with more diagrams commuting. This makes $g(N)$ a cone from φ, so since $\operatorname{colim}\varphi$ is initial we find that the maps above factor as

$$\varphi(s) \to \operatorname{colim}\varphi \to g(N).$$

Applying adjunction again, we get canonical maps $f(\operatorname{colin}\varphi) \to N$. $\square$

Exercise: Do the right adjoints/ limits version.

Maps from a colimit are just the same as maps from each object in the image of the diagram, plus commuting maps. Maps to a limit have a similarly easy description. Hence:

Moreover, two things from the same side of the diagram tend to commute with one another. Nb. the things at the bottom are not (co)limits as defined so far, so it seems a bit silly to put them there. However:
- it is still easy to characterise maps from/to them;
-more generally, it is possible to define colimits as left adjoints to a certain functor, and similarly limits as right adjoints. Hence the columns do fit together.

There is a whole formal theory generalising limits and colimits, which will tell you immediately which things will commute 'for formal reasons'. However, before you get around to learning that, the rough principals outlined in the above table should give you a handy guide to which of the functors you should expect to commute for 'formal reasons', and which you should expect either not to commute, or that you will have to do some real work to prove that they do commute in your situation.

References

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[Sta13] The Stacks Project Authors. Stacks project. `http://stacks.math.columbia.edu`, 2013.