Valuativisations and Néron models

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Abstract

Notes for a talk at Garnet Akeyr's log geometry seminar. The contents are work in progress. Statements are certainly not optimal, and proofs may not be correct. Comments welcome!

1 Introduction

The fundamental reference for the valuativisation is [Kat89].

2 Valuativisations

2.1 Recap of Zariski and étale log structures

We work the standard definition of a log scheme as a scheme $(X, \mathcal{O}_X)$ together with a sheaf of monoids M_X on the small *étale* site X_{et}, and a map $\alpha\colon M \to \mathcal{O}_X$. We are mainly interested in fine saturated (fs) log schemes.

In this note we will also need to work with *Zariski* log schemes. Write X_{Zar} for the small Zariski site on X, and $\epsilon\colon X_{et} \to X_{Zar}$ for the projection. One approach is to define a Zariski log scheme in the same way as the above except that M_X is a sheaf of monoids on X_{Zar}. But this is not what we will do; rather, we say a log scheme X is *Zariski* if the natural map $\epsilon^*\epsilon_* M_X \to M_X$ is an isomorphism (cf. [Niz06, after prop 2.4]).

2.2 Log locally ringed spaces

One can define a log locally ringed space (log LRS) in a very similar to the way we define a log scheme; a *pre-log* LRS is a LRS $(X, \mathcal{O}_X)$ together with a sheaf of monoids M on X, and a map of sheaves of monoids $\alpha\colon M \to \mathcal{O}_X$, and it is a *log*

LRS if the restricted map $\alpha^{-1}\mathcal{O}_X^\times \to \mathcal{O}_X^\times$ is an isomorphism. The forgetful map from log LRS to pre-log LRS has a left adjoint just as for schemes.

There is one key difference in the above from what we do with schemes, namely that the sheaf of monoids is only defined on the small Zariski site of X, whereas for schemes we work on the small étale site. This is of course because LRS don't have an étale site!

They way that log LRS will arise for us is as limits of towers of log blowups of a log scheme. Now as long as one makes a 'large enough' blowup, then this log scheme will automatically become Zariski, so this does not cause problems for us. This will be expanded upon below.

2.3 The schematic locus of a (log) LRS

If X is a locally ringed space, we write X^{sch} for the largest open subspace which is a scheme. This makes sense; being a scheme is a property of a LRS, and one checks easily that if U, $V \subseteq X$ are schemes then so is $U \cup V$. The same discussion applies to log LRS, and we use the same notation (note that in this case X^{sch} will be a Zariski log scheme).

2.4 Valuative monoids and valuative log LRS

Let M be a monoid. Recall that M is *integral* if the map $M \to M^{gp}$ is injective. We say M is *valuative* if M is integral and if for every $m \in M^{gp}$, either $m \in M$ or $-m \in M$.

Example 2.1. The monoid $\mathbb{N}^a \times \mathbb{Z}^b$ is valuative if and only if $a \leq 1$.

We say a log LRS X is valuative if all its stalks are valuative.

Example 2.2. The toric scheme $\mathbb{A}^a \times \mathbb{G}_m^b$ (with toric log structure) is valuative if and only if $a \leq 1$.

2.5 The valuativisation X^{val} and its largest schematic open $X^{val-sch}$

Let X be a log scheme. We define $\underline{Val-LRS}_X$ to be the full subcategory of $\underline{LRS}_X$ whose objects are valuative log *LRS*.

Definition 2.3. The *valuativisation* X^{val} is the terminal object of $\underline{Val-LRS}_X$, if one exists.

Theorem 2.4 (K. Kato). *Assume X is quasi-coherent and Zariski. The valuativisation X^{val} exists.*

Proof. We follow the proof of Kato [Kat89]. Since the valuativisation is defined by a universal property, it is unique up to unique isomorphism if it exists, so it is enough to construct it Zariski locally (by the trivial observation that Zariski descent is effective for log LRS over X). Since X is quasicoherent Zariski it has charts Zariski locally, so we reduce to the case where X admits a strict map to $\mathrm{Spec}(P \to \mathbb{Z}[P])$ for some monoid P. Moreover, it is not hard to check that forming the valuativisation commutes with strict pullback whenever it exists, so we reduce to the case where $X = \mathrm{Spec}(P \to \mathbb{Z}[P])$. We may moreover assume that P is integral, see [Kat89, 1.2.9].To conclude the proof we will need the notion of log blowups, see section 2.5.1. The proof is then immediate from lemma 2.7. □

2.5.1 Log blowups

Remark 2.5.

1. We define log blowups or schemes of the form $\mathrm{Spec}(P \to \mathbb{Z}[P])$, and define them to commute with strict base-change.

2. There are more modern/refined notions of log blowups for fine (= integral + coherent) or fs log schemes, coming with a suitable universal property (see eg Ogus' notes, or [Niz06]. But this version feels very concrete, and the proof of lemma lemma 2.7 gives a nice illustration of how to work with valuative monoids. Also, this version is slightly more general (no fineness assumptions needed).

Let P be a monoid and let $X = \mathrm{Spec}(P \to \mathbb{Z}[P])$. An *ideal* of P is a subset $I \subseteq P$ so that $IP \subseteq I$. If I and J are ideals of P, we writ IJ for the ideal generated by their product. Let $\phi = \phi_P$ be the set of all finitely generated non-empty ideals of P. We make ϕ into a directed system by saying that $I' \geq I$ if and only if there exists $J \in \phi$ such that $I' = IJ$.

Let $I \in \phi$. As a scheme, we define the *blowup* X_I of X at I to be the usual blowup of X at the ideal sheaf generated by images of elements of I. Equivalently, $X_I = \mathrm{Proj} \oplus_{n \geq 0} I^n$.

We want to endow X_I with a log structure. Of course, we could for example just require that $X_I \to X$ be strict, but this log structure will not behave as we want. Instead we will do this by working on affine patches of X_I. Given an element $a \in I$, define $P[a^{-1}I]$ to be the submonoid of P^{gp} generated by set $a^{-1}I = \{^{-1}x : x \in I\}$. Then $X_{a^{-1}I} = \mathrm{Spec}\, \mathbb{Z}[P[a^{-1}I]]$ is in a natural way an affine patch of the blowup X_I, and the $X_{a^{-1}I}$ cover X_I, and this remains true if we restrict a to running over a generating set of I. Now $X_{a^{-1}I}$ is defined as the spectrum of a monoid ring, and hence has a natural log structure. These glue together as a runs over a generating set for) I, and this gives a log structure to the blowup X_I.

One checks without difficulty that, if $I' \geq I$ in ϕ, then we have a natural map $X_{I'} \to X_I$ over X, and in this way the X_I form a directed system.

2.5.2 Toric blowups

In the case where the log scheme we start with is $\mathbb{A}^n$ and the ideal we blow up at is radical, there is a very nice combinatorial picture (the non-radical case can be treated similarly but is more complicated). The wonders of log schemes allow us to trivially apply this picture whenever the log scheme comes from a NCD. Let's describe the picture.

The fan of $\mathbb{A}^n$ is $P \coloneqq \mathbb{N}^n$. Let I be a finitely generated ideal. Then the blowup is covered by affine patches of the form $P[a^{-1}I]$, the submonoid of P^{gp} generated by $a^{-1}I$. It is enough to let a run over generators of I.

What does this look like on the fan? Draw the ideal in P, and let a be a point in the ideal. Then $a^{-1}I$ is given by translating a to the origin, and the monoid $P[a^{-1}I]$ is the submonoid of $\mathbb{Z}^2$ generated by this set.

There is an important distinction between the radical and non-radical cases. If the ideal I is radical (in the obvious sense) then the monoid $P[a^{-1}I]$ corresponds to a (r.p.) cone. But in the general case it is just a finitely generated submonoid - see the pictures on the next page for examples.

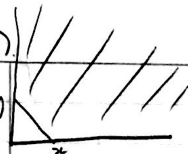
In the radical case, the corresponding dual cone is $P[a^{-1}I]^\vee$. To obtain the fan, it is enough to run over generators of the ideal.

Remark 2.6. The above discussion of radical vs non-radical should more generally be seen in terms of f vs fs blowups. The log blowup can/should be defined by a universal property ('principalising log ideal sheaves'), just as in the classical case. If one starts with an fs log scheme (such as $\mathbb{A}^n$), one can then choose whether to apply the universal property in the category of fine log schemes ot that of fs log schemes. In general the resulting objects will be different, and the difference is essentially whether on e saturates the ideal before blowing up (equivalently, normalises the blowup). So fs blowups of $\mathbb{A}^n$ will always be (normal) toric varieties, but f blowups (as we consider) need not be.

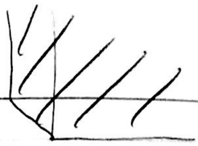
We are not assuming our log schemes are even fine, so this general machinery does not quite apply in this setting, though for the applications I have in mind it would be harmless. However, it is also nice to see how these things work in the language that Kato uses - everything is very concrete and explicit, while also being very general.

Blowing up $\mathbb{A}^2$. $\quad P = \mathbb{N}^2$

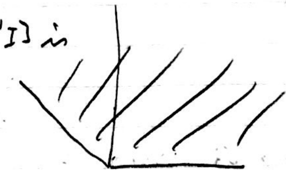
eg $I = (x, y)$



$a = x$, then $a^{-1}I$ is given by



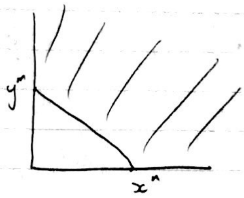
So $P[a^{-1}I]$ is



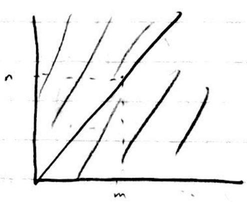
And $P[a^{-1}I]^\vee$ is

eg $I = (x^2, xy, y^2)$ Then get exactly the same $P[a^{-1}I]^\vee$.

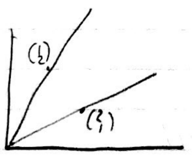
eg $I = (x^n, y^m)$.
m, n coprime.



Fan is



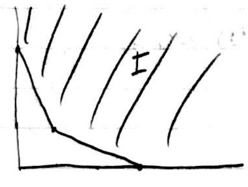
eg To make fan



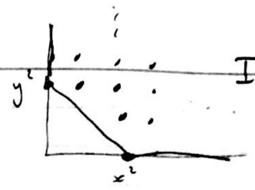
need slopes

and

Many choices for I, but simplest is (x^3, xy, y^3):

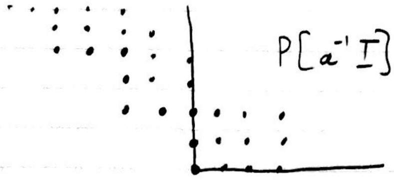


Non saturated picture



Say I blow up at $I = (x^2, y^2)$.

Taking eg. $a = x^2$, I get



– note a bunch of points 'missing' from the line on the left.

Now $P[a^{-1}I]$ is still a perfectly good fin. gen. submonoid of $\mathbb{Z}^2$, (eg. gen. by x, y, $\frac{y^2}{x^2}$), but does not correspond to a cone.

2.5.3 Limits of log LRS

In the notation of the above section, we want to look at the limit of the X_I as I runs over ϕ. What exactly does this mean? As a topological space, $\lim_{I\in\phi} X_I$ is the limit of the topological spaces of the X_I. The structure sheaf and sheaf of monoids on $\lim_{I\in\phi} X_I$ are obtained by taking the colimits of the (pullbacks of the) corresponding sheaves on the X_I.

Note that ϕ is a directed system. In general, pullbacks of open subsets only form a sub-basis of open subsets of the limit, but one checks easily that in the directed case they give an actual basis.

2.5.4 Finishing the proof of the existence of the valuativisation

Lemma 2.7. $\lim_{I\in\phi} X_I$ *is the valuativisation of* X*.*

Proof. Kato leaves this as an exercise. Indeed it is not hard, but it gives a nice illustration of how valuative monoids can behave like valuation rings, so we give some details. We need to check two things: (1) that $\lim_{I\in\phi} X_I$ is valuative, and (2) that map $T \to X$ from a valuative log LRS factors uniquely via $\lim_{I\in\phi} X_I$.

First we show (1), that $\lim_{I\in\phi} X_I$ is valuative. Let $x \in \lim_{I\in\phi} X_I$ be any point, and M_p the stalk of the monoid at x. Since colimits commute, this stalk M_x is just the colimit of the stalks of the images $x_I \in X_I$ of x as I runs over ϕ. Note moreover that all these stalks are submonoids of P^{gp}, and the colimit can be taken as the union of the submonoids.

Let $m \in M_x^{gp} = P^{gp}$; we must show that either $m \in M_x$ or $m^{-1} \in M_x$. Write $m = a/b$ for some a, $b \in P$. Then the ideal $I := (a, b)$ of P is in Φ, and after pullback to X_I the ideal (a, b) becomes principal. Since everything is integral, we find that either $a = bc$ of $b = ac$ for some element c of the stalk M' of X_I at x_I. This shows that either m (in the first case) or m^{-1} (in the second case) lies in the stalk M', and hence in M_x as required.

Next we must show (2), so let $T \to X$ be a map from a valuative log LRS. Let $I \in \phi$ be an ideal of P. We want to construct a factorisation of $T \to X$ via the blowup X_I. By quasicoherence (or something) we can assume that $T = \operatorname{Spec} \mathbb{Z}[M]$ for some monoid M, and the map $T \to X$ corresponds to a map of monoids $f\colon P \to M$. From our description above o the charts of a log blowup, it is enough to find some $a \in I$ such that the map $P \to M$ extends to a map $P[a^{-1}I] \to M$ (uniqueness is clear since P is integral). Well, let $I = (x_1, \dots, x_n)$. Order the x_i using M; we say $x_i \le x_j$ iff $f(x_j)/f(x_i)$ lies in $M \subseteq M^{gp}$. Valuativity of M makes this a total ordering, so choose a to be a minimum element among these generators. Then easy to see that this choice of a works. This argument illustrates how a valuative monoid can behave like a valuation ring, in the sense that it can give us a way to compare the 'sizes' of elements. □

Example: affine plane with toric log structure

2.6 Functoriality of the formation of X^{val} and $X^{val-sch}$

Definition 2.8. Let X be a quasi-coherent Zariski log scheme, with valuativisation X^{val}. The *schematic locus* of the valuativisation we write as $X^{val-sch}$ - it is a log scheme over X (necessarily Zariski). The *locally-finitely-presented locus* of the valuativisation we write as $X^{val-lfp}$ - it is a log scheme (whose underlying scheme is) locally of finite presentation over X (again necessarily Zariski).

Let $f\colon X \to Y$ be a map of log schemes. Then X^{val} is valuative, and so we obtain a canonical map $X^{val} \to Y^{val}$. Can we say something similar for the schematic/lfp loci? About the schematic locus I am not sure, but about the lfp I can say something. We have maps $X^{val-sch} \to X^{val} \to Y^{val}$; the question is whether the composite $X^{val-sch} \to Y^{val}$ factors via the open immersion $Y^{val-sch} \hookrightarrow Y^{val}$? Same question for $X^{val-lfp}$?

Not always:

Example 2.9. Let $\mathbb{N}^2 \to \mathbb{N}$ be projection onto one factor. On level of log schemes, I am including one coordinate axis into the affine plane, write $X \rightarrowtail Y$. Then X is valuative. And we know the valuativisation of Y. And we also know that the map from $X = X^{val} = X^{val-sch} = X^{val-lfp}$ lifts to Y^{val}, but it does not factor via $Y^{val-sch} = Y^{val-lfp}$.

So we need some extra assumptions, about non-degeneracy of the map.

Theorem 2.10. *Let $f\colon X \to Y$ be a separated morphism of integral Zariski quasi-coherent log schemes. Assume:*

1. *The log structure on $\underline{Y}$ comes from a NCD;*
2. *X is locally noetherian and the log structure is trivial on some dense open;*
3. *There exists an open cover $U_i \to X$ such that each $X^{val} \times_X U_i$ is irreducible;*
4. *Let Y_{triv} be the largest open where the log structure is trivial. Then $f^{-1}U$ is dense in X.*

Then the composite $X^{val-lfp} \hookrightarrow X^{val} \to Y^{val}$ factors via $Y^{val-lfp} \hookrightarrow Y^{val}$.

The theorem may hold rather more generally than stated. Note however that some assumptions are needed; for example, as discussed above, this fails if we take $X \to Y$ to be the map on log schemes corresponding to the projection map $\mathbb{N}^2 \to \mathbb{N}$.

Proof. We reduce immediately to the case where X is noetherian and X^{val} is irreducible. Let $x \in X^{val-lfp}$. We want to show that the image of x in Y^{val} lands inside the open $Y^{val-lfp}$. By definition, $X^{val-lfp}$ is locally of finite presentation over X. Moreover, $X^{val} \to X$ is surjective [Kat89, 1.3.8] and X^{val} (and hence $X^{val-lfp}$) is irreducible by our assumption, and the non-emptyness of $X^{val-lfp}$). Hence the preimage of Y_{triv} in $X^{val-lfp}$ is dense. Hence by [GD61, 7.1.9] there exists a trait T and a map $t\colon T \to \underline{X^{val-lfp}}$ mapping the closed point to x and the generic point to a point in the preimage of Y_{triv}.

Composing, we obtain a map $T \to Y$, sending the generic point to Y_{triv}. By separatedness, it is enough to show that this lifts to a map $T \to Y^{val-lfp}$ (noting that $Y^{val-lfp} \to Y$ is an isomorphism over Y_{triv}). This is a local problem, so we can just consider the case where $Y = \operatorname{Spec}(\mathbb{N}^r \to \mathbb{Z}[\mathbb{N}^r])$. Then the orders of vanishing of the coordinate divisors along T yields a vector of r integers, hence a ray in $\mathbb{N}^r$. Let F be a subdivision of $\mathbb{N}^r$ which includes this ray, and let I be the log ideal whose blowup leads to that subdivision. Then $T \to Y$ lifts to a map $T \to Y_I$, and it lands within the valuative locus. Since forming the valuativisation commutes with base change, we see that subsequence blowups will induce isomorphisms in an open neighbourhood of the image of T, and so $T \to Y^{val}$ will factor via $Y^{val-lfp}$. □

2.7 Extension to log stacks

LRS and algebraic stacks do not very naturally live together in some (2)-category, so it is a bit awkward to extend the notion of the valuativisation to the case of log stacks or log algebraic spaces. This is done in [Sar16], but is not necessary for our present purposes. In contract, the pro-valuativisation and the schematic locus of the valuativisation extend in a very straightforward way to stacks. For $X^{val-sch}$, this is simply the fact that it is unique up to unique isomorphism, and hence descends to an algebraic space over X. For the pro-valuativisation (limit of log blowups) one must think slightly more, but one just ends up with a pro-algebraic space over X — see [Sar16].

3 The Néron model on the valuativisation

The starting point for this section is the following result from [Hol17].

Corollary 3.1. *Let S be a regular scheme, $X \rightarrowtail$ a closed subscheme of codimension 1, with Z also regular. Let $C \to S$ be a nodal curve (i.e. proper, flat, finite presentation, algebraic space, fibrewise at worst odp singularities) which is smooth over $U := S \setminus Z$. Let J be the jacobian of C_U/U. Then U has a Néron model over S.*

There is actually a weaker condition which suffices for (is even equivalent to) the existence of Néron model, but it takes longer to state.

Actually, Giulio tells us we can do better: we don't need a family of jacobians, but rather just a family of abelian varieties with semistable reduction (at least in characteristic zero)! He also has a more fancy criterion, such that the 'regular boundary' case is a special case.

Theorem 3.2. *Let S be a regular scheme and C/S a nodal curve. Let U be the largest open over which C is smooth, and give S the divisorial log structure coming from the divisor $S \setminus U$, ,which we assume NCD. Let J be the jacobian of C_U/U. Then J has a Néron model over $S^{val-sch}$.*

Proof. By corollary 3.1, it is enough to check that the locus in $S^{val-lfp}$ over which the pullback of C is not smooth is regular (when equipped with reduced scheme structure). Smoothness is stable under base-change, so it is equivalent to check that the pullback of $Z := \setminus U$ to $S^{val-sch}$ is regular (with reduced induced scheme structure as usual).

By the cofinality of saturated blowups, we can reduce to the saturated case. Moreover, every saturated(=toric) blowup is dominated by a regular one, so we can reduce to the case of blowups that are regular. Then it should hold that, for an ideal I, the log structure on the blowup S_I is exactly the divisorial log structure coming from the pullback of Z, which is NCD (think toricly to see why).

Then suppose the boundary is not regular. By our NCD assumption, we will have two branches going through some point p, and the stalk at that point will be $\mathbb{N}^2$, which is not valuative. In other word, the divisorial log structure from a NCD is valuative if and only if that divisor is regular, if and only if every connected component is irreducible. □

4 Realising $X^{val-lfp}$ as a colimit

Let X be a log scheme coming from an NCD, and I a saturated log ideal. In the blowup X_I, write $X_{I,val}$ for the open subscheme of points at whose stalks the monoid is valuative. Since $X_{I,val}$ is valuative, it comes with a canonical map to X^{val}. Taking the union over all such I we obtain a map

$$f\colon \bigcup_{I \in \phi_{sat}} X_{I,val} \to X^{val}.$$

where ϕ_{sat} is the subset of ϕ consisting of saturated (i.e. radical) ideals.

Theorem 4.1. *The map f is open, and its image is exactly $X^{val-lfp}$.*

Proof. We immediately reduce to the case where $X = \mathbb{A}^n$ with toric log structure. The saturated log blowups are cofinal in all log blowups.

All log ideals of $X_{I,val}$ are principal, and so all log blowups of $X_{I,val}$ are isomorphisms. Log blowups of X dominating X_I are the same as log blowups of X_I. So we see that, for any $J \geq I$, the map $X_J \to X_I$ is an isomorphism over $X_{I,val}$. Hence there is a unique X-morphism $X_{I,val} \to X_J$ for every $J \geq I$. By the universal property of the limit (and the cofinality of $\{J : J \geq I\}$) this gives us a map $X_{I,val}$ (the same as the map one obtains from the universal property of X^{val} as the valuativisation, but by constructing the map in this way we see that it is open (since $X_{I,val} \to X_J$ is open for every $J \geq I$, and the image of $X_{I,val}$ in X^{val} maps isomorphically to its image in any of these X_J).

Clearly each $X_{I,val}$ is locally of finite presentation over X, so the open map f factors via the inclusion of $X^{val-lfp}$. It remains to verify that the induced map to $X^{val-lfp}$ is surjective. But the above discussion, it is enough to prove that, for any point $p \in X^{val-lfp}$, there exist a saturated log ideal I such that the image of p in X_I lands in $X_{I,val}$.

Now by the structure of the limit topology, there exist an open neighbourhood $p \in U \subseteq X^{val-lfp}$, a saturated ideal I, and an open subset $V \subseteq X_I$, such that V pulls back to U. Now $X_I^{val} = X_{val}$, and forming the valuativisation commutes with strict open immersions, so we see that $U = V^{val}$. But U is locally of finite presentation over V, so by the next lemma we see that $U = V$ and hence that V is valuative, so $V \subseteq X_{I,val}$ and we are done. □

Lemma 4.2. *Let X be a normal fs log scheme with valuativisation X^{val}, and such that the log structure is trivial on some dense open. Assume that X^{val} is a scheme and $X^{val} \to X$ is locally of finite presentation. Then there exists a log blowup X_I of X such that $X^{val} = X_I$.*

In particular, if X has log structure coming from an NCD then the following are equivalent:

1. *X^{val} is an lfp scheme;*

2. *$X^{val} = X$;*

3. *X^{val} is a blowup of X ;*

4. *the boundary divisor in X is regular.*

Proof. First, note that $X^{val} \to X$ satisfies the valuative criterion for properness, since all the $X_I \to X$ do and by the universal property of the limit. We will show that there is some saturated ideal I such that $X^{val} \to X_I$ is affine, so since it is birational it is an isomorphism.

We reduce immediately to the case where X is affine. If $U \subseteq X^{val}$ is affine open then there exists X_I and an open subset $V \subseteq X_I$ such that V pulls back to U. Since X^{val} is quasi-compact it can be covered by finitely many affine opens, hence there exists some I such that $X^{val} \to X_I$ is affine. We may take I to be saturated (by cofinality), so this X_I is normal [?].

Now $X^{val} \to X_I$ is still proper (for example, repeat the above argument about valuation rings, and use that it is quasi-compact [Kato] and lfp [by assumption]). Since X_I is normal, the map $X^{val} \to X_I$ is an isomorphism.

For the second part, note that an NCD log structure is valuative iff the boundary is regular, and this is stable under log blowups, so X^I is valuative if and only if X itself is. □

Corollary 4.3. *If X is NCD, then $X^{val-lfp}$ can be realised as the colimit of the 'flipped system' of $X_{I,val}$ as I runs over ϕ or ϕ_{sat}.*

5 Root stacks and the derived equivalence

Maybe this is redundant - it is done nicely in paper of Sibilla et al. Maybe just discuss, don't duplicate their stuff.

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