Analytic continuation and functional equations

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References

Some notes I found useful were:
Hindry, 'Introduction to zeta and L functions from arithmetic geometry and some applications' (overview, some details)
Cogdell, 'Converse theorems, functoriality and applications' (overview, few details)
Rösch, 'Hecke's converse theorem' (short and simple, gives a proof)
Math overflow, various questions and answers useful for getting a picture of what might be true.
Corvallis proceedings, classic reference but seem very hard to read at this point.
Google to find them, or links may be on the course website soon...?

Introduction

Recall the basic idea: take some 'number theoretic object' and extract a sequence (a_n) of numbers from it. Write down a Dirichlet series $L(s) \stackrel{\text{def}}{=} \sum_{n \geq 1} \frac{a_n}{n^s}$. If the a_n are $O(n^\nu)$ for some $\nu > 0$, then $L(s)$ converges absolutely on the right half-plane $re(s) > \nu + 1$.

If you extracted the a_n in a clever way then you may hope to extract number-theoretic information from $L(s)$ — later we will talk about the 'right way'— for example, the Riemann zeta function allows you to prove the prime number theorem, and the L-function of an elliptic curve appears in the BSD conjecture - obtaing the rank from local invariants. In both these cases, you need the value (residue etc) of the L-function outside the region of convergence for the Dirichlet series, and so you need to analytically continue a bit to the left. This is actually easy and elementary for the Riemann zeta function (not for BSD).

It turns out that you can continue the L-function to the whole complex plane, and moreover that there is a line of symmetry. Why does this happen, and why should we care?

Prototypical example

Set

$$\Lambda(s) = \pi^{-s}\Gamma(\frac{s}{2})\zeta(s) \tag{1}$$

where ζ denotes the Riemann zeta function, and

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}. \tag{2}$$

Then have meromorphic continuation and FE:

$$\Lambda(s) = \Lambda(1-s). \tag{3}$$

This generalises vastly, as we will see.

Remark

The 'Γ-factor' is NOT a 'fudge factor' — it is a sign that we defined ζ incorrectly, see later...

What are AC+FE good for?

NOT proving the prime number theorem — while this does need L-functions to be defined outside the region of convergence for the Dirichlet series, it is in fact elementary and easy to extend the L-series ‘a bit to the left’ and prove PNT.

I think the point is:

Theorem (Converse theorems)

“AC+FE(+boundedness) for an L-series (and possibly twists) $\implies$ the L-series is MODULAR”

Modularity is a big deal — see Fermat etc.
The other obvious question is: why do functional equations hold? This seems very mysterious, and will be mumbled about at the end of the talk...

Mellin transform:
The Mellin transform of a function f is

$$\left\{\mathcal{M}f\right\}(s) = \varphi(s) = \int_0^\infty x^{s-1} f(x) dx. \tag{4}$$

The inverse transform is

$$\left\{\mathcal{M}^{-1}\varphi\right\}(x) = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\varphi(s)\, ds. \tag{5}$$

Conditions under which this inversion is valid are given in the Mellin inversion theorem.

'Proof' of FE+AC for Riemann zeta

See [Hindry, p4] for details.
We begin the proof with theta functions. We will end there too...
Definition:

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{-\pi z n^2} \tag{6}$$

Notes:
— very useful for constructing projective embeddings of Abelian varieties.
— fundamental solutions to the heat equation.
— FE: $\theta(1/z) = \sqrt{z}\theta(z)$ (prove using Poisson summation).
— 1/2-integral weight modular forms (see next week?)

Start by taking a 'sort of Mellin transform'(see Wikipedia for def'n):

$$\Lambda(s) = \int_0^\infty \left(\frac{\theta(z) - 1}{2}\right) z^{s/2} \frac{dz}{d}. \tag{7}$$

Using FE of θ, we obtain:

$$\Lambda(s) = \int_1^\infty \left(\frac{\theta(z) - 1}{2}\right) \left(z^{s/2} + z^{\frac{1-s}{2}}\right) \frac{dz}{z} + \frac{1}{s-1} - \frac{1}{s}. \tag{8}$$

Now $(\theta(z) - 1)/2 = O(e^{-\pi z})$, so this gives an entire function. It is clearly symmetric wrt $s \leftrightarrow 1 - s$. Hence we have AC+FE for $\Lambda(s)$ as desired.

We can do something similar for $\zeta_k(s)$ where k is a number field, but this gets pretty messy.

Converse theorems

I think these are why we care about AC+FE for our L-series:

Theorem (Hecke, 1936, simplified)

Let $k \geq 2$ be an even integer, and let $f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$ be a holomorphic function on $\mathcal{H}$ such that there exists $\nu > 0$ with $a_n = O(n^\nu)$. TFAE:
1) $f(z)$ is a modular form of weight k (for $SL_2(\mathbb{Z})$).
2) the function $\Lambda(s, f) \stackrel{def}{=} (2\pi)^{-s}\Gamma(s)\sum_{n\geq 1} \frac{a_n}{n^s}$ has meromorphic continuation + FE:

$$\Lambda(s, f) = (-1)^{k/2}\Lambda(k - s, f) \tag{9}$$

and moreover the function

$$\Lambda(s, f) + \frac{a_0}{s} + \frac{(-1)^{k/2}a_0}{k - s} \tag{10}$$

is holomorphic on $\mathbb{C}$ and bounded on any vertical strip.

What if we want $f(z)$ to have higher level, for example we want it to be a cusp form for $\Gamma_0(N)$? Then we want a stronger theorem, eg Weil's converse theorem. Idea is that we should insist that the conditions (2) above hold for all *twists* of L-functions by Dirichlet characters:

$$\Lambda(f\chi, s) \stackrel{\text{def}}{=} \sum_{n\geq 0} \chi(n)\frac{a_n}{n^s} \tag{11}$$

as χ runs over 'all' primitive Dirichlet characters for 'all' conductors coprime to N.

What motivated Weil to prove this converse theorem?

Let E be an elliptic curve, and set $a_p = p - \#E(\mathbb{F}_p) + 1$. Let

$$L(E,s) \approx \prod_{p \nmid 2\Delta} (1 - a_p p^{-s} + p^{1-2s})^{-1} \tag{12}$$

which converges for $re(s) > 3/2$.

Conjecture (Taniyama-Shimura)

$L(E,s)$ transforms to a cusp form of weight 2.

Weil's converse theorem $\implies$ it suffices to prove AC+FE + boundedness for $L(E,\chi,s)$ for 'all' primitive χ. Did Wiles do that? No idea!

If we apply Hecke's converse theorem to the Riemann zeta function, we get back θ as the modular form (the version of Hecke I gave is not quite enough).

The local case: Weil conjectures

For details, see [Hindry, p17] Let X be a variety over $\mathbb{F}_p$, set

$$Z(X,T) \stackrel{\text{def}}{=} \exp\left(\sum_{M\geq 1} \frac{\#X(\mathbb{F}_{p^m}}{m} T^m\right), \tag{13}$$

$$\zeta(X,s) \stackrel{\text{def}}{=} Z(X,p^{-s}). \tag{14}$$

<u>Theorem</u> [Dwork, <u>Grothendieck</u>, Deligne, 'Weil conjectures']:
Let X be a smooth projective variety over $\mathbb{F}_p$ of dimension n.

1) (Rationality) There are polynomials $p_j(X,T) = \prod_{i=1}^{b_j}(1-\alpha_{j,i}T) \in \mathbb{Z}[T]$ for $j = 0, \ldots 2n$ such that

$$Z(X,T) = \frac{P_1(X,T)\ldots P_{2n-1}(X,T)}{P_0(X,T)\ldots P_{2n}(X,T)}. \tag{15}$$

Further, $b_0 = b_{2n} = 1$ and $P_0(X,T) = 1 - T$ and $P_{2n}(X,T) = 1 - p^n T$.
2) (FE) Let $\chi(X) = \sum_{j=0}^{2n}(-1)^j b_j$ be the Euler-Poincare characteristic, then

$$Z(X,T) = \pm p^{n\chi(X)/2} T^{\chi(X)} Z\left(X, \frac{1}{p^n T}\right). \tag{16}$$

3) (Riemann hypothesis) The algebraic integers $\alpha_{i,j}$ satisfy $|\alpha_{i,j}| = p^{j/2}$.
4) The b_j are the complex Betti numbers (under some assumptions).

The functional equation here is a fairly formal consequence of Poincare duality.
eg:
$X = \mathbb{P}^n$, $Z(X,T) = \frac{1}{(1-T)(1-pT)\cdots(1-p^nT)}$.
$X = \mathbb{A}^n$, $Z(X,T) = \frac{1}{1-p^nT}$ - no FE.

Local to global

Lefschetz trace formula:

$$\#X(\mathbb{F}_{p^m}) = \sum_{j=0}^{2n} (-1)^j \mathit{Trace}\left(\mathit{Frob}_p^m | H^j_{et}(X \times_{\mathbb{F}_p} \mathbb{F}_p{}^{\mathrm{sep}}, \mathbb{Q}_l)\right). \tag{17}$$

Set

$$L_p(H^j(X), s) \stackrel{\text{def}}{=} \exp\left(\sum_{m \geq 1} \mathit{Trace}(\mathit{Frob}_p^m | H^j_{et}(X \times_{\mathbb{F}_p} \mathbb{F}_p{}^{\mathrm{sep}}, \mathbb{Q}_p) p^{-ms}/m \right). \tag{18}$$

Do the same for $p = \infty$: let X now be a smooth projective variety over $\mathbb{Q}$, Hodge decomposition

$$\mathrm{H}^i\left(X(\mathbb{C}), \mathbb{C}\right) = \oplus_{p+q=i} H^{p,q} \tag{19}$$

with 'Frobenius at infinity' $F_\infty =$ complex conjugation, and define $H^{p,\pm} \stackrel{\text{def}}{=} \{c \in H^{p,p} | F_\infty(c) = \pm c\}$. Define the Gamma factors as:

$$L_\infty(H^i(X), s) = \begin{cases} \prod_{p<q, p+q=i}((2\pi)^{p-s}\Gamma(s-p))^{\dim(H^{p,q})} & i \text{ odd} \\ \text{similar, messier with } \dim(H^{p,\pm}) & i \text{ even} \end{cases} \tag{20}$$

We should also give similar factors at places of bad reduction for a variety over $\mathbb{Z}$, but you can look this up if you need details.
Now we can state a big conjecture:

Conjecture (due to ???)

Let X be a smooth projective variety over $\mathbb{Q}$. The function $L(H^j(X), s) \stackrel{def}{=} \prod_{p<\infty} L_p(H^j(X), s)$ extends analytically to the whole complex plane except, when j is even, for a pole at $s = 1 + j/2$. Further, there exists an integer N (the conductor of the Galois rep on $H^j(X)$) such that defining

$$\Lambda(H^j(X), s) \stackrel{def}{=} N^{s/2} L_\infty(H^j(X), s) L(H^j(X), s), \tag{21}$$

we have the functional equation

$$\Lambda(H^j(X), s) = \pm\Lambda(H^j(X), j + 1 - s). \tag{22}$$

Eg. if X is an elliptic curve defined over $\mathbb{Q}$, we write $L(X, s)$ instead of $L(H^1(X), s)$ and then have

$$\Lambda(X, s) \stackrel{\text{def}}{=} N_X^{s/2}(2\pi)^s \Gamma(s) L(X, s) \text{ and } \Lambda(X, s) = \pm\Lambda(X, 2 - s). \tag{23}$$

This is now a theorem due to Wiles et al.

Why does AC+FE hold?

1) The Dirichlet series we consider come from ‘geometric sources’($\mathcal{O}_k$ for k a number field, an elliptic curve, ...), and so there should be an underlying automorphic representation. This satisfies periodicity conditions, which are ‘sent to’ the FE under the Mellin transform.

2) In the local case (Weil conjectures), recall that FE was an easy consequence of Poincare duality. However, it is not easy to ‘glue’ these to get a global FE (don’t know why).

3)In order to have FE, our geometric object must be ‘compact’. In the local case, this appeared as requiring our variety over $\mathbb{F}_p$ to be projective (no idea if proper will suffice). In the global case, we interpreted the Gamma factors in terms of the trace of Frobenius on the cohomology of the complex manifold corresponding to infinite places of k. cf ideas about Spec($\mathcal{O}_k$) being a non-compact curve, and the places at infinity serving to compactify it; Arakelov theory.