Compactifications of Siegel Modular Varieties

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We will restrict to the case of principal polarisations and level 1, because it will ease notation and the ideas are generally all there.

1 What and why?

Recall that last time Diane defined

$$\mathbb{H}_g = \{\tau \in M_{g\times g}(\mathbb{C}) | \tau = \tau^T \text{ and } \Im(\tau) > 0\}, \tag{1}$$

and gave an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ by

$$\begin{gathered} \mathrm{Sp}_{2g}(\mathbb{Z}) \curvearrowright \mathbb{H}_g \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau \mapsto (A\tau + B)(C\tau + D)^{-1} \end{gathered} \tag{2}$$

Then define

$$A_g = \mathrm{Sp}_{2g}(\mathbb{Z}) \diagdown \mathbb{H}_g. \tag{3}$$

Note:
- this is not compact as a topological space
- it is in fact an algebraic variety, but this is a subtle fact. Probably the easiest proof in this context is to use modular forms, which I will define later. However, for now let us take it for granted. Then:

Definition 1. *A compactification of A_g is a complex projective variety with a dense open subset isomorphic as algebraic varieties to A_g.*

Why?
- if you want to talk about the fundamental group, canonical class etc it will help.
- compactifications and their moduli interpretations are (essentially the same as) the study of dengenerations of abelian varieties in families, which has loads of applications. See later for more discussion of this.

2 The dimension 1 case

Recall from Diane's talk that we have the j function, an isomorphism

$$j : A_1 \to \mathbb{C}. \tag{4}$$

In the notes it is referred to as an iso of algebraic varieties, but in an attempt to be self contained (i.e. avoid the language of representable functors) I will use this as the definition of the algebraic structure on A_1.

Re-writing as $j : A_1 \to \mathbb{A}^1_{\mathbb{C}}$, we see an obvious compactification as $\mathbb{P}^1_{\mathbb{C}}$, and indeed this is the unique smooth compactification (in fact, it is the Satake compactification).

2.1 A topological viewpoint on the Satake compactification

To obtain the Satake compactification, we just added a single point (outside level 1, we would add a finite collection of points). From a topological point of view this seems pretty simple. Can we construct the compact topological space directly? More precisely:

Definition 2. *We define a partial compactification of $\mathbb{H}_g$ to be a topological space $\overline{\mathbb{H}_g}$ which contains $\mathbb{H}_g$ as a dense open subset, and to which the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ extends in such a way that the quotient*

$$\mathrm{Sp}_{2g}(\mathbb{Z})\diagdown\overline{\mathbb{H}}_g \tag{5}$$

is a compact Hausdorff topological space (in which A_g is dense open - is this automatic?)

In dimension 1, a set $\overline{\mathbb{H}_1}$ is easy to construct, but the topology is a bit more subtle. In general g, both become harder.

2.1.1 The set $\overline{\mathbb{H}}_1$

Define the Cayley Transform

$$\begin{aligned} \phi : \mathbb{H}_g &\to \mathbb{C} \\ \tau &\mapsto \frac{\tau - i}{\tau + i} \end{aligned} \tag{6}$$

which sends the upper half space to the disk. Then as a set, $\overline{\mathbb{H}}_1$ is $\phi(\mathbb{H} \cup \mathbb{Q}) \cup \{1\}$.

2.2 The topology on $\overline{\mathbb{H}}_1$

First, note that the wrong thing to do is to give $\overline{\mathbb{H}}_1$ the subspace topology from $\mathbb{C}$; the resulting quotient will not be Hausdorff, since any neighbourhood of any point on the boundary will contain entire translates of the fundamental domain.

Instead, use the Satake/horocycle topology, for which a basis of open neighbourhoods of p on the boundary is $\{D_\epsilon \cup p : \epsilon > 0\}$, where D_ϵ is an open disk of radius ϵ touching the boundary of $\phi(\mathbb{H}_1)$ at p.

With this topology, the quotient $\mathrm{Sp}_2(\mathbb{Z})\diagdown\overline{\mathbb{H}}_1$ is Hausdorff, and has dense open subset isomorphic to A_1.

Notes:

- from this point of view, it is far from clear what algebraic structure to give this compactification. In fact, the answer comes from modular forms.
- this compactification is in fact the same as that obtained above using modular forms.

3 Modular forms ($g > 1$)

Definition 3. *A modular form of weight k and level 1 is a holomorphic function*

$$F : \mathbb{H}_g \to \mathbb{C} \tag{7}$$

such that for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$ *we have*

$$F(M\tau) = \det(C\tau + D)^k F(\tau). \tag{8}$$

(if $g = 1$, we would also need growth condition at cusps.)

Fact:

- elements of $\mathrm{Sp}_{2g}(\mathbb{Z})$ which have fixed points are torsion.
- the order of all torsion elements in $\mathrm{Sp}_{2g}(\mathbb{Z})$ is bounded.

Thus:

Fix $k \geq 1$. Then for some $n > 0$, the space $M(nk)$ of modular forms of weight nk is equal to the set of global sections of a very-ample line bundle, so we obtain an immersion

$$A_g \to \mathbb{P}(M(nk))^v. \tag{9}$$

(This is a little unsatisfactory, since we would like to use the modular forms to prove that A_g has an algebraic structure. However, this is easily amended).

Definition 4. *The Satake compactification $\overline{A_g}$ of A_g is the closure of A_g in $\mathbb{P}(M(nk))^v$ for any nk as above. It is independent of n and k; this is not so hard to see, since any two pairs n_1k_1 and n_2k_2 yield the same immersions after composition with appropriate Segre embeddings.*

Facts:
- $\overline{A_g}$ is normal, but in general it is (very) singular.
- As sets, $\overline{A_g} = \sqcup_{n \leq g} A_n$.

4 A partial compactification

There is a partial compactification for this general version of the Satake compactification, generalising the dimension 1 version. We will describe the set; the topology would take too long.

Step 1) Define the Cayley Transform

$$\begin{aligned} \phi : \mathbb{H}_g &\to \mathrm{Sym}^g(\mathbb{C}) = \{\tau \in M_{g\times g}(\mathbb{C}) | \tau = \tau^T\} \\ \tau &\mapsto \frac{\tau - i.I_g}{\tau + i.I_g} \end{aligned} \tag{10}$$

Call the image $\mathcal{D}_g$.

Step 2) A real affine hyperplane $H \subset \mathrm{Sym}^g(\mathbb{C})$ is:
- a <u>supporting hyperplane</u> if $\mathcal{D}_g \cap H = \emptyset$ and $\overline{\mathcal{D}_g} \cap H \neq \emptyset$.
- <u>rational</u> if it is spanned by rational vectors.

Let H be a rational supporting hyperplane, and let $\overline{F} = H \cap \overline{\mathcal{D}_g}$. Let L be the smallest affine subspace of $\mathrm{Sym}^g(\mathbb{C}$ containing $\overline{F}$, and let F be the interior if $\overline{F}$ in L. Then F is a rational boundary component of $\mathcal{D}_g$.

There are countably many rational boundary components.

Step 3) Set $\overline{\mathbb{H}}_g$ to equal $\mathbb{H}_g$ union the set of rational boundary components (as a set).

Exercise:
check that the rational boundary components of $\mathcal{D}_1$ are exactly 1 and the image of $\mathbb{Q}$ under the Cayley transform.

Fact:
again, this compactification coincides with the one obtained using modular forms!

5 The toroidal compactification

The toroidal compactification is a resolution of singularities of the Satake compactification. It can be constructed in a manner similar to the 'partial compactification' approach given here. It is not unique, but depends on a choice of combinatorial data.

6 Compactifications and degenerations

At this point I begin to regret that I restricted myself to level 1, since now I cannot assume that I have a fine moduli space. However I shall ignore this; perhaps everything the follows should be interpreted on the level of algebraic stacks.

Universal abelian varieties:
A_g comes (almost) with a universal abelian variety $U_g \to A_g$, universal in the sense that for 'any' family $Y \to X$ of abelian varieties, we obtain a unique cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & U_g \\ \downarrow & & \downarrow \\ X & \longrightarrow & A_g. \end{array} \tag{11}$$

A good compactification $\overline{A}_g$ should come with a proper map $\overline{U}_g \to \overline{A}_g$ such that

$$\overline{U}_g \times_{\overline{A}_g} A_g = U_g. \tag{12}$$

(In particular, this holds for the toroidal compactification, whereupon $\overline{U}_g$ will be a semi-abelian scheme (see Faltings-Chai).)

By the valuative criterion for properness, this determines how to extend any family of abelian varieties over $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ to the whole of $\mathbb{A}^1_{\mathbb{C}}$ (the fibre over 0 won't in general be an abelian variety!) If we use the toroidal compactification, we will obtain an extension to a semi-abelian scheme. (For $\mathbb{A}^1_{\mathbb{C}}$ read a strictly hensellian local ring of residue characteristic zero, and for $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ read its generic point).

6.1 How to interpret singularities on the boundary of our compactification?

6.2 The dimension 1 case

There are no singularities. The only (semistable) thing you can degenerate to is $\mathbb{G}_m$.; this corresponds to a nodal elliptic curve. If your family appeared to degenerate to a cusp, you should
1) make sure your total space is regular, and if that doesn't do it then
2) make a ramified extension.

In level 1 (one marked point), the fibre of the universal abelian variety over the cusp (point you added in) witll just be $\mathbb{G}_m$ with the unit marked.

In higher level, each cusp will have the same scheme over it (again, $\mathbb{G}_m$), but the level structure can degenerate to different things...

If in higher level you took the 'one point compactification' (equivalently you glued all the cusps together), then

- you would get something singular (!)

- this would equate to forgetting the level structure on the degenerations.

Posible moral: singularities of compactifications of moduli spaces result from choosing degenerations which are 'too coase'

6.3 the higher dimension case

Thanks to Damiano Testa for telling me this example (which didn't go in the actual talk, but is here for fun).

Let $C/\mathbb{C}$ be a smooth curve of genus g. If we glue together any two points on C we get a singular curve of arithmetic genus $g + 1$ Smoothing out one of these curves gives us a (generically smooth) family whosh smooth fibres clearly have geometric (and arithmetic) genus $g + 1$. Thus we obtain a family of degenerations of curves, parameterised by pairs of points on C.

Take the Jacobian of this picture away from the degenerating locus - this gives an abelian variety of relative dimension $g + 1$. The 'correct' thing to do over the whole locus is to take the generalised Jacobian, a semi-abelian variety. The 'silly' thing to do is to take the fibre over a closed point to be the Jacobian of the normalisation of the fibre over that point. This yields an ableian variety of dimension 1 less over the degenerate locus, and this corresponds to the Satake compactification.