Néron models of jacobians over base schemes of dimension > 1.

Thanks to organisers: Merci aux organisateurs pour l'invitation.

~~[illegible]~~

Basic Situation: · S regular, integral, separated scheme;
· $U \subseteq S$ dense open;
· C/S semistable curve, smooth over U.
  flat, proper, geom. fibres connected & with only ODP sings. eg. α, not $\measuredangle$.
· $J := \mathrm{Jac}(C_U/U)$, an abelian scheme over U

Basic question: how to make a 'good' model of J over S?

What do we mean by 'good model'? A candidate:

Def: let S reg. scheme, $U \subseteq S$ dense open, A/U an abelian scheme.
A Néron model for A/S is a smooth, separated (group) algebraic space N/S, together with an isomorphism $A \xrightarrow{\sim} N_{\times_S U}$, with the following universal property:

Given a smooth morphism $\phi: T \to S$ of algebraic spaces, & a U-morphism $f: T_U \to A$, there exists a unique S-morphism $F: T \to N$ s.t. $F|_{T_U} = f$.

Rks: · In particular, any section in $A(U)$ extends;
· If A/S is abelian then A is the NM of $A|_U$;
· Thm [Néron, 65]: If $\dim S = 1$, then a NM always exists (even as a scheme).

eg: $S = \mathrm{Spec}\, \mathbb{C}[[t]]$, (dimension 1), $U = S \setminus \{t=0\}$.

$E : y^2 = x(x-t)(x-1)$ elliptic curve $\subset \mathbb{P}^2_S$
- semistable, smooth $/U$.

~~Set $E^{sm} = E \setminus (x=y=t=0)$.~~

$J := \mathrm{Jac}(E_{U/U}) \xrightarrow{\cong} E_U$. section $\Rightarrow$ canonical map.

Néron's thm $\Rightarrow$ J has a Néron Model. What is it?

Set $E^{sm} = E \setminus \{x=y=t=0\}$. Then $\exists$ canonical gp law on E^{sm}, & it is the NM. In particular, every section in $E(U)$ extends across $E^{sm}(S)$.

cf. Jilong's talk.

Basic questions: Does J always admit a NM?
- If not, what about after blowingup/altering

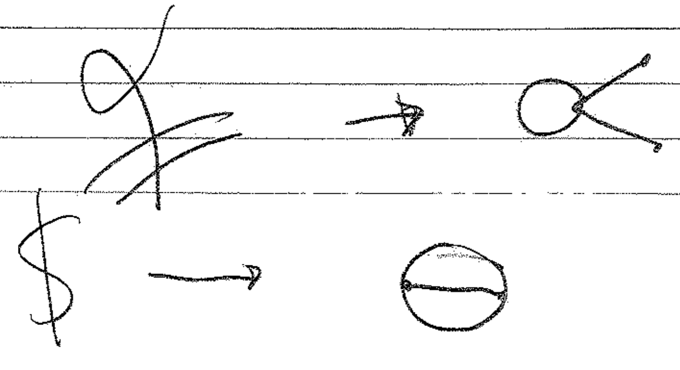
To answer, we need more definitions.

Def: let C/k a s-stable curve over a sep. cl field. The dual graph Γ of C has:

- 1 vertex for each irred comp of C;
- 1 edge for each sing. pt of C;

An edge e goes between the (one or 2) irreducible components containing it.

eg.



Next, we want to put labels on the edges of the dual graph Γ. We need:

Lemma: ~~S a loc. Noeth scheme, C → S semistable curve~~ Usual setup. Let $\bar{c} \in C$ a non-smooth geometric pt, lying over geom. pt. $\bar{s} \in S$. Then $\exists\ \alpha \in \mathfrak{m}_{\mathcal{O}_{S,\bar{s}}}^{et}$ and an isomorphism of complete local rings

$$\frac{\widehat{\mathcal{O}}^{et}_{S,\bar{s}}[[u,v]]}{(uv-\alpha)} \xrightarrow{\sim} \widehat{\mathcal{O}}^{et}_{C,\bar{c}}.$$

Moreover, this α is unique up to unit in $\mathcal{O}^{et}_{S,\bar{s}}$; we call $\alpha \cdot \mathcal{O}^{et}_{S,\bar{s}}$ the *singular ideal* of c.

Def: ~~S loc Noeth, C/S semistable,~~ Usual setup. $\bar{s} \in S$ geom pt. The *labelled reduction graph* $\Gamma_{\bar{s}}$ is the dual graph of $C_{\bar{s}}$, with each edge e labelled by the singular ideal of the corresp. non-smooth pt in C lying over $\bar{s}$.

⇝ edges labelled by principal ideals of $\mathcal{O}^{et}_{S,\bar{s}}$

Def: ~~Usual setup. We say C/S is aligned at a geom. pt $\bar{s} \in S$ if for every 2-vertex-connected subgraph $H \subseteq \Gamma$, $\exists\ \alpha_H \in \mathcal{O}^{et}_{S,\bar{s}}$ such that for all edges $e \in H$ $\exists\ n_e \in \mathbb{Z}_{>0}$ s.t.~~

$$\text{label}(e) = (\alpha_H)^{n_e}$$

We say C/S *aligned* if aligned at $\bar{s}$ $\forall$ geom pts of S.

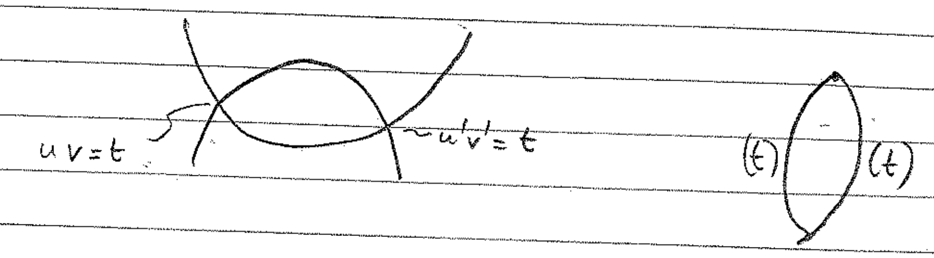
i.e. 'every 2-v-c has all its labels gen. by one elt'

Usual setup. We say C/S is *aligned* at a geom. pt $\bar{s} \in S$ if for every circuit (non-self-intersecting loop) $\gamma \in \Gamma$, & for every two edges e, e' on γ, $\exists$ integers $n, n' > 0$ s.t.

$$(\text{label}(e))^{n} = (\text{label}(e'))^{n'}.$$

$S = \mathrm{Spec}\, \mathbb{C}[[s,t]]$, $s =$ cl. pt.

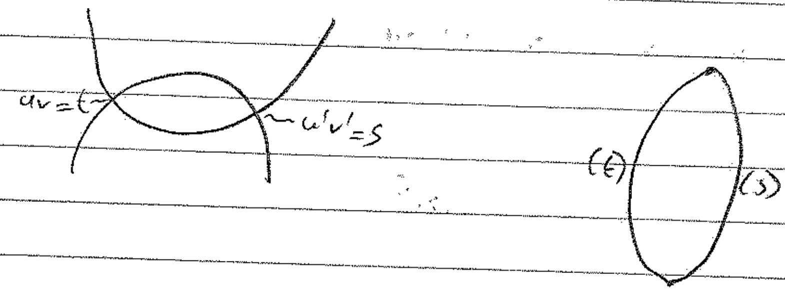
eg: $y^2 = (x+1)(x+1+t)(x-1)(x-1-t) \subseteq \mathbb{P}^2_S(1,1,2)$.



aligned.

eg: S, s as before,

$$y^2 = (x+1)\underline{(x+1+s)}(x-1)(x-1-t) \quad \subseteq \mathbb{P}^2_S(1,1,2)$$



NOT aligned

eg: compact type $\Rightarrow$ aligned.

eg $\dim S = 1 \Rightarrow$ aligned.

Now we can answer questions:

Thm [H]: Usual setup.

1) If J has a NM over S, then C/S is aligned.

2) If C/S is aligned & C is regular, then J has a NM over

- Resolving sings is not too hard if eg ~~C/S~~ U is complement of
- Can give a more precise cond. that doesn't need C reg, if U comp. of NCD

Idea of pf:

- We actually show that being aligned & C reg is $\equiv$ to $\bar{E}$ in $\mathrm{Pic}_{C/S}$ being flat over S. If it is, then the quotient $\mathrm{Pic}^0_{C/S}/\bar{E}$ is the Néron model. Conversely, if NM exists then can show $\bar{E}$ flat.

Thm [H]: ~~Let~~ Usual setup, S' regular, $S' \twoheadrightarrow S$ proper surjective (eg. ~~blowing up at smooth centre~~ alteration).

Then $C/_S$ aligned $\Leftrightarrow$ $C\times_S S'/_S$ aligned

Cor: If ~~C reg & C/S~~ $C/_S$ not aligned, then J does not admit a NM, even after blowup / alteration / proper surj cover of S.

Cor: ~~Usual setup~~ S normal, loc. Noeth, sep, $U \subseteq S$ d. open, (reg. in codim 1) $C/_S$ semistable, smooth$/_U$. $J = \mathrm{Jac}(C_U/_U)$.

Then $\exists U \subseteq V \subseteq S$ open s.t. $\mathrm{codim}_S(S \setminus V) \geq 2$ & s.t.

J has a NM over V.

- applications to 'algebraic Lear extensions', ht. jumping in Hodge theory, (cases of) Hain's conjectures.

PTO for universal case

What about the universal case?

- Let $g, n \in \mathbb{Z}_{\geq 0}$ s.t. $2g-2+n > 0$.
- Let $M_{g,n}$ = DM stack of smooth proper curves of genus g w. n distinct ordered marked pts,
- $\overline{M}_{g,n}$ = DMK cpct.
- $C := \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ universal curve, stable, smooth over $M_{g,n}$
- Set $J_{g,n} = \mathrm{Jac}\left(C_{M_{g,n}}/M_{g,n}\right)$, ab. sch over $M_{g,n}$.

In general, $J_{g,n}$ has no ~~NM~~ NM over $\overline{M}_{g,n}$.

<u>Def</u>: Let T reg. alg. stack, $f: T \to \overline{M}_{g,n}$ morphism s.t. $f^{-1} M_{g,n}$ dense in T. We say f is 'NM admitting' if $f^* J_{g,n}$ has a NM over T.

<u>Thm [H]</u>: The 2-cat of NM admitting morphisms to $\overline{M}_{g,n}$ has a terminal object, denoted

$$\widetilde{M}_{g,n} \xrightarrow{\beta} \overline{M}_{g,n}.$$

~~β is l.f. pres, in general not q-cpct in general not q-cpct.~~

<u>IOW</u>: $\exists$ ~~β~~ $\beta: \widetilde{M}_{g,n} \to \overline{M}_{g,n}$ & a NM of $J_{g,n}$ over $\widetilde{M}_{g,n}$, & moreover for any $f: T \to \overline{M}_{g,n}$ NM admitting morphism, we have that f factors uniquely (up to ! isom) via β.

Silverman – Tate – Green theorem [Jt w. R de Jong]

Using these techniques we can also give a new, more 'effective' proof of

Thm: K global field, $S/_K$ rel-curve (not necess proper),

$A/_S$ ab-scheme, $\sigma \in A(S)$ section of ∞ order.

~~Then $\forall d \in \mathbb{Z}_{\geq 1}$,~~

$\{ s \in S(K) \mid \sigma_s \text{ is torsion} \}$ is finite (also version over fin-ext of bdd degree).

Pf Idea: • Reduce to case of Jacobians of S-S curves.

• Relate local ht pairings to electrical resistance on 'circuits' coming from labelled graphs.