Stability of Traveling Waves for Reaction-Diffusion Equations with Multiplicative Noise

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Abstract

We consider reaction-diffusion equations that are stochastically forced by a small multiplicative noise term. We show that spectrally stable traveling wave solutions to the deterministic system retain their orbital stability if the amplitude of the noise is sufficiently small.

By applying a stochastic phase-shift together with a time-transform, we obtain a semilinear SPDE that describes the fluctuations from the primary wave. We subsequently develop a semigroup approach to handle the nonlinear stability question in a fashion that is closely related to modern deterministic methods.

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1 Introduction

In this paper we consider stochastically perturbed versions of a class of reaction-diffusion equations that includes the bistable Nagumo equation

$$u_t = u_{xx} + f_{\mathrm{cub}}(u) \tag{1.1}$$

and the Fitzhugh-Nagumo equation

$$\begin{aligned} u_t &= u_{xx} + f_{\mathrm{cub}}(u) - v \\ v_t &= v_{xx} + \varrho[u - \gamma v] \end{aligned} \tag{1.2}$$

Here we take $\varrho > 0$, $\gamma > 0$ and consider the standard bistable nonlinearity

$$f_{\mathrm{cub}}(u) = u(1-u)(u-a). \tag{1.3}$$

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It is well-known [15, 46] that (1.1) admits spectrally stable traveling front solutions

$$u(x,t) = \frac{1}{2}\big[1 + \tanh(\frac{1}{2}\sqrt{2}(x - ct))\big] \tag{1.4}$$

that travel with speed

$$c = \sqrt{2}(a - \frac{1}{2}). \tag{1.5}$$

In addition, the existence of traveling pulse solutions to (1.2) with $0 < \varrho \ll 1$ was established recently [8] using variational methods. Using the Maslov index, a proof for the spectral stability of these waves has recently been obtained in [10, 11].

Our main results show that these spectrally stable wave solutions survive in a suitable sense upon adding a small pointwise multiplicative noise term to the underlying PDE. This noise term is assumed to be globally Lipschitz and must vanish for the asymptotic values of the waves. For example, our results cover the scalar Stochastic Partial Differential Equation (SPDE)

$$dU = \big[U_{xx} + f_{\mathrm{cub}}(U)\big]dt + \sigma\chi(U)U(1-U)d\beta_t \tag{1.6}$$

together with the two-component SPDE

$$\begin{aligned} dU &= \big[U_{xx} + f_{\mathrm{cub}}(U) - V\big]dt + \sigma\chi(U)U(1-U)d\beta_t, \\ dV &= \big[V_{xx} + \varrho\big(U - \gamma V\big)\big]dt + \sigma(U - \gamma V)d\beta_t, \end{aligned} \tag{1.7}$$

both for small $|\sigma|$, in which (β_t) is a Brownian motion and $\chi(U)$ is a cut-off function with $\chi(U) = 1$ for $|U| \leq 2$. The presence of this cut-off is required to enforce the global Lipschitz-smoothness of the noise term. In this regime, one can think of (1.6) and (1.7) as versions of the PDEs (1.1)-(1.2) where the parameters a and ϱ are replaced by $a + \sigma\dot{\beta}_t$ respectively $\varrho + \sigma\dot{\beta}_t$.

Many additional multi-component reaction-diffusion PDEs such as the Gray-Scott [35], Rinzel-Keller [45], Tonnelier-Gerstner [51] and Lotka-Volterra systems [26] are also known to admit spectrally stable traveling waves in the equal-diffusion setting [18, 21, 57]. This allows our results to be applied to these waves after appropriately truncating the deterministic nonlinearities (in regimes that are far away from the interesting dynamics).

Such cut-offs are not necessary when considering equal-diffusion three-component FitzHugh-Nagumo-type systems such as those studied in [40, 52]. Such equations were first used by Purwins to study the formation of patterns during gas discharges [47]. However, in the equal-diffusion setting there is at present only numerical evidence to suggest that spectrally stable waves exist for the underlying deterministic equation. Analytical approaches to prove such facts typically use methods from singular perturbation theory, but these often require the diffusive length scales to be strictly separated.

Noisy patterns Stochastic forcing of PDEs has become an important tool for modellers in a large number of fields, ranging from medical applications such as neuroscience [6, 7] and cardiology [58] to finance [13] and meteorology [16]. While a rather general existence theory for solutions to SPDEs has been developed over the past decades [9, 19, 41, 42], the study of patterns such as stripes, spots and waves in such systems is less well-developed.

Preliminary results for specific equations such as Ginzburg-Landau [5, 17] and Swift-Hohenberg [32] are available. Kühn and Gowda [20] analyzed both these equations in the linear regime before the onset of the Turing bifurcation. They obtained scaling laws for the natural covariance operators that can be used as early-warnings signs to predict the appearance of patterns.

In addition, several numerical studies have been initiated to study the impact of noise on patterns, see e.g. [37, 48, 54]. The results in [37] relating to (1.6) are particularly interesting from our perspective. Indeed, they clearly show that traveling wave solutions persist under the stochastic forcing, but the speed decreases linearly in σ^2 and the wave becomes steeper.

Rigorous results concerning the impact of stochastic forcing on deterministic waves are still relatively scarce. However, some important contributions have already been made, focusing on two important issues that need to be addressed. The first of these is that one needs to identify appropriate mechanisms to identify the phase, speed and shape of a stochastic wave. The second issue is that one needs to control the influence of the nonlinear terms by using the decay properties of the linear terms.

Phase tracking An appealing intuitive idea is to define the phase $\vartheta(u)$ of a solution profile u relative to the deterministic traveling wave Φ by writing

$$\vartheta(u) = \mathrm{argmin}_{\vartheta \in \mathbb{R}} \left\| u - \Phi(\cdot + \vartheta) \right\|_{L^2}, \tag{1.8}$$

which picks the closest translate of Φ. Inspired by this idea, Stannat [49, 50] obtained orbital stability results for a class of systems including (1.6) by appending an ODE to track the position of the wave. This is done via a gradient-descent technique, whereby the phase is updated continuously in the direction that lowers the norm in (1.8). A slight drawback of this method is that the phase is always lagging in a certain sense. In particular, it is not immediately clear how to define a stochastic speed and relate it with its deterministic counterpart.

This gradient-descent approach has been extended to neural field equations with additive noise [33, 34]. In order to clarify the dynamic effects caused by the noise, the authors employed a perturbative approach and expanded the phase of the wave and the shape of the perturbations in powers of the noise strength σ. By taking the infinite update-speed limit, the authors were able to eliminate the phase lag mentioned above. At lowest order they roughly recovered the diffusive wandering of the phase that was predicted by Bressloff and Webber [7]. This perturbative expansion can be maintained on finite time intervals, which increase in length to infinity as the noise size σ is decreased. However, one needs separate control on the deviations of the phase and the shape from the deterministic wave, which are both required to stay small.

Inglis and MacLaurin take a directer approach in [27] by using a stochastic differential equation for the phase that forces (1.8) to hold. For equations with additive noise, they obtain results that allow waves to be tracked over finite time intervals. As above, this tracking time increases to infinity as $\sigma \downarrow 0$. The main issue here is that global minima do not necessarily behave in a continuous fashion. This means that (1.8) can become multi-valued at times, leading to sudden jumps of the phase. However, under a (restrictive) technical condition the extension of the tracking time can be performed uniformly in σ.

Nonlinear effects In order to control the nonlinear terms over long time intervals one needs the linear flow to admit suitable decay properties. We write $S(t)$ for the semigroup generated by the linear operator $\mathcal{L}_{\mathrm{tw}}$ associated to the linearization of the PDEs above around their traveling wave Φ. A direct consequence of the translational invariance is that $\mathcal{L}_{\mathrm{tw}}\Phi' = 0$ and hence $S(t)\Phi' = \Phi'$ for all $t > 0$. In order to isolate this neutral mode, we write P for the spectral projection onto Φ', together with its complement $Q = I - P$. Assuming a standard spectral gap condition on the remainder of the spectrum of $\mathcal{L}_{\mathrm{tw}}$, one can subsequently obtain the estimate

$$\left\| S(t)Q \right\|_{L^2 \to L^2} \leq M e^{-\beta t} \tag{1.9}$$

for some constants $\beta > 0$ and $M \geq 1$; see for example [55, Lem. 5.1.2].

The common feature in all the approaches described above is that they require the identity $M = 1$ to hold. In this special case the linear flow is immediately contractive in the direction orthogonal to the translational eigenfunction. This identity certainly holds if one can obtain an estimate of the form

$$\langle \mathcal{L}_{\mathrm{tw}} v, v \rangle \leq -\beta \left\| v \right\|_{H^1}^2 + \kappa \left\| P v \right\|^2 \tag{1.10}$$

for some $\kappa > 0$, since one can then use the commutation property $PS(t) = S(t)P$ to compute

$$\frac{d}{dt} \|S(t)Qv\|_{L^2}^2 = 2\langle \mathcal{L}_{\mathrm{tw}} S(t)Qv, S(t)Qv \rangle_{L^2} \leq -2\beta \|S(t)Qv\|_{H^1}^2 \leq -2\beta \|S(t)Qv\|_{L^2}^2 . \qquad (1.11)$$

In the deterministic case, coercive estimates of this type can be used to obtain similar differential inequalities for the L^2-norm of perturbations from the phase-adjusted traveling wave. Using the Itô formula this can be generalized to the stochastic case [33, 34, 49, 50], allowing stability estimates to be obtained that do not need any control over the H^1-norm of these perturbations. The approach developed in [27] proceeds directly from (1.9) using a renormalisation method. Similar L^2-stability results can be obtained in this fashion, again crucially using the fact that $M = 1$; see [27, (6.15)].

In light of the discussion above, a considerable effort is underway to identify systems for which the immediate contractivity condition $M = 1$ indeed holds. This has been explicitly verified for the Nagumo PDE (1.6) and several classes of one-component systems [34, 49, 50]. However, these computations are very delicate and typically proceed on an ad-hoc basis. For example, it is unclear (and doubtful) whether such a condition holds for the FitzHugh-Nagumo PDE (1.7). We refer to [53, §1] for an informative discussion on this issue.

In the case $M > 1$ the semigroup is still eventually contractive on the range of Q, but it can cause transient dynamics that grow on short timescales. Such dynamics play an important role and need to be tracked over temporal intervals of intermediate length. In this case the nonlinearities cannot be immediately dominated by the linear terms as above. To control these terms it is hence crucial to understand the H^1-norm of perturbations, which poses some challenging regularity issues in the stochastic setting.

Semigroup approach In this paper we take a step towards harnessing the power of modern deterministic nonlinear stability techniques for use in the stochastic setting. In particular, inspired by the informative expository paper [59], we abandon any attempt to describe the phase of the wave via a priori geometric conditions. Instead, we initiate a semigroup approach based on the stochastic variation of constants formula. This leads to a stochastic evolution equation for the phase that follows naturally from technical considerations. More specifically, we use the phase to neutralize the dangerous non-decaying terms in our evolution equation. Our tracking mechanism is robust and allows us to focus solely on the behaviour of the perturbation from the phase-shifted wave. This allows us to track solutions up to the point where this perturbation becomes too large as a result of the stochastic forcing, which resembles an Ornstein-Uhlenbeck process and hence is unbounded almost certainly. In particular, we do not need to impose restrictions on the size of the phaseshift as in [31, 33].

The first main advantage of our approach is that it provides orbital stability results without requiring the immediate contractivity condition described above. Indeed, we are able to track the H^1-norm of perturbations and not merely the L^2-norm, which allows us to have $M > 1$ in (1.9). This significantly broadens the class of systems that can be understood and aligns the relevant spectral assumptions with those that are traditionally used in deterministic settings.

The second main advantage is that we are (in some sense) able to isolate the drift-like contributions to the shape and speed of the wave that are caused by the noise term. This becomes fully visible in our analysis of (1.6), where the noise term is specially tailored to the deterministic wave Φ in the sense that it is proportional to the neutral mode Φ'. In this case we are able to obtain an exponential stability result for a modified waveprofile Φ_σ that propagates with a modified speed c_σ and exists for all positive time. This allows us to rigorously understand the changes to the waveprofile and speed that were numerically observed for (1.6) in [37]. In general, if the $\mathbb{R}^n$-orbit of the traveling wave of an n-component reaction-diffusion equation contains no self-intersections, our results allow special forcing terms to be constructed for which the modified waves remain exponentially stable.

However, the need to use stochastic calculus causes several delicate technical complications that are not observed in the deterministic setting. For example, the Itô Isometry is based on L^2 norms.

At times, this forces us to square the natural semigroup decay rates, which leads to short-term regularity issues. Indeed, the heat semigroup $S(t)$ behaves as $\|S(t)\|_{\mathcal{L}(L^2;H^1)} \sim t^{-1/2}$, which is in $L^1(0,1)$ but not in $L^2(0,1)$. This precludes us from obtaining supremum control on the H^1-norm of our solutions. Instead, we obtain bounds on square integrals of the H^1-norm. For this reason, we need to carefully track how the cubic behaviour of $f_{\rm cub}(u)$ propagates through our arguments.

A second major complication is that stochastic phase-shifts lead to extra nonlinear diffusive terms. By contrast, deterministic phase-shifts lead to extra convective terms, which are of lower order and hence less dangerous. As a consequence, we encounter quasi-linear equations in our analysis that do not immediately fit into a semigroup framework. We solve this problem by using a suitable stochastic time-transform to scale out the extra diffusive terms. The fact that we need the diffusion coefficients in (1.2) to be identical is a direct consequence of this procedure.

Outlook Let us emphasize that we view the present paper merely as a proof-of-concept result for a pure semigroup-based approach. For example, in a companion paper [22] we show how the severe restriction on the diffusion coefficients of (1.2) can be removed by exploiting the block-structure of the semigroup.

In addition, our results here use the variational framework developed by Liu and Röckner [36] in order to ensure that our SPDE has a well-defined global weak solution. In future work, we intend to replace this procedure by constructing local mild solutions directly based on fixed-point arguments.

Finally, we are interested in more delicate spectral stability scenarios, which allow one or more branches of essential spectrum to touch the imaginary axis in a quadratic tangency. Situations of this type are encountered when analyzing the two-dimensional stability of traveling planar waves [4, 23, 24, 30] or when studying viscous shocks in the context of conservation laws [2, 3, 39].

Organization This paper is organized as follows. We formulate our phase-tracking mechanism and state our main results in §2. In §3 we obtain preliminary estimates on our nonlinearities, which are used in §4 to fit our coupled SPDE into the theory outlined in [36, 42]. This guarantees that our SPDE has well-defined solutions, to which we apply a stochastic phase-shift in §5 followed by a stochastic time-transform in §6. These steps lead to a stochastic variation of constants formula.

In §7 we develop two fixed-point arguments that capture the modifications to the waveprofile and speed that arise from the stochastic forcing. These modifications allow us to obtain suitable estimates on the nonlinearities in the variation of constants formula in §8, which allow us to pursue a nonlinear-stability argument in §9.

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2 Main results

In this paper we are interested in the stability of traveling wave solutions to SPDEs of the form

$$dU = \big[A_* U + f(U)\big]dt + \sigma g(U) d\beta_t. \tag{2.1}$$

Here we take $U = U(x,t) \in \mathbb{R}^n$ with $x \in \mathbb{R}$ and $t \geq 0$.

In §2.1 we formulate several conditions on the nonlinearity f and the diffusion operator A_*, which imply that in the deterministic case $\sigma = 0$ the system (2.1) has a variational structure and admits a spectrally stable traveling wave solution. In §2.2 we impose several standard conditions on the noise term in (2.1), which guarantee that (2.1) is covered by the variational framework developed in [36]. In addition, we couple an extra SDE to our SPDE that will serve as a phase-tracking mechanism. Finally, in §2.3 and §2.4 we formulate and discuss our main results concerning the impact of the noise term on the deterministic traveling wave solutions.

2.1 Deterministic setup

We start here by stating our conditions on the form of A_* and f. These conditions require A_* to be a diffusion operator with identical diffusion coefficients and restrict the growth-rate of f to be at most cubic.

(HA) For any $u \in C^2(\mathbb{R};\mathbb{R}^n)$ we have $A_* u = \rho I_n u_{xx}$, in which $\rho > 0$ and I_n is the $n \times n$-identity matrix.

(Hf) We have $f \in C^3(\mathbb{R}^n;\mathbb{R}^n)$ and there exist $u_\pm \in \mathbb{R}^n$ for which $f(u_-) = f(u_+) = 0$. In addition, there exists a constant $K_f > 0$ so that the bound

$$\left|D^3 f(u)\right| \le K_f \tag{2.2}$$

holds for all $u \in \mathbb{R}^n$.

We now demand that the deterministic part of (2.1) has a traveling wave solution that connects the two equilibria $u_\pm$ (which are allowed to be equal). This traveling wave should approach these equilibria at an exponential rate.

(HTw) There exists a waveprofile $\Phi_0 \in C^2(\mathbb{R};\mathbb{R}^n)$ and a wavespeed $c_0 \in \mathbb{R}$ so that the function

$$u(x,t) = \Phi_0(x - c_0 t) \tag{2.3}$$

satisfies the deterministic PDE

$$u_t = A_* u + f(u) \tag{2.4}$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}$. In addition, there is a constant $K > 0$ together with exponents $\nu_\pm > 0$ so that the bound

$$|\Phi_0(\xi) - u_-| + |\Phi_0'(\xi)| \le K e^{-\nu_- |\xi|} \tag{2.5}$$

holds for all $\xi \le 0$, while the bound

$$|\Phi_0(\xi) - u_+| + |\Phi_0'(\xi)| \le K e^{-\nu_+ |\xi|} \tag{2.6}$$

holds for all $\xi \ge 0$.

Throughout this paper, we will use the shorthands

$$L^2 = L^2(\mathbb{R};\mathbb{R}^n), \qquad H^1 = H^1(\mathbb{R};\mathbb{R}^n), \qquad H^2 = H^2(\mathbb{R};\mathbb{R}^n). \tag{2.7}$$

Linearizing the deterministic PDE (2.4) around the traveling wave (Φ_0, c_0), we obtain the linear operator

$$\mathcal{L}_{\mathrm{tw}} : H^2 \to L^2 \tag{2.8}$$

that acts as

$$[\mathcal{L}_{\mathrm{tw}} v](\xi) = c_0 v'(\xi) + [A_* v](\xi) + Df\big(\Phi_0(\xi)\big) v(\xi). \tag{2.9}$$

The formal adjoint

$$\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} : H^2 \to L^2 \tag{2.10}$$

of this operator acts as

$$[\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} w](\xi) = -c_0 w'(\xi) + [A_* w](\xi) + Df\big(\Phi_0(\xi)\big) w(\xi). \tag{2.11}$$

Indeed, one easily verifies that

$$\langle \mathcal{L}_{\mathrm{tw}} v, w \rangle_{L^2} = \langle v, \mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} w \rangle_{L^2} \tag{2.12}$$

whenever $(v,w) \in H^2 \times H^2$. Here $\langle \cdot, \cdot \rangle_{L^2}$ denotes the standard inner-product on L^2.

We now impose a standard spectral stability condition on the wave. In particular, we require that the standard translational eigenvalue at zero is a simple eigenvalue. In addition, the remainder of the spectrum of $\mathcal{L}_{\mathrm{tw}}$ must be strictly bounded to the left of the imaginary axis.

(HS) There exists $\beta > 0$ so that the operator $\mathcal{L}_{\mathrm{tw}} - \lambda$ is invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$ that have $\mathrm{Re}\,\lambda \geq -2\beta$, while $\mathcal{L}_{\mathrm{tw}}$ is a Fredholm operator with index zero. In addition, we have the identities

$$\mathrm{Ker}(\mathcal{L}_{\mathrm{tw}}) = \mathrm{span}\{\Phi_0'\}, \qquad \mathrm{Ker}(\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}}) = \mathrm{span}\{\psi_{\mathrm{tw}}\} \tag{2.13}$$

for some $\psi_{\mathrm{tw}} \in H^2$ that has

$$\langle \Phi_0', \psi_{\mathrm{tw}} \rangle_{L^2} = 1. \tag{2.14}$$

We conclude by imposing a standard monotonicity condition on f, which ensures that the SPDE (2.1) fits into the variational framework of [36]. We remark here that we view this condition purely as a technical convenience, since it guarantees that solutions to (2.1) do not blow up. However, it does not a play a key role in the heart of our computations, where we restrict our attention to solutions that remain small in some sense.

(HVar) There exists $K_{\mathrm{var}} > 0$ so that the one-sided inequality

$$\langle f(u_A) - f(u_B), u_A - u_B \rangle_{\mathbb{R}^n} \leq K_{\mathrm{var}} \left| u_A - u_B \right|^2 \tag{2.15}$$

holds for all pairs $(u_A, u_B) \in \mathbb{R}^n \times \mathbb{R}^n$.

2.2 Stochastic setup

Our first condition here states that the noise term in (2.1) is driven by a standard Brownian motion. Let us emphasize that we made this choice purely to enhance the readability of our arguments. Indeed, our results can easily be generalized to the situation where the noise is driven by cylindrical Q-Wiener processes.

(Hβ) The process $(\beta_t)_{t \geq 0}$ is a Brownian motion with respect to the complete filtered probability space

$$\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\Big). \tag{2.16}$$

We require the function Dg to be globally Lipschitz and uniformly bounded. While the former condition is essential in our analysis to ensure that our cut-offs only depend on L^2-norms, the latter condition is only used to fit (2.1) into the framework of [36].

(Hg) We have $g \in C^2(\mathbb{R}^n; \mathbb{R}^n)$ with $g(u_-) = g(u_+) = 0$. In addition, there is $K_g > 0$ so that

$$|Dg(u)| \leq K_g \tag{2.17}$$

holds for all $u \in \mathbb{R}^n$, while

$$|g(u_A) - g(u_B)| + |Dg(u_A) - Dg(u_B)| \leq K_g \, |u_A - u_B| \tag{2.18}$$

holds for all pairs $(u_A, u_B) \in \mathbb{R}^n \times \mathbb{R}^n$.

We remark here that it is advantageous to view SPDEs as evolutions on Hilbert spaces, since powerful tools are available in this setting. However, in the case where $u_- \neq u_+$, the waveprofile Φ_0 does not lie in the natural statespace L^2. In order to circumvent this problem, we use Φ_0 as a reference function that connects u_- to u_+, allowing us to measure deviations from this function in the Hilbert spaces H^1 and L^2.

In order to highlight this dual role and prevent any confusion, we introduce the duplicate notation

$$\Phi_{\mathrm{ref}} = \Phi_0 \tag{2.19}$$

and emphasize the fact that $\Phi_{\rm ref}$ remains fixed in the original frame, unlike the wave-solution (2.3). We also introduce the sets

$$\mathcal{U}_{L^2} = \Phi_{\rm ref} + L^2, \qquad \mathcal{U}_{H^1} = \Phi_{\rm ref} + H^1, \qquad \mathcal{U}_{H^2} = \Phi_{\rm ref} + H^2, \tag{2.20}$$

which we will use as the relevant state-spaces to capture the solutions U to (2.1).

We now set out to append a phase-tracking SDE to (2.1). In the deterministic case, we would couple the PDE to a phase-shift γ that solves an ODE of the form

$$\dot{\gamma}(t) = c_0 + \mathcal{O}\Big(U(t) - \Phi_0\big(\cdot - \gamma(t)\big)\Big). \tag{2.21}$$

By tuning the forcing function it is possible to remove the non-decaying terms in the original PDE, which act in the direction of $\Phi_0'\big(\cdot - \gamma(t)\big)$. This allows a nonlinear stability argument to be closed; see e.g. [59].

In this paper we extend this procedure by introducing a phase-shift Γ that experiences the stochastic forcing

$$d\Gamma = \Big[c_\sigma + \mathcal{O}\Big(U(t) - \Phi_\sigma\big(\cdot - \Gamma(t)\big)\Big)\Big]dt + \mathcal{O}(\sigma)d\beta_t. \tag{2.22}$$

By choosing the function Φ_σ, the scalar c_σ and the two forcing functions in an appropriate fashion, the dangerous neutral terms can be eliminated from the original SPDE. These are hence purely technical considerations, but in §2.4 we discuss how these choices can be related to quantities that are interesting from an applied point of view.

In order to define our forcing functions in a fashion that is globally Lipschitz continuous, we introduce the constant

$$K_{\rm ip} = \big[\, \|g(\Phi_0)\|_{L^2} + 2K_g\big]\, \|\psi_{\rm tw}\|_{L^2}\,. \tag{2.23}$$

In addition, we pick two C^∞-smooth non-decreasing cut-off functions

$$\chi_{\rm low} : \mathbb{R} \to [\frac{1}{4}, \infty), \qquad \chi_{\rm high} : \mathbb{R} \to [-K_{\rm ip} - 1, K_{\rm ip} + 1] \tag{2.24}$$

that satisfy the identities

$$\chi_{\rm low}(\vartheta) = \frac{1}{4} \text{ for } \vartheta \le \frac{1}{4}, \qquad \chi_{\rm low}(\vartheta) = \vartheta \text{ for } \vartheta \ge \frac{1}{2}, \tag{2.25}$$

together with

$$\chi_{\rm high}(\vartheta) = \vartheta \text{ for } |\vartheta| \le K_{\rm ip}, \qquad \chi_{\rm high}(\vartheta) = \mathrm{sign}(\vartheta)\big[K_{\rm ip} + 1\big] \text{ for } |\vartheta| \ge K_{\rm ip} + 1. \tag{2.26}$$

For any $u \in \mathcal{U}_{H^1}$ and $\psi \in H^1$, this allows us to introduce the functions

$$\begin{array}{lcl} b(u, \psi) & = & -\Big[\chi_{\rm low}\big(\langle \partial_\xi u, \psi\rangle_{L^2}\big)\Big]^{-1} \chi_{\rm high}\big(\langle g(u), \psi\rangle_{L^2}\big), \\ \kappa_\sigma(u, \psi) & = & 1 + \frac{1}{2\rho}\sigma^2 b(u, \psi)^2. \end{array} \tag{2.27}$$

In addition, for any $u \in \mathcal{U}_{H^1}$, $c \in \mathbb{R}$ and $\psi \in H^1$ we define the expression

$$\mathcal{J}_\sigma(u, c, \psi) \quad = \quad \kappa_\sigma(u, \psi)^{-1}\Big[f(u) + cu' + \sigma^2 b(u, \psi)\partial_\xi[g(u)]\Big], \tag{2.28}$$

while for any $u \in \mathcal{U}_{H^1}$, $c \in \mathbb{R}$ and $\psi \in H^2$ we write

$$a_\sigma(u, c, \psi) \quad = \quad -\kappa_\sigma(u, \psi)\Big[\chi_{\rm low}\big(\langle \partial_\xi u, \psi\rangle_{L^2}\big)\Big]^{-1}\Big[\langle u, A_* \psi\rangle_{L^2} + \langle \mathcal{J}_\sigma(u, c, \psi), \psi\rangle_{L^2}\Big]. \tag{2.29}$$

Finally, we introduce the right-shift operators

$$[T_\gamma u](\xi) = u(\xi - \gamma) \tag{2.30}$$

that act on any function $u : \mathbb{R} \to \mathbb{R}^n$.

With these ingredients in hand, we are ready to introduce the main SPDE that we analyze in this paper. We formally write this SPDE as the skew-coupled system[1]

$$\begin{array}{rcl} dU & = & \big[A_* U + f(U)\big]dt + \sigma g(U) d\beta_t, \\ d\Gamma & = & \big[c + a_\sigma\big(U, c, T_\Gamma \psi_{\mathrm{tw}}\big)\big]dt + \sigma b\big(U, T_\Gamma \psi_{\mathrm{tw}}\big)\, d\beta_t, \end{array} \tag{2.31}$$

noting that we seek solutions with $\big(U(t), \Gamma(t)\big) \in \mathcal{U}_{H^1} \times \mathbb{R}$. Observe that the first equation is the same as (2.1).

In order to make this precise, we introduce the spaces

$$\begin{array}{rcl} \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); \mathcal{H}\big) & = & \{X \in L^2\big([0,T] \times \Omega; dt \otimes \mathbb{P}; \mathcal{H}\big) : \\ & & \qquad X \text{ has a } (\mathcal{F}_t)\text{-progressively measurable version}\}, \end{array} \tag{2.32}$$

where we allow $\mathcal{H} \in \{\mathbb{R}, L^2, H^1\}$. We note that we follow the convention of [42, 43] here by requiring progressive measurability instead of the usual stronger notion of predictability. Since we are exclusively dealing with Brownian motions, this choice suffices to construct stochastic integrals.

Our first result clarifies what we mean by a solution to (2.31). We note that (i) and (ii) in Proposition 2.1 imply that (X, Γ) is an $L^2 \times \mathbb{R}$-valued continuous $(\mathcal{F}_t)$-adapted process. We remark that in the integral equation (2.42) we interpret the diffusion operator A_* as an element of $\mathcal{L}(H^1; H^{-1})$, where H^{-1} is the dual of H^1 under the standard embeddings

$$H^1 \hookrightarrow L^2 \cong \big[L^2\big]^* \hookrightarrow H^{-1} = \big[H^1\big]^*. \tag{2.33}$$

We note that the set (H^1, L^2, H^{-1}) is commonly referred to as a Gelfand triple; see e.g. [14, §5.9] for a more detailed explanation. For $(v, w) \in H^{-1} \times H^1$ we write $\langle v, w \rangle_{H^{-1};H^1}$ to refer to the duality pairing between H^1 and H^{-1}. If in fact $v \in L^2$, then we have

$$\langle v, w \rangle_{H^{-1};H^1} = \langle v, w \rangle_{L^2}. \tag{2.34}$$

Proposition 2.1 (see §4)**.** *Suppose that (HA), (Hf), $(HVar)$, (HTw), (HS), (Hg) and $(H\beta)$ are all satisfied and fix $T > 0$, $c \in \mathbb{R}$ and $0 \le \sigma \le 1$. In addition, pick an initial condition*

$$(X_0, \Gamma_0) \in L^2 \times \mathbb{R}. \tag{2.35}$$

Then there are maps

$$X : [0,T] \times \Omega \to L^2, \qquad \Gamma : [0,T] \times \Omega \to \mathbb{R} \tag{2.36}$$

that satisfy the following properties.

(i) For almost all $\omega \in \Omega$, the map

$$t \mapsto \big(X(t,\omega), \Gamma(t,\omega)\big) \tag{2.37}$$

is of class $C([0,T]; L^2 \times \mathbb{R})$.

(ii) For all $t \in [0,T]$, the map

$$\omega \mapsto \big(X(t,\omega), \Gamma(t,\omega)\big) \in L^2 \times \mathbb{R} \tag{2.38}$$

is $(\mathcal{F}_t)$-measurable.

[1] Note here that formally $b(U, T_\Gamma \psi_{\mathrm{tw}})$ is a multiplication operator from $\mathbb{R} \to \mathbb{R}$, hence a number. If we generalize β_t to a cylindrical $Q-$Wiener process on a space H then the term involving b becomes a functional from H to $\mathbb{R}$. Fortunately, all the relevant technical machinery that we use has been generalized to this setting. In particular, our approach carries over but the notation becomes significantly more convoluted.

(iii) We have the inclusion

$$X \in L^6\big(\Omega, \mathbb{P}; C([0,T]; L^2)\big), \tag{2.39}$$

together with

$$\begin{array}{lcl} X & \in & \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); H^1\big), \\ \Gamma & \in & \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); \mathbb{R}\big) \end{array} \tag{2.40}$$

and

$$\begin{array}{lcl} g(X + \Phi_{\mathrm{ref}}) & \in & \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); L^2\big), \\ b\big(X + \Phi_{\mathrm{ref}}, T_\Gamma \psi_{\mathrm{tw}}\big) & \in & \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); \mathbb{R}\big). \end{array} \tag{2.41}$$

(iv) For almost all $\omega \in \Omega$, the identities

$$\begin{array}{lcl} X(t) & = & X_0 + \int_0^t A_*[X(s) + \Phi_{\mathrm{ref}}]\, ds + \int_0^t f\big(X(s) + \Phi_{\mathrm{ref}}\big)\, ds \\ & & \quad + \sigma \int_0^t g\big(X(s) + \Phi_{\mathrm{ref}}\big) d\beta_s \end{array} \tag{2.42}$$

and

$$\begin{array}{lcl} \Gamma(t) & = & \Gamma_0 + \int_0^t \big[c + a_\sigma\big(X(s) + \Phi_{\mathrm{ref}}, c, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big)\big]\, ds \\ & & \quad + \sigma \int_0^t b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big) d\beta_s \end{array} \tag{2.43}$$

hold[2] *for all $0 \le t \le T$.*

(v) Suppose that the pair $(\tilde{X}, \tilde{\Gamma}) : [0,T] \times \Omega \to L^2 \times \mathbb{R}$ also satisfies (i)-(iv). Then for almost all $\omega \in \Omega$, we have

$$(\tilde{X}, \tilde{\Gamma})(t) = (X, \Gamma)(t) \qquad \text{for all } 0 \le t \le T. \tag{2.44}$$

2.3 Wave stability

By inserting the traveling wave Ansatz (2.3) into the deterministic PDE (2.4), we observe that

$$A_* \Phi_0 + \mathcal{J}_0(\Phi_0, c_0, \psi_{\mathrm{tw}}) = 0, \tag{2.45}$$

which means that $a_0(\Phi_0, c_0, \psi_{\mathrm{tw}}) = 0$. Our first result here shows that this can be extended into a branch of profiles and speeds for which

$$a_\sigma(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}) = 0. \tag{2.46}$$

Roughly speaking, this means that the adjusted phase $\Gamma(t) - ct$ will (instantaneously) feel only stochastic forcing if one takes $c = c_\sigma$ and $U = T_{\Gamma(t)}\Phi_\sigma$ in (2.31).

Proposition 2.2 (see §7)**.** *Suppose that (HA), (Hf), (HTw), (HS) and (Hg) are all satisfied and pick a sufficiently large constant $K > 0$. Then there exists $\delta_\sigma > 0$ so that for every $0 \le \sigma \le \delta_\sigma$, there is a unique pair*

$$(\Phi_\sigma, c_\sigma) \in \mathcal{U}_{H^2} \times \mathbb{R} \tag{2.47}$$

that satisfies the system

$$A_* \Phi_\sigma + \mathcal{J}_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big) = 0 \tag{2.48}$$

and admits the bound

$$\|\Phi_\sigma - \Phi_0\|_{H^2} + |c_\sigma - c_0| \le K\sigma^2. \tag{2.49}$$

We are interested in solutions to (2.31) with an initial condition for U that is close to Φ_σ. We will use the remaining degree of freedom to pick the initial phase Γ in such a way that the orthogonality condition described in the following result is enforced.

[2]Note that this equation initially only holds as an identity in H^{-1}. Inclusion (2.39) makes that we can interpret the integrals in L^2. We have $X \in \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); H^1\big)$ but this does not mean that $X(t) \in H^1$ pointwise.

Proposition 2.3 (see §7)**.** *Suppose that (HA), (Hf), (HTw), (HS) and (Hg) are all satisfied. Then there exist constants* $\delta_0 > 0$*,* $\delta_\sigma > 0$ *and* $K > 0$ *so that the following holds true. For every* $0 \le \sigma \le \delta_\sigma$ *and any* $u_0 \in \mathcal{U}_{L^2}$ *that has*

$$\|u_0 - \Phi_\sigma\|_{L^2} < \delta_0, \tag{2.50}$$

there exists $\gamma_0 \in \mathbb{R}$ *for which the function*

$$v_{\gamma_0} = T_{-\gamma_0}[u_0] - \Phi_\sigma \tag{2.51}$$

satisfies the identity

$$\langle v_{\gamma_0}, \psi_{\rm tw}\rangle_{L^2} = 0 \tag{2.52}$$

together with the bound

$$|\gamma_0| + \|v_{\gamma_0}\|_{L^2} \le K\,\|u_0 - \Phi_\sigma\|_{L^2}\,. \tag{2.53}$$

If in fact $u_0 \in \mathcal{U}_{H^1}$*, then we also have the estimate*

$$|\gamma_0| + \|v_{\gamma_0}\|_{H^1} \le K\,\|u_0 - \Phi_\sigma\|_{H^1}\,. \tag{2.54}$$

Let us now pick any $u_0 \in \mathcal{U}_{H^1}$ for which (2.50) holds. We write (X_{u_0}, Γ_{u_0}) for the process described in Proposition 2.1 with the initial condition

$$(X_0, \Gamma_0) = \big(u_0 - \Phi_{\rm ref}, \gamma_0\big), \tag{2.55}$$

in which γ_0 is the initial phase defined in Proposition 2.3. We then define the process

$$V_{u_0}(t) = T_{-\Gamma_{u_0}(t)}\big[X_{u_0}(t) + \Phi_{\rm ref}\big] - \Phi_\sigma, \tag{2.56}$$

which can be thought of as the deviation of the solution U of (2.31) from the stochastic wave Φ_σ shifted to the position $\Gamma_{u_0}(t)$.

In order to measure the size of the perturbation, we pick $\varepsilon > 0$ and introduce the scalar function

$$N_{\varepsilon;u_0}(t) = \|V_{u_0}(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)}\,\|V_{u_0}(s)\|_{H^1}^2\,ds. \tag{2.57}$$

For each $T > 0$ we now define a probability

$$p_\varepsilon(T, \eta, u_0) = P\Big(\sup_{0\le t\le T} N_{\varepsilon;u_0}(t) > \eta\Big). \tag{2.58}$$

Our first main result shows that the probability that $N_{\varepsilon;u_0}$ remains small on timescales of order σ^{-2} can be pushed arbitrarily close to one by restricting the strength of the noise and the size of the initial perturbation.

Theorem 2.4 (see §9)**.** *Suppose that* (HA)*,* (Hf)*,* $(HVar)$*,* (HTw)*,* (HS)*,* (Hg) *and* $(H\beta)$ *are all satisfied and pick sufficiently small constants* $\varepsilon > 0$*,* $\delta_0 > 0$*,* $\delta_\eta > 0$ *and* $\delta_\sigma > 0$*. Then there exists a constant* $K > 0$ *so that for every* $T > 1$*, any* $0 \le \sigma \le \delta_\sigma T^{-1/2}$*, any* $u_0 \in \mathcal{U}_{H^1}$ *that satisfies (2.50) and any* $0 < \eta \le \delta_\eta$*, we have the inequality*

$$p_\varepsilon(T, \eta, u_0) \le \eta^{-1} K\Big[\,\|u_0 - \Phi_\sigma\|_{H^1}^2 + \sigma^2 T\Big]. \tag{2.59}$$

Our second main result concerns the special case where the noise pushes the stochastic wave Φ_σ in a rigid fashion. This is the case when

$$g(\Phi_0) = \vartheta_0 \Phi_0' \tag{2.60}$$

for some proportionality constant $\vartheta_0 \in \mathbb{R}$. It is easy to verify that (2.60) with $\vartheta_0 = -\sqrt{2}$ holds for (1.6).

In this setting we expect the perturbation V to decay exponentially on timescales of order σ^{-2} with large probability. In order to formalize this, we pick small constants $\varepsilon > 0$ and $\alpha > 0$ and introduce the scalar function

$$N_{\varepsilon,\alpha;u_0}(t) = e^{\alpha t} \|V_{u_0}(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V_{u_0}(s)\|_{H^1}^2 \, ds, \tag{2.61}$$

together with the associated probabilities

$$p_{\varepsilon,\alpha}(T, \eta, u_0) = P\Big(\sup_{0 \le t \le T} N_{\varepsilon,\alpha;u_0}(t) > \eta \Big). \tag{2.62}$$

Theorem 2.5 (see §9)**.** *Suppose that* (HA), (Hf), $(HVar)$, (HTw), (HS), (Hg) *and* $(H\beta)$ *are all satisfied . Suppose furthermore that (2.60) holds and pick sufficiently small constants* $\varepsilon > 0$, $\delta_0 > 0$, $\alpha > 0$, $\delta_\eta > 0$ *and* $\delta_\sigma > 0$. *Then there exists a constant* $K > 0$ *so that for any* $T > 1$, *every* $0 \le \sigma \le \delta_\sigma T^{-1/2}$, *any* $u_0 \in \mathcal{U}_{H^1}$ *that satisfies (2.50) and any* $0 < \eta \le \delta_\eta$, *we have the inequality*

$$p_{\varepsilon,\alpha}(T, \eta, u_0) \le \eta^{-1} K \|u_0 - \Phi_\sigma\|_{H^1}^2 . \tag{2.63}$$

2.4 Interpretation

In §5 we show that the pair $(V, \Gamma) = (V_{u_0}, \Gamma_{u_0})$ defined in §2.3 satisfies the SPDE

$$\begin{aligned} dV &= \mathcal{R}_\sigma(V)\, dt + \sigma \mathcal{S}_\sigma(V)\, d\beta_t, \\ d\Gamma &= \big[c_\sigma + a_\sigma\big(\Phi_\sigma + V, c_\sigma, \psi_{\mathrm{tw}}\big)\big]\, dt + \sigma b\big(\Phi_\sigma + V, \psi_{\mathrm{tw}}\big)\, d\beta_t, \end{aligned} \tag{2.64}$$

in which the nonlinearities satisfy the identities

$$a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big) = 0, \qquad \mathcal{R}_\sigma(0) = 0, \qquad \mathcal{S}_\sigma(0) = g(\Phi_\sigma) + b(\Phi_\sigma)\Phi_\sigma', \tag{2.65}$$

together with the asymptotics

$$D_1 a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big) = \mathcal{O}(\sigma^2), \qquad D\mathcal{R}_\sigma(0) = \mathcal{O}(\sigma^2). \tag{2.66}$$

For our discussion here we take $V(0) = 0$ and $\Gamma(0) = 0$, which corresponds with the initial condition condition $U(0) = \Phi_\sigma$ for the original system (2.1).

The identities (2.65) imply that $V(t)$ and $\Gamma(t) - c_\sigma t$ experience no deterministic forcing at $t = 0$. We now briefly discuss the consequences of this observation on the behaviour of (2.64) in the two regimes covered by Theorems 2.4 and 2.5.

Exponential stability Our results are easiest to interpret in the special case

$$g(\Phi_0) = \vartheta_0 \Phi_0' \tag{2.67}$$

where Theorem 2.5 applies. Remarkably, the modified profiles and speeds (Φ_σ, c_σ) can be computed explicitly in this setting.

Proposition 2.6 (see §7)**.** *Consider the setting of Proposition 2.2 and suppose that (2.60) holds. Then for all sufficiently small* $0 \le \sigma \le \delta_\sigma$ *we have the identities*

$$\begin{aligned} \Phi_\sigma(\xi) &= \Phi_0\Big(\big[1 + \tfrac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{1/2}\xi\Big), \\ c_\sigma &= \big[1 + \tfrac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{-1/2} c_0, \end{aligned} \tag{2.68}$$

together with

$$g(\Phi_\sigma) = \big[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{-1/2} \vartheta_0 \Phi_\sigma' = -b\big(\Phi_\sigma, \psi_{\mathrm{tw}}\big)\Phi_\sigma'. \tag{2.69}$$

A direct consequence of (2.69) is that the identity

$$\mathcal{S}_\sigma(0) = 0 \tag{2.70}$$

can be added to the list (2.65). In particular, we obtain the explicit solution

$$\begin{array}{rcl} (V, \Gamma) & = & \Big(0, c_\sigma t + \sigma b(\Phi_\sigma, \psi_{\rm tw})\beta_t\Big) \\ & = & \Big(0, c_\sigma t - \sigma\big[1 + \frac{1}{2\rho}\sigma^2 \vartheta_0^2\big]^{-1/2}\vartheta_0 \beta_t\Big) \end{array} \tag{2.71}$$

for the system (2.64). This corresponds to the solution

$$U(t) = \Phi_\sigma(\cdot + \Gamma(t)) \tag{2.72}$$

for (2.31), which exists for all $t \geq 0$.

We hence see that the shape Φ_σ of the stochastic profile remains fixed, while the phase $\Gamma(t)$ of the wave performs a scaled Brownian motion around the position $c_\sigma t$. Since the identities (2.68) imply that the waveprofile is steepened while the speed is slowed down, our results indeed confirm the numerical observations from [37] that were discussed in §1.

Any small perturbation in the V component will decay exponentially fast with high probability on account of Theorem 2.5. Intuitively, the leading order behaviour for V resembles a geometric Brownian motion, as the noise term is proportional to V while the deterministic forcing leads to exponential decay. In particular, we expect that our approach can keep track of the wave for timescales that are far longer than the $\mathcal{O}(\sigma^{-2})$ bounds stated in our results.

Orbital stability In general we have $\mathcal{S}_\sigma(0) \neq 0$, which prevents us from solving (2.64) explicitly. Indeed, Theorem 2.4 states that $V(t)$ will remain small with high probability, but the stochastic forcing will preclude it from converging to zero. However, our construction does guarantee that $\langle V(t), \psi_{\rm tw}\rangle = 0$ as long as V stays small. Since $\langle \Phi_0', \psi_{\rm tw}\rangle = 1$, this still allows us to interpret $\Gamma(t)$ as the position of the wave. In particular, if the expression $t^{-1}\Gamma(t)$ converges in a suitable sense as $t \to \infty$ then it is natural to use this limit as a proxy for the notion of a wavespeed.

In order to explore this, we introduce the formal expansion

$$V(t) = \sigma V_\sigma^{(1)}(t) + \mathcal{O}(\sigma^2) \tag{2.73}$$

and use the mild formulation developed in §6 to obtain

$$V_\sigma^{(1)}(t) = \int_0^t S(t-s)\mathcal{S}_\sigma(0)\, d\beta_t. \tag{2.74}$$

Here S denotes the semigroup generated by $\mathcal{L}_{\rm tw}$, which by construction decays exponentially when applied to $\mathcal{S}_\sigma(0)$. In particular, for any bilinear map $B : H^1 \times H^1 \to \mathbb{R}$ we can use the Itô isometry to obtain

$$\begin{array}{rcl} EB\big[V_\sigma^{(1)}(t), V_\sigma^{(1)}(t)\big] & = & \int_0^t B\big[S(t-s)\mathcal{S}_\sigma(0), S(t-s)\mathcal{S}_\sigma(0)\big]\, ds \\ & = & \int_0^t B\big[S(s)\mathcal{S}_\sigma(0), S(s)\mathcal{S}_\sigma(0)\big]\, ds, \end{array} \tag{2.75}$$

which converges in the limit $t \to \infty$.

Introducing the formal expansion

$$\Gamma(t) = c_\sigma t + \sigma \Gamma_\sigma^{(1)}(t) + \sigma^2 \Gamma_\sigma^2(t) + \mathcal{O}(\sigma^3), \tag{2.76}$$

the first bound in (2.66) implies that

$$\Gamma_\sigma^{(1)}(t) \quad = \quad b(\Phi_\sigma, \psi_{\rm tw})\beta_t \tag{2.77}$$

together with

$$\Gamma_\sigma^{(2)}(t) = \int_0^t D_1^2 a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big)\big[V_\sigma^{(1)}(s), V_\sigma^{(1)}(s)\big]\, ds + D_1 b\big(\Phi_\sigma, \psi_{\mathrm{tw}}\big) \int_0^t V_\sigma^{(1)}(s)\, d\beta_s. \tag{2.78}$$

Since $EV_\sigma^{(1)}(t) = 0$ we obtain

$$E\Gamma_\sigma^{(1)}(t) = 0 \tag{2.79}$$

together with

$$\begin{aligned} E\Gamma_\sigma^{(2)}(t) &= \int_0^t \int_0^s D_1^2 a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big)\big[S(s')\mathcal{S}_\sigma(0), S(s')\mathcal{S}_\sigma(0)\big]\, ds'\, ds \\ &= \int_0^t (t-s) D_1^2 a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big)\big[S(s)\mathcal{S}_\sigma(0), S(s)\mathcal{S}_\sigma(0)\big]\, ds. \end{aligned} \tag{2.80}$$

Upon writing

$$c_\infty^{(2)} = c_\sigma + \sigma^2 \int_0^\infty D_1^2 a_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big)\big[S(s)\mathcal{S}_\sigma(0), S(s)\mathcal{S}_\sigma(0)\big]\, ds, \tag{2.81}$$

we hence conjecture that the expected limiting value of the wavespeed behaves as $c_\infty^{(2)} + \mathcal{O}(\sigma^3)$. Since $c_\sigma = c_0 + \mathcal{O}(\sigma^2)$ this would mean that the stochastic contributions to the wavespeed are second order in σ.

We remark that computations of this kind resemble the multi-scale approach initiated by Lang in [33] and Stannat and Krüger in [31]. However, our approach does allow us to consider limiting expressions such as (2.81), for which one needs the exponential decay of the semigroup. Indeed, (2.74) resembles a mean-reverting Ornstein-Uhlenbeck process, which has a variation that can be globally bounded in time, despite the fact that the individual paths are unbounded.

As above, we expect to be able to track the wave for timescales that are longer than the $\mathcal{O}(\sigma^{-2})$ bounds stated in our results. The key issue is that the mild version of the Burkholder-Davis-Gundy inequality that we use is not able to fully incorporate the mean-reverting effects of the semigroup. We emphasize that even the standard scalar Ornstein-Uhlenbeck process requires sophisticated probabilistic machinery to uncover statistics concerning the behaviour of the running supremum [1, 44]. We plan to explore these issues in more detail in a forthcoming paper. For the moment however, we note that our initial numerical experiments seem to confirm that the expression (2.81) indeed captures the leading order stochastic correction to the wavespeed.

3 Preliminary estimates

In this section we derive several preliminary estimates for the functions f, g, $\mathcal{J}_0$, b and κ_σ. We will write the arguments $(u, \overline{c}) \in \mathcal{U}_{H^1} \times \mathbb{R}$ as

$$u = \Phi + v, \qquad \overline{c} = c + d, \tag{3.1}$$

in which we take $(\Phi, c) \in \mathcal{U}_{H^1} \times \mathbb{R}$ and $(v, d) \in H^1 \times \mathbb{R}$. We do not restrict ourselves to the case where $(\Phi, c) = (\Phi_0, c_0)$, but impose the following condition.

(hPar) The conditions (HTw) and (HS) hold and the pair $(\Phi, c) \in \mathcal{U}_{H^1} \times \mathbb{R}$ satisfies the bounds

$$\|\Phi - \Phi_0\|_{H^1} \le \min\{1, [4\,\|\psi_{\mathrm{tw}}\|_{L^2}]^{-1}\}, \qquad |c - c_0| \le 1. \tag{3.2}$$

In §3.1 we obtain global and Lipschitz bounds for the functions f and g. These bounds are subsequently used in §3.2 to analyze the auxiliary functions $\mathcal{J}_0$, b and κ_σ. Throughout this paper we use the convention that all numbered constants appearing in proofs are strictly positive and have the same dependencies as the constants appearing in the statement of the result.

3.1 Bounds on f and g

The conditions (Hf) and (Hg) allow us to obtain standard cubic bounds on f and globally Lipschitz bounds on g. We also consider expressions of the form $\partial_\xi g(u)$, which give rise to quadratic bounds.

Lemma 3.1. *Suppose that (Hf) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bounds*

$$\begin{array}{rcl} \|f(\Phi + v)\|_{L^2} & \leq & K\big[1 + \|v\|_{H^1}^2 \|v\|_{L^2}\big], \\ |\langle f(\Phi + v), \psi\rangle_{L^2}| & \leq & K\big[1 + \|v\|_{H^1} \|v\|_{L^2}^2\big] \|\psi\|_{H^1}, \end{array} \tag{3.3}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$, the expressions

$$\begin{array}{rcl} \Delta_{AB} f & = & f(\Phi + v_A) - f(\Phi + v_B), \\ \Delta_{AB}\langle f, \cdot\rangle_{L^2} & = & \langle f(\Phi + v_A), \psi_A\rangle_{L^2} - \langle f(\Phi + v_B), \psi_B\rangle_{L^2} \end{array} \tag{3.4}$$

satisfy the estimates

$$\begin{array}{rcl} \|\Delta_{AB} f\|_{L^2} & \leq & K \|v_A - v_B\|_{L^2} \\ & & +K\Big(\|v_A\|_{H^1} \|v_A\|_{L^2} + \|v_B\|_{H^1} \|v_B\|_{L^2} \Big) \|v_A - v_B\|_{H^1}, \\ |\Delta_{AB}\langle f, \cdot\rangle_{L^2}| & \leq & K \|v_A - v_B\|_{L^2} \|\psi_A\|_{H^1} \\ & & +K \|v_A - v_B\|_{H^1} \big(\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2 \big) \|\psi_A\|_{H^1} \\ & & +K\Big[1 + \|v_B\|_{H^1} \|v_B\|_{L^2}^2\Big] \|\psi_A - \psi_B\|_{H^1}. \end{array} \tag{3.5}$$

Proof. Exploiting (Hf) we obtain

$$\left|D^2 f(u)\right| \leq C_1[1 + |u|], \tag{3.6}$$

together with

$$|Df(u)| \leq C_1[1 + |u|^2] \tag{3.7}$$

for all $u \in \mathbb{R}^n$. In particular, (hPar) yields the pointwise Lipschitz bound

$$|f(\Phi + v_A) - f(\Phi + v_B)| \leq C_2[1 + |v_A|^2 + |v_B|^2] \, |v_A - v_B|. \tag{3.8}$$

Using the Sobolev embedding $\|\cdot\|_\infty \leq C_3 \|\cdot\|_{H^1}$ this immediately implies the first estimate in (3.5). Applying this estimate with $v_A = 0$ and $v_B = \Phi_0 - \Phi$ we find

$$\begin{array}{rcl} \|f(\Phi)\|_{L^2} & \leq & \|f(\Phi_0)\|_{L^2} + \|f(\Phi) - f(\Phi_0)\|_{L^2} \\ & \leq & C_4. \end{array} \tag{3.9}$$

Exploiting

$$\|f(\Phi + v)\|_{L^2} \leq \|f(\Phi)\|_{L^2} + \|f(\Phi + v) - f(\Phi)\|_{L^2}, \tag{3.10}$$

we hence obtain

$$\|f(\Phi + v)\|_{L^2} \quad \leq \quad C_5\big[1 + \|v\|_{L^2} + \|v\|_{H^1}^2 \|v\|_{L^2}\big]. \tag{3.11}$$

The first estimate in (3.3) now follows by noting that $\|v\|_{L^2} \leq \|v\|_{H^1}^2 \|v\|_{L^2}$ for $\|v\|_{L^2} \geq 1$.

Turning to the inner products, (3.8) allows us to compute

$$\begin{array}{rcl} |\langle f(\Phi + v_A) - f(\Phi + v_B), \psi_A\rangle_{L^2}| & \leq & C_2 \|v_A - v_B\|_{L^2} \|\psi_A\|_{L^2} \\ & & +C_2\big[\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2 \big] \|v_A - v_B\|_{H^1} \|\psi_A\|_{H^1}. \end{array} \tag{3.12}$$

Exploiting (3.9), the second estimate in (3.3) hence follows from the bound

$$|\langle f(\Phi+v),\psi\rangle_{L^2}| \quad \le \quad |\langle f(\Phi),\psi\rangle_{L^2}| + |\langle f(\Phi+v)-f(\Phi),\psi\rangle_{L^2}|, \tag{3.13}$$

using a similar observation as above to eliminate the $\|v\|_{L^2}\|\psi\|_{L^2}$ term. Finally, the second estimate in (3.5) can be obtained by applying (3.12) and (3.3) to the splitting

$$\begin{aligned} |\langle f(\Phi+v_A),\psi_A\rangle_{L^2} - \langle f(\Phi+v_B),\psi_B\rangle_{L^2}| \quad &\le \quad |\langle f(\Phi+v_A)-f(\Phi+v_B),\psi_A\rangle_{L^2}| \\ &\qquad + |f(\Phi+v_B),\psi_A-\psi_B\rangle_{L^2}|. \end{aligned} \tag{3.14}$$

□

Lemma 3.2. *Suppose that (Hg) and (hPar) are satisfied. Then there exists a constant $K>0$, which does not depend on the pair (Φ,c), so that the following holds true. For any $v\in H^1$ we have the bounds*

$$\begin{aligned} \|g(\Phi+v)\|_{L^2} \quad &\le \quad \|g(\Phi_0)\|_{L^2} + K_g(1+\|v\|_{L^2}) \\ &\le \quad K[1+\|v\|_{L^2}], \\ \|\partial_\xi g(\Phi+v)\|_{L^2} \quad &\le \quad K[1+\|v\|_{H^1}], \end{aligned} \tag{3.15}$$

while for any pair $(v_A,v_B)\in H^1\times H^1$ we have the estimates

$$\begin{aligned} \|g(\Phi+v_A)-g(\Phi+v_B)\|_{L^2} \quad &\le \quad K\,\|v_A-v_B\|_{L^2}, \\ \|\partial_\xi[g(\Phi+v_A)-g(\Phi+v_B)]\|_{L^2} \quad &\le \quad K\big[1+\|v_A\|_{H^1}\big]\,\|v_A-v_B\|_{H^1}. \end{aligned} \tag{3.16}$$

Proof. The Lipschitz estimate on g implies that

$$\|g(\Phi+v_A)-g(\Phi+v_B)\|_{L^2} \le K_g\,\|v_A-v_B\|_{L^2}. \tag{3.17}$$

Applying this inequality with $v_A=v$ and $v_B=\Phi_0-\Phi$ we obtain

$$\|g(\Phi+v)\|_{L^2} \quad \le \quad \|g(\Phi_0)\|_{L^2} + K_g\big[\,\|\Phi-\Phi_0\|_{L^2}+\|v\|_{L^2}\,\big], \tag{3.18}$$

which in view of (hPar) yields the first line of (3.15).

The uniform bound

$$|Dg(\Phi+v)| \le K_g \tag{3.19}$$

together with the identity

$$\partial_\xi g(\Phi+v) = Dg(\Phi+v)\big(\Phi'+v'\big) \tag{3.20}$$

immediately imply the second estimate in (3.15). Finally, using

$$|Dg(\Phi+v)-Dg(\Phi+w)| \le K_g\,|v-w| \tag{3.21}$$

and the identity

$$\begin{aligned} \partial_\xi\big[g(\Phi+v_A)-g(\Phi+v_B)\big] \quad &= \quad \big[Dg(\Phi+v_A)-Dg(\Phi+v_B)\big]\big(\Phi'+v_A'\big) \\ &\qquad + Dg(\Phi+v_B)\big(v_A'-v_B'\big), \end{aligned} \tag{3.22}$$

we obtain

$$\begin{aligned} \big\|\partial_\xi\big[g(\Phi+v_A)-g(\Phi+v_B)\big]\big\|_{L^2} \quad &\le \quad K_g\,\|v_A-v_B\|_\infty\big[\,\|\Phi'\|_{L^2}+\|v_A'\|_{L^2}\,\big] \\ &\qquad + K_g\,\|v_A'-v_B'\|_{L^2}. \end{aligned} \tag{3.23}$$

The second estimate in (3.16) now follows easily. □

Lemma 3.3. *Suppose that (Hg) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bounds*

$$\begin{array}{lcl} |\langle g(\Phi + v), \psi \rangle_{L^2}| & \leq & K[1 + \|v\|_{L^2}] \, \|\psi\|_{L^2} \, , \\ |\langle \partial_\xi g(\Phi + v), \psi \rangle_{L^2}| & \leq & K[1 + \|v\|_{L^2}] \, \|\psi\|_{H^1} \, , \end{array} \tag{3.24}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$ we have the estimates

$$\begin{array}{lcl} |\langle g(\Phi + v_A), \psi_A \rangle_{L^2} - \langle g(\Phi + v_B), \psi_B \rangle_{L^2}| & \leq & K \, \|v_A - v_B\|_{L^2} \, \|\psi_A\|_{L^2} \\ & & +K[1 + \|v_B\|_{L^2}] \, \|\psi_A - \psi_B\|_{L^2} \, , \\ |\langle \partial_\xi [g(\Phi + v_A)], \psi_A \rangle_{L^2} - \langle \partial_\xi [g(\Phi + v_B)], \psi_B \rangle_{L^2}| & \leq & K \, \|v_A - v_B\|_{L^2} \, \|\psi_A\|_{H^1} \\ & & +K[1 + \|v_B\|_{L^2}] \, \|\psi_A - \psi_B\|_{H^1} \, . \end{array} \tag{3.25}$$

Proof. The estimates (3.24) follow immediately from the bound $\|g(\Phi + v)\|_{L^2} \leq K\big[1 + \|v\|_{L^2}\big]$. The first bound in (3.25) can be obtained from Lemma 3.2 by noting that

$$\begin{array}{lcl} |\langle g(\Phi + v_A), \psi_A \rangle_{L^2} - \langle g(\Phi + v_B), \psi_B \rangle_{L^2}| & \leq & |\langle g(\Phi + v_A) - g(\Phi + v_B), \psi_A \rangle_{L^2}| \\ & & + |\langle g(\Phi + v_B), \psi_A - \psi_B \rangle_{L^2}| \, . \end{array} \tag{3.26}$$

The final bound can be obtained by transferring the derivative to the functions ψ_A and ψ_B. □

3.2 Bounds on $\mathcal{J}_0$, b and κ_σ

We are now ready to obtain global and Lipschitz bounds on the functions $\mathcal{J}_0$, b and κ_σ. In addition, we show that it suffices to impose an a priori bound on $\|v\|_{L^2}$ in order to avoid hitting the cut-offs in the definition of b. This is crucial for the estimates in §9, where we have uniform control on $\|v\|_{L^2}$, but only an integrated form of control on $\|v\|_{H^1}$.

Lemma 3.4. *Suppose that (Hf) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $(v, d) \in H^1 \times \mathbb{R}$ and $\psi \in H^1$ we have the bounds*

$$\begin{array}{lcl} \|\mathcal{J}_0(\Phi + v, c + d)\|_{L^2} & \leq & K(1 + |d|)\big[1 + \|v\|_{H^1} + \|v\|_{H^1}^2 \, \|v\|_{L^2}\big], \\ |\langle \mathcal{J}_0(\Phi + v, c + d), \psi \rangle_{L^2}| & \leq & K(1 + |d|)\big[1 + \|v\|_{H^1} \, \|v\|_{L^2}^2\big] \, \|\psi\|_{H^1} \, . \end{array} \tag{3.27}$$

In addition, for any set of pairs $(v_A, v_B) \in H^1 \times H^1$, $(d_A, d_B) \in \mathbb{R} \times \mathbb{R}$ and $(\psi_A, \psi_B) \in H^1 \times H^1$, the expressions

$$\begin{array}{lcl} \Delta_{AB} \mathcal{J}_0 & = & \mathcal{J}_0(\Phi + v_A, c + d_A) - \mathcal{J}_0(\Phi + v_B, c + d_B), \\ \Delta_{AB} \langle \mathcal{J}_0, \cdot \rangle_{L^2} & = & \langle \mathcal{J}_0(\Phi + v_A, c + d_A), \psi_A \rangle_{L^2} - \langle \mathcal{J}_0(\Phi + v_B, c + d_B), \psi_B \rangle_{L^2} \end{array} \tag{3.28}$$

satisfy the estimates

$$\begin{array}{lcl} \|\Delta_{AB} \mathcal{J}_0\|_{L^2} & \leq & K[\|v_A\|_{H^1} \, \|v_A\|_{L^2} + \|v_B\|_{H^1} \, \|v_B\|_{L^2}] \, \|v_A - v_B\|_{H^1} \\ & & +[1 + \|v_A\|_{H^1}] \, |d_A - d_B| \\ & & +K(1 + |d_B|) \, \|v_A - v_B\|_{H^1} \, , \\ |\Delta_{AB} \langle \mathcal{J}_0, \cdot \rangle_{L^2}| & \leq & K[\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2] \, \|v_A - v_B\|_{H^1} \, \|\psi_A\|_{H^1} \\ & & +[1 + \|v_A\|_{L^2}] \, |d_A - d_B| \, \|\psi_A\|_{H^1} \\ & & +K(1 + |d_B|) \, \|v_A - v_B\|_{L^2} \, \|\psi_A\|_{H^1} \\ & & +K(1 + |d_B|)\big[1 + \|v_B\|_{H^1} \, \|v_B\|_{L^2}^2\big] \, \|\psi_A - \psi_B\|_{H^1} \, . \end{array} \tag{3.29}$$

Proof. We first note that the terms in (3.3)-(3.5) can be absorbed in (3.27)-(3.29), so it suffices to study the function

$$\mathcal{J}_{0;II}(u, \overline{c}) = \overline{c}u'. \tag{3.30}$$

Recalling that (hPar) implies

$$|c| + \|\Phi'\|_{L^2} \leq C_1, \tag{3.31}$$

we find

$$\|\mathcal{J}_{0;II}(\Phi + v, c + d)\|_{L^2} \leq C_2(1 + |d|)(1 + \|v\|_{H^1}), \tag{3.32}$$

together with

$$|\langle \mathcal{J}_{0;II}(\Phi + v, c + d), \psi \rangle_{L^2}| \leq C_2(1 + |d|)(1 + \|v\|_{L^2}) \|\psi\|_{H^1}, \tag{3.33}$$

which can be absorbed in (3.27).

In addition, writing

$$\begin{array}{lcl} \Delta_{AB}\mathcal{J}_{0;II} & = & \mathcal{J}_{0;II}(\Phi + v_A, c + d_A) - \mathcal{J}_{0;II}(\Phi + v_B, c + d_B), \\ \Delta_{AB}\langle \mathcal{J}_{0;II}, \cdot \rangle_{L^2} & = & \langle \mathcal{J}_{0;II}(\Phi + v_A, c + d_A), \psi_A \rangle_{L^2} - \langle \mathcal{J}_{0;II}(\Phi + v_B, c + d_B), \psi_B \rangle_{L^2}, \end{array} \tag{3.34}$$

we compute

$$\Delta_{AB}\mathcal{J}_{0;II} = (d_A - d_B)(\Phi' + v_A') + (c + d_B)(v_A' - v_B'). \tag{3.35}$$

This yields

$$\|\Delta_{AB}\mathcal{J}_{0;II}\|_{L^2} \leq C_3 |d_A - d_B| (1 + \|v_A\|_{H^1}) + C_3(1 + |d_B|) \|v_A - v_B\|_{H^1}, \tag{3.36}$$

which establishes the first estimate in (3.29).

In a similar fashion, we obtain

$$|\langle \Delta_{AB}\mathcal{J}_{0;II}, \psi \rangle_{L^2}| \leq C_3 |d_A - d_B| (1 + \|v_A\|_{L^2}) \|\psi\|_{H^1} + C_3(1 + |d_B|) \|v_A - v_B\|_{L^2} \|\psi\|_{H^1}. \tag{3.37}$$

The remaining estimate now follows from the inequality

$$\begin{array}{lcl} |\Delta_{AB}\langle \mathcal{J}_{0;II}, \cdot \rangle_{L^2}| & \leq & |\langle \Delta_{AB}\mathcal{J}_{0;II}, \psi_A \rangle_{L^2}| \\ & & \quad + |\langle \mathcal{J}_{0;II}(\Phi + v_B, c + d_B), \psi_A - \psi_B \rangle_{L^2}|. \end{array} \tag{3.38}$$

□

Lemma 3.5. *Assume that (hPar) is satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bound*

$$|\langle \partial_\xi(\Phi + v), \psi \rangle_{L^2}| \leq K\big[1 + \|v\|_{L^2}\big] \|\psi\|_{H^1}, \tag{3.39}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$ we have the estimate

$$\begin{array}{lcl} |\langle \partial_\xi[\Phi + v_A], \psi_A \rangle_{L^2} - \langle \partial_\xi[\Phi + v_B], \psi_B \rangle_{L^2}| & \leq & K \|v_A - v_B\|_{L^2} \|\psi_A\|_{H^1} \\ & & \quad + K\big[1 + \|v_B\|_{L^2}\big] \|\psi_A - \psi_B\|_{H^1}. \end{array} \tag{3.40}$$

Proof. The desired bounds follow from the identity

$$|\langle \partial_\xi(\Phi + v), \psi \rangle_{L^2}| = |\langle \Phi + v, \partial_\xi \psi \rangle_{L^2}|, \tag{3.41}$$

together with the estimate

$$\begin{array}{lcl} |\langle \partial_\xi[\Phi + v_A], \psi_A \rangle_{L^2} - \langle \partial_\xi[\Phi + v_B], \psi_B \rangle_{L^2}| & \leq & |\langle v_A - v_B, \partial_\xi \psi_A \rangle_{L^2}| + |\langle \partial_\xi \Phi, \psi_A - \psi_B \rangle_{L^2}| \\ & & \quad + |\langle v_B, \partial_\xi[\psi_A - \psi_B] \rangle_{L^2}|. \end{array} \tag{3.42}$$

□

Lemma 3.6. *Suppose that (Hg) and (hPar) are satisfied. Then there exists constants $K_b > 0$ and $K > 0$, which do not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bound*

$$|b(\Phi + v, \psi)| \quad \leq \quad K_b, \tag{3.43}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$ we have the estimate

$$\begin{aligned} |b(\Phi + v_A, \psi_A) - b(\Phi + v_B, \psi_B)| \quad \leq \quad & K \|v_A - v_B\|_{L^2} \|\psi_A\|_{H^1} \\ & + K\big[1 + \|v_B\|_{L^2}\big] \|\psi_A - \psi_B\|_{H^1}. \end{aligned} \tag{3.44}$$

Proof. The uniform bound (3.43) follows directly from the properties of the cut-off functions. Upon introducing the function

$$\tilde{b}(x, y) = -\chi_{\mathrm{low}}(x)^{-1}\chi_{\mathrm{high}}(y), \tag{3.45}$$

the global Lipschitz smoothness of the cut-off functions implies that

$$\left|\tilde{b}(x_A, y_A) - \tilde{b}(x_B, y_B)\right| \leq C_1\big[\,|x_B - x_A| + |y_B - y_A|\,\big]. \tag{3.46}$$

Using the identity

$$b(u, \psi) = \tilde{b}\Big(\langle \partial_\xi u, \psi\rangle_{L^2}, \langle g(u), \psi\rangle_{L^2}\Big), \tag{3.47}$$

the desired bound (3.44) follows from Lemma's 3.3 and 3.5. □

Lemma 3.7. *Assume that (Hg) and (hPar) are satisfied. Then for any $v \in H^1$ that has*

$$\|v\|_{L^2} \leq \min\{1, [4\|\psi_{\mathrm{tw}}\|_{H^1}]^{-1}\}, \tag{3.48}$$

we have the identity

$$b(\Phi + v, \psi_{\mathrm{tw}}) = -\big[\langle \partial_\xi[\Phi + v], \psi_{\mathrm{tw}}\rangle_{L^2}\big]^{-1}\langle g(\Phi + v), \psi_{\mathrm{tw}}\rangle_{L^2}. \tag{3.49}$$

Proof. Using (3.15) and recalling the definition (2.23), we find that

$$|\langle g(\Phi + v), \psi_{\mathrm{tw}}\rangle_{L^2}| \leq \Big[\|g(\Phi_0)\|_{L^2} + 2K_g\Big] \|\psi_{\mathrm{tw}}\|_{L^2} = K_{\mathrm{ip}}. \tag{3.50}$$

In addition, we note that (hPar) and the normalisation (2.14) imply that

$$\langle \partial_\xi \Phi, \psi_{\mathrm{tw}}\rangle_{L^2} = \langle \partial_\xi \Phi_0, \psi_{\mathrm{tw}}\rangle_{L^2} + \langle \partial_\xi[\Phi - \Phi_0], \psi_{\mathrm{tw}}\rangle_{L^2} \geq 1 - \frac{1}{4} = \frac{3}{4}. \tag{3.51}$$

This allows us to estimate

$$\langle \partial_\xi(\Phi + v), \psi_{\mathrm{tw}}\rangle_{L^2} \geq \frac{3}{4} - \langle v, \partial_\xi \psi_{\mathrm{tw}}\rangle_{L^2} \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \tag{3.52}$$

which shows that the cut-off functions do not modify their arguments. □

Lemma 3.8. *Suppose that (Hg) and $(hPar)$ are satisfied. Then there exists a constant $K_\kappa > 0$, which does not depend on the pair (Φ, c), so that for any $0 \leq \sigma \leq 1$, any $v \in H^1$ and any $\psi \in H^1$, we have the bound*

$$|\kappa_\sigma(\Phi + v, \psi)| + \left|\kappa_\sigma(\Phi + v, \psi)^{-1}\right| + \left|\kappa_\sigma(\Phi + v, \psi)^{-1/2}\right| \leq K_\kappa. \tag{3.53}$$

Proof. This follows directly from the bound

$$1 \leq \kappa_\sigma(\Phi + v, \psi) \leq 1 + \frac{1}{2\rho}\sigma^2 K_b^2. \tag{3.54}$$

□

In order to state our final result, we introduce the functions

$$\begin{array}{rcl} \nu_\sigma^{(1)}(u,\psi) & = & \kappa_\sigma(u,\psi) - 1, \\ \nu_\sigma^{(-1)}(u,\psi) & = & \kappa_\sigma(u,\psi)^{-1} - 1, \\ \nu_\sigma^{(-1/2)}(u,\psi) & = & \kappa_\sigma(u,\psi)^{-1/2} - 1, \end{array} \tag{3.55}$$

which isolate the σ-dependence in κ_σ.

Lemma 3.9. *Suppose that* (Hg) *and* $(hPar)$ *are satisfied and pick* $\vartheta \in \{-1, -\frac{1}{2}, 1\}$*. Then there exist constants* $K_\nu > 0$ *and* $K > 0$*, which do not depend on the pair* (Φ, c)*, so that the following holds true. For any* $0 \le \sigma \le 1$*, any* $v \in H^1$ *and any* $\psi \in H^1$ *we have the bound*

$$\left|\nu_\sigma^{(\vartheta)}(\Phi + v, \psi)\right| \le \sigma^2 K_\nu, \tag{3.56}$$

while for any $0 \le \sigma \le 1$ *and any set of pairs* $(v_A, v_B) \in H^1 \times H^1$ *and* $(\psi_A, \psi_B) \in H^1 \times H^1$*, we have the estimate*

$$\begin{array}{rcl} \left|\nu_\sigma^{(\vartheta)}(\Phi + v_A, \psi_A) - \nu_\sigma^{(\vartheta)}(\Phi + v_B, \psi_B)\right| & \le & K\sigma^2 \left\|v_A - v_B\right\|_{L^2} \left\|\psi_A\right\|_{H^1} \\ & & +K\sigma^2 \big[1 + \left\|v_B\right\|_{L^2}\big] \left\|\psi_A - \psi_B\right\|_{H^1}. \end{array} \tag{3.57}$$

Proof. As a preparation, we observe that for any $x \ge 0$ and $y \ge 0$ we have the inequality

$$\left|\frac{1}{1+x} - \frac{1}{1+y}\right| = \frac{|y-x|}{(1+x)(1+y)} \le |y-x|, \tag{3.58}$$

together with

$$\left|\frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1+y}}\right| = \frac{|y-x|}{\sqrt{(1+x)(1+y)}(\sqrt{1+x}+\sqrt{1+y})} \le \frac{1}{2}|y-x|. \tag{3.59}$$

Applying these bounds with $y = 0$, we obtain

$$\left|\nu_\sigma^{(\vartheta)}(\Phi + v, \psi)\right| \le \frac{1}{2\rho}\sigma^2 \left|b(\Phi + v, \psi)\right|^2 \le \frac{1}{2\rho}\sigma^2 K_b^2, \tag{3.60}$$

which yields (3.56). In addition, we may compute

$$\begin{array}{rcl} \left|\nu_\sigma^{(\vartheta)}(\Phi + v_A, \psi_A) - \nu_\sigma^{(\vartheta)}(\Phi + v_B, \psi_B)\right| & \le & \frac{1}{2\rho}\sigma^2 \left|b(\Phi + v_A, \psi_A)^2 - b(\Phi + v_B, \psi_B)^2\right| \\ & = & \frac{1}{2\rho}\sigma^2 \left|b(\Phi + v_A, \psi_A) + b(\Phi + v_B, \psi_B)\right| \\ & & \times \left|b(\Phi + v_A, \psi_A) - b(\Phi + v_B, \psi_B)\right|. \end{array} \tag{3.61}$$

In particular, the bounds (3.57) follow from Lemma 3.6. □

4 Variational solution

In this section we set out to establish Proposition 2.1. Our strategy is to fit the first component of (2.31) into the framework of [36]. Indeed, the conditions (H1)-(H4) in this paper are explicitly verified in Lemma 4.1 below. The second line of (2.31) can subsequently be treated as an SDE for Γ with random coefficients. In Lemma 4.3 below we show that this SDE fits into the framework that was developed in [42, Chapter 3] to handle such equations.

Lemma 4.1. *Suppose that (HA), (Hf), (HTw), (HS), (HVar) and (Hg) are all satisfied. Then there exist constants $K > 0$ and $\vartheta > 0$ so that the following properties hold true.*

(i) For any triplet $(v_A, v_B, v) \in H^1 \times H^1 \times H^1$, the map

$$s \mapsto \langle A_*[v_A + sv_B], v\rangle_{H^{-1};H^1} + \langle f(\Phi_{\mathrm{ref}} + v_A + sv_B), v\rangle_{L^2} \tag{4.1}$$

is continuous.

(ii) For every pair $(v_A, v_B) \in H^1 \times H^1$, we have the inequality

$$\begin{aligned} K\left\|v_A - v_B\right\|_{L^2}^2 \quad \geq \quad & 2\langle A_*(v_A - v_B), v_A - v_B\rangle_{H^{-1};H^1} \\ & +2\langle f(\Phi_{\mathrm{ref}} + v_A) - f(\Phi_{\mathrm{ref}} + v_B), v_A - v_B\rangle_{L^2} \\ & + \left\|g(\Phi_{\mathrm{ref}} + v_A) - g(\Phi_{\mathrm{ref}} + v_B)\right\|_{L^2}^2 . \end{aligned} \tag{4.2}$$

(iii) For any $v \in H^1$ we have the inequality

$$2\langle A_* v, v\rangle_{H^{-1};H^1} + 2\langle f(\Phi_{\mathrm{ref}} + v), v\rangle_{L^2} + \left\|g(\Phi_{\mathrm{ref}} + v)\right\|_{L^2}^2 + \vartheta\left\|v\right\|_{H^1}^2 \leq K\big[1 + \left\|v\right\|_{L^2}^2\big]. \tag{4.3}$$

(iv) For any $v \in H^1$ we have the bound

$$\left\|A_* v\right\|_{H^{-1}}^2 + \left\|f(\Phi_{\mathrm{ref}} + v)\right\|_{H^{-1}}^2 \leq K\big[1 + \left\|v\right\|_{H^1}^2\big]\big[1 + \left\|v\right\|_{L^2}^4\big]. \tag{4.4}$$

Proof. Item (i) follows from the linearity of A_* and the Lipschitz bound (3.5). In addition, writing

$$\mathcal{I} = \langle f(\Phi_{\mathrm{ref}} + v_A) - f(\Phi_{\mathrm{ref}} + v_B), v_A - v_B\rangle_{L^2}, \tag{4.5}$$

(HVar) implies the one-sided inequality

$$\begin{aligned} \mathcal{I} \quad & = \quad \langle f(\Phi_{\mathrm{ref}} + v_A) - f(\Phi_{\mathrm{ref}} + v_B), \Phi_{\mathrm{ref}} + v_A - (\Phi_{\mathrm{ref}} + v_B)\rangle_{L^2} \\ & \leq \quad C_1\left\|v_A - v_B\right\|_{L^2}^2 . \end{aligned} \tag{4.6}$$

Item (ii) hence follows from the Lipschitz bound (3.16) together with the bound

$$\langle A_* v, v\rangle_{H^{-1},H^1} \leq -\rho\left\|v\right\|_{H^1}^2 . \tag{4.7}$$

A second consequence of (HVar) is that

$$\begin{aligned} \langle f(\Phi_{\mathrm{ref}} + v), v\rangle_{L^2} \quad & = \quad \langle f(\Phi_{\mathrm{ref}} + v) - f(\Phi_{\mathrm{ref}}), (\Phi_{\mathrm{ref}} + v) - \Phi_{\mathrm{ref}}\rangle_{L^2} \\ & \qquad +\langle f(\Phi_{\mathrm{ref}}), v\rangle_{L^2} \\ & \leq \quad C_1\left\|v\right\|_{L^2}^2 + \left\|f(\Phi_{\mathrm{ref}})\right\|_{L^2}\left\|v\right\|_{L^2} \\ & \leq \quad C_2\big[1 + \left\|v\right\|_{L^2}^2\big]. \end{aligned} \tag{4.8}$$

In particular, we may obtain (iii) by combining (4.7) with (3.15).

Finally, for any $v \in H^1$ and $\psi \in H^1$ we may use (3.3) to compute

$$\begin{aligned} \langle f(\Phi_{\mathrm{ref}} + v), \psi\rangle_{H^{-1};H^1} \quad & = \quad \langle f(\Phi_{\mathrm{ref}} + v), \psi\rangle_{L^2} \\ & \leq \quad C_3\Big[1 + \left\|v\right\|_{H^1}\left\|v\right\|_{L^2}^2\Big]\left\|\psi\right\|_{H^1} . \end{aligned} \tag{4.9}$$

In other words, we see that

$$\left\|f(\Phi_{\mathrm{ref}} + v)\right\|_{H^{-1}} \leq C_3\big[1 + \left\|v\right\|_{H^1}\left\|v\right\|_{L^2}^2\big] \leq C_3(1 + \left\|v\right\|_{H^1})(1 + \left\|v\right\|_{L^2}^2), \tag{4.10}$$

which yields (iv). □

Lemma 4.2. *Suppose that (HA), (Hf), (Hg) and (hPar) are all satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following properties hold true for any $0 \le \sigma \le 1$.*

(i) For any $v \in H^1$ and any $\psi \in H^2$ with $\|\psi\|_{H^2} \le 2 \|\psi_{\rm tw}\|_{H^2}$, we have the bound

$$|a_\sigma(\Phi + v, c, \psi)| \le K\Big[1 + \|v\|_{H^1} \|v\|_{L^2}^2\Big]. \tag{4.11}$$

(ii) For any $v \in H^1$ and any pair $(\psi_A, \psi_B) \in H^2 \times H^2$ for which $\|\psi_A\|_{H^2} \le 2 \|\psi_{\rm tw}\|_{H^2}$ and $\|\psi_B\|_{H^2} \le 2 \|\psi_{\rm tw}\|_{H^2}$, the difference

$$\Delta_{AB} a_\sigma = a_\sigma(\Phi + v, c, \psi_A) - a_\sigma(\Phi + v, c, \psi_B) \tag{4.12}$$

satisfies the bound

$$|\Delta_{AB} a_\sigma| \le K\Big[1 + \|v\|_{H^1} (1 + \|v\|_{L^2}^3)\Big] \|\psi_A - \psi_B\|_{H^1}. \tag{4.13}$$

Proof. We first compute

$$\begin{aligned} \kappa_\sigma(u,\psi)\langle \mathcal{J}_\sigma(u,c,\psi), \psi\rangle_{L^2} &= \langle f(u) + cu' + \sigma^2 b(u,\psi)\partial_\xi[g(u)], \psi\rangle_{L^2} \\ &= \langle \mathcal{J}_0(u,c), \psi\rangle_{L^2} + \sigma^2 b(u,\psi)\langle \partial_\xi[g(u)], \psi\rangle_{L^2}. \end{aligned} \tag{4.14}$$

Upon defining

$$\begin{aligned} \mathcal{E}_I(u,c,\psi) &= \langle \mathcal{J}_0(u,c), \psi\rangle_{L^2}, \\ \mathcal{E}_{II}(u,\psi) &= \sigma^2 b(u,\psi)\langle \partial_\xi g(u), \psi\rangle_{L^2}, \\ \mathcal{E}_{III}(u,\psi) &= \kappa_\sigma(u,\psi)\langle u, A_* \psi\rangle_{L^2}, \end{aligned} \tag{4.15}$$

we hence see that

$$a_\sigma(u,c,\psi) = -\Big[\chi_{\rm low}\big(\langle \partial_\xi u, \psi\rangle_{L^2}\big)\Big]^{-1}\big[\mathcal{E}_I(u,c,\psi) + \mathcal{E}_{II}(u,\psi) + \mathcal{E}_{III}(u,\psi)\big]. \tag{4.16}$$

For $\# \in \{I, II, III\}$, we define

$$\Delta_{AB}\mathcal{E}_\# = \mathcal{E}_\#(\Phi + v, c, \psi_A) - \mathcal{E}_\#(\Phi + v, c, \psi_B). \tag{4.17}$$

We note that Lemma's 3.3, 3.4 and 3.6 yield the bounds

$$\begin{aligned} |\mathcal{E}_I(\Phi + v, c, \psi)| &\le C_1\big[1 + \|v\|_{H^1} \|v\|_{L^2}^2\big], \\ |\mathcal{E}_{II}(\Phi + v, \psi)| &\le C_1\big[1 + \|v\|_{L^2}\big], \end{aligned} \tag{4.18}$$

together with

$$\begin{aligned} |\Delta_{AB}\mathcal{E}_I| &\le C_1\big[1 + \|v\|_{H^1} \|v\|_{L^2}^2\big] \|\psi_A - \psi_B\|_{H^1}, \\ |\Delta_{AB}\mathcal{E}_{II}| &\le C_1[1 + \|v\|_{L^2}]^2 \|\psi_A - \psi_B\|_{H^1} \\ &\qquad + C_1[1 + \|v\|_{L^2}] \|\psi_A - \psi_B\|_{H^1}. \end{aligned} \tag{4.19}$$

A direct estimate using the a-priori bound on $\|\psi\|_{H^2}$ and (3.53) yields

$$\begin{aligned} |\mathcal{E}_{III}(\Phi + v, \psi)| &\le K_\kappa\big[\,|\langle \Phi, A_*\psi\rangle_{L^2}| + |\langle v, A_*\psi\rangle_{L^2}|\,\big] \\ &\le C_2\big[1 + \|v\|_{L^2}\big]. \end{aligned} \tag{4.20}$$

By transferring one of the derivatives in A_*, we also obtain using Lemma 3.9 the bound

$$\begin{array}{rcl} |\Delta \mathcal{E}_{III}| & \leq & |\kappa_\sigma(\Phi + v, \psi_A) - \kappa_\sigma(\Phi + v, \psi_B)| \, |\langle \Phi + v, A_* \psi_A \rangle_{L^2}| \\ & & \quad + |\kappa_\sigma(\Phi + v, \psi_B)| \, |\langle \Phi + v, A_*[\psi_A - \psi_B] \rangle_{L^2}| \\ & \leq & C_3 (1 + \|v\|_{L^2})^2 \, \|\psi_A - \psi_B\|_{H^1} \\ & & \quad + C_3 \big[1 + \|v\|_{H^1}\big] \, \|\psi_A - \psi_B\|_{H^1} . \end{array} \tag{4.21}$$

Upon writing

$$\begin{array}{rcl} \mathcal{E}(u, c, \psi) & = & \mathcal{E}_I(u, c, \psi) + \mathcal{E}_{II}(u, \psi) + \mathcal{E}_{II}(u, \psi), \\ \Delta_{AB} \mathcal{E} & = & \Delta_{AB} \mathcal{E}_I + \Delta_{AB} \mathcal{E}_{II} + \Delta_{AB} \mathcal{E}_{III}, \end{array} \tag{4.22}$$

we hence conclude that

$$\begin{array}{rcl} |\mathcal{E}(\Phi + v, c, \psi)| & \leq & C_4 \big[1 + \|v\|_{H^1} \, \|v\|_{L^2}^2 \big], \\ |\Delta_{AB} \mathcal{E}| & \leq & C_4 \big[1 + \|v\|_{H^1}\big] \big[1 + \|v\|_{L^2}^2\big] \, \|\psi_A - \psi_B\|_{H^1} . \end{array} \tag{4.23}$$

Item (i) follows immediately from the first bound, since $\chi_{\rm low}(\cdot)^{-1}$ is globally bounded. To obtain (ii), we compute

$$\begin{array}{rcl} |\Delta_{AB} a_\sigma| & \leq & C_5 \, |\langle \partial_\xi(\Phi + v), \psi_A \rangle_{L^2} - \langle \partial_\xi(\Phi + v), \psi_B \rangle_{L^2}| \, |\mathcal{E}(\Phi + v, \psi_A)| \\ & & \quad + C_5 \, |\Delta_{AB} \mathcal{E}| \\ & \leq & C_6 [1 + \|v\|_{L^2}] \big[1 + \|v\|_{H^1} \, \|v\|_{L^2}^2\big] \, \|\psi_A - \psi_B\|_{H^1} \\ & & \quad + C_6 \big[1 + \|v\|_{H^1}\big] \big[1 + \|v\|_{L^2}^2\big] \, \|\psi_A - \psi_B\|_{H^1} \\ & \leq & C_7 \Big[1 + \|v\|_{H^1} \, (1 + \|v\|_{L^2}^3)\Big] \, \|\psi_A - \psi_B\|_{H^1} , \end{array} \tag{4.24}$$

in which we used several estimates of the form

$$\|v\|_{L^2} \leq C_8 \big[1 + \|v\|_{L^2}^4\big] \leq C_8 \big[1 + \|v\|_{H^1} \, \|v\|_{L^2}^3\big]. \tag{4.25}$$

□

Upon introducing the shorthands

$$\begin{array}{rcl} p(v, \gamma) & = & c + a_\sigma(\Phi_{\rm ref} + v, c, T_\gamma \psi_{\rm tw}), \\ q(v, \gamma) & = & b(\Phi_{\rm ref} + v, T_\gamma \psi_{\rm tw}), \end{array} \tag{4.26}$$

the second line of (2.31) can be written as

$$d\Gamma = p\big(X(t), \Gamma(t)\big) dt + \sigma q\big(X(t), \Gamma(t)\big) d\beta_t. \tag{4.27}$$

Taking the view-point that $X(t) = X(t, \omega)$ is known upon picking $\omega \in \Omega$, (4.27) can be viewed as an SDE with random coefficients. Our next result relates directly to the conditions of [42, Thm 3.1.1], which is specially tailored for equations of this type.

Lemma 4.3. *Suppose that (HA), (Hf), (HTw), (HS) and (Hg) are all satisfied and fix $c \in \mathbb{R}$ together with $0 \leq \sigma \leq 1$. Then there exists $K > 0$ so that the following properties are satisfied.*

(i) For any $v \in H^1$ and any pair $(\gamma_A, \gamma_B) \in \mathbb{R}^2$, we have the inequality

$$\begin{array}{rcl} K \big[1 + \|v\|_{H^1} \, (1 + \|v\|_{L^2}^3)\big] \, |\gamma_A - \gamma_B|^2 & \geq & 2 \big[\gamma_A - \gamma_B\big] \big[p(v, \gamma_A) - p(v, \gamma_B)\big] \\ & & \quad + |q(v, \gamma_A) - q(v, \gamma_B)|^2 . \end{array} \tag{4.28}$$

(ii) For any $v \in H^1$ and $\gamma \in \mathbb{R}$, we have the inequality

$$2\gamma p(v,\gamma) + |q(v,\gamma)|^2 \le K\big[1 + \|v\|_{H^1}\,\|v\|_{L^2}^2\,\big]\big[1+\gamma^2\big]. \tag{4.29}$$

(iii) For any $v \in H^1$ and $\gamma \in \mathbb{R}$, we have the bound

$$|p(v,\gamma)| + |q(v,\gamma)|^2 \le K\big[1 + \|v\|_{H^1}\,\|v\|_{L^2}^2\,\big]. \tag{4.30}$$

Proof. The exponential decay of ψ'_{tw} and ψ''_{tw} implies that

$$\|T_{\gamma_A}\psi_{\mathrm{tw}} - T_{\gamma_B}\psi_{\mathrm{tw}}\|_{H^1} \le C_1\,|\gamma_A - \gamma_B|\,. \tag{4.31}$$

Using Lemma's 3.6 and 4.2, we hence find the bounds

$$\begin{array}{rcl} |p(v,\gamma)| & \le & C_2\big[1 + \|v\|_{H^1}\,\|v\|_{L^2}^2\,\big], \\ |q(v,\gamma)| & \le & K_b, \end{array} \tag{4.32}$$

together with

$$\begin{array}{rcl} |p(v,\gamma_A) - p(v,\gamma_B)| & \le & C_3\big[1 + \|v\|_{H^1}\,(1+\|v\|_{L^2}^3)\big]\,|\gamma_A - \gamma_B|\,, \\ |q(v,\psi_A) - q(v,\psi_B)| & \le & C_3[1 + \|v\|_{L^2}]\,|\gamma_A - \gamma_B|\,. \end{array} \tag{4.33}$$

Items (i), (ii) and (iii) can now be verified directly. □

Proof of Proposition 2.1. The existence of the $dt \otimes \mathbb{P}$ version of X that is $(\mathcal{F}_t)$-progressively measurable as a map into H^1, follows from [42, Ex. 4.2.3].

We remark that the conditions (H1) through (H4) appearing in [36] correspond directly with items (i)-(iv) of Lemma 4.1. In particular, we may apply the main result from this paper with $\alpha = 2$ and $\beta = 4$ to verify the remaining statements concerning X.

Finally, we note that items (i)-(iii) of Lemma 4.3 allow us to apply [42, Thm. 3.1.1], provided that the function

$$t \mapsto \big[1 + \|X(t)\|_{H^1}\,(1 + \|X(t)\|_{L^2}^3)\big] \tag{4.34}$$

is integrable on $[0,T]$ for almost all $\omega \in \Omega$. This however follows directly from the inclusions

$$X \in L^6\big(\Omega,\mathbb{P}; C([0,T];L^2)\big) \cap \mathcal{N}^2\big([0,T];(\mathcal{F}_t);H^1\big), \tag{4.35}$$

allowing us to verify the statements concerning Γ. The remaining inclusions (2.41) follow directly from the bounds in Lemma 3.2 and 3.6. □

5 The stochastic phase-shift

In this section we consider the process (X,Γ) described in Proposition 2.1 and define the new process

$$V(t) = T_{-\Gamma(t)}[X(t) + \Phi_{\mathrm{ref}}] - \Phi \tag{5.1}$$

for some $\Phi \in \mathcal{U}_{H^1}$. In addition, we introduce the nonlinearity

$$\begin{array}{rcl} \mathcal{R}_{\sigma;\Phi,c}(v) & = & \kappa_\sigma(\Phi+v,\psi_{\mathrm{tw}})A_*[\Phi+v] \\ & & \quad + f(\Phi+v) + \sigma^2 b(\Phi+v,\psi_{\mathrm{tw}})\partial_\xi[g(\Phi+v)] \\ & & \quad + \Big[c + a_\sigma\big(\Phi+v,c,\psi_{\mathrm{tw}}\big)\Big][\Phi'+v'] \\ & = & \kappa_\sigma(\Phi+v,\psi_{\mathrm{tw}})\Big[A_*[\Phi+v] + \mathcal{J}_\sigma(\Phi+v,c,\psi_{\mathrm{tw}})\Big] + a_\sigma\big(\Phi+v,c,\psi_{\mathrm{tw}}\big)[\Phi'+v'], \end{array} \tag{5.2}$$

together with

$$\mathcal{S}_{\Phi}(v) \quad = \quad g(\Phi + v) + b(\Phi + v, \psi_{\mathrm{tw}})[\Phi' + v']. \tag{5.3}$$

Our main result states that the shifted process V can be interpreted as a weak solution to the SPDE

$$dV \quad = \quad \mathcal{R}_{\sigma;\Phi,c}(V)\, dt + \sigma \mathcal{S}_{\Phi}(V) d\beta_t. \tag{5.4}$$

Proposition 5.1. *Consider the setting of Proposition 2.1 and suppose that (hPar) is satisfied. Then the map*

$$V : [0,T] \times \Omega \to L^2 \tag{5.5}$$

defined by (5.1) satisfies the following properties.

(i) For almost all $\omega \in \Omega$, the map $t \mapsto V(t,\omega)$ is of class $C([0,T]; L^2)$.

(ii) For all $t \in [0,T]$, the map $\omega \mapsto V(t,\omega) \in L^2$ is $(\mathcal{F}_t)$-measurable.

(iii) We have the inclusion

$$V \quad \in \quad \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); H^1\big) \tag{5.6}$$

together with

$$\mathcal{S}_{\sigma;\Phi}(V) \quad \in \quad \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); L^2\big). \tag{5.7}$$

(iv) For almost all $\omega \in \Omega$, we have the inclusion

$$\mathcal{R}_{\sigma;\Phi,c}\big(V(\cdot,\omega)\big) \in L^1([0,T]; H^{-1}). \tag{5.8}$$

(v) For almost all $\omega \in \Omega$, the identity

$$V(t) \quad = \quad V(0) + \int_0^t \mathcal{R}_{\sigma;\Phi,c}\big(V(s)\big)\, ds + \sigma \int_0^t \mathcal{S}_{\Phi}\big(V(s)\big) d\beta_s \tag{5.9}$$

holds for all $0 \le t \le T$.

Taking derivatives of translation operators typically requires extra regularity of the underlying function, which prevents us from applying an Itô formula directly to (5.1). In order to circumvent this technical issue, we pick a test function $\zeta \in C_c^{\infty}(\mathbb{R};\mathbb{R}^n)$ and consider the two maps

$$\phi_{1;\zeta} : H^{-1} \times \mathbb{R} \to \mathbb{R}, \qquad \phi_{2;\zeta} : \mathbb{R} \to \mathbb{R} \tag{5.10}$$

that act as

$$\begin{aligned} \phi_{1;\zeta}\big(x,\gamma\big) \quad &= \quad \langle x, T_{\gamma}\zeta \rangle_{H^{-1};H^1}, \\ \phi_{2;\zeta}\big(\gamma\big) \quad &= \quad \langle T_{-\gamma}\Phi_{\mathrm{ref}} - \Phi, \zeta \rangle_{H^{-1};H^1} \\ &= \quad \langle T_{-\gamma}\Phi_{\mathrm{ref}} - \Phi, \zeta \rangle_{L^2}. \end{aligned} \tag{5.11}$$

These two maps do have sufficient smoothness for our purposes here.

Lemma 5.2. *Consider the setting of Proposition 2.1. Then for almost all $\omega \in \Omega$ the identity*

$$\begin{aligned} \phi_{1;\zeta}\big(X(t),\Gamma(t)\big) \quad = \quad & \phi_{1;\zeta}\big(X(0),\Gamma(0)\big) \\ & + \int_0^t \langle A_*[X(s) + \Phi_{\mathrm{ref}}] + f(X(s) + \Phi_{\mathrm{ref}}), T_{\Gamma(s)}\zeta \rangle_{H^{-1};H^1}\, ds \\ & - \int_0^t \big[c + a_{\sigma}\big(X(s) + \Phi_{\mathrm{ref}}, c, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big)\big] \langle X(s), T_{\Gamma(s)}\zeta' \rangle_{L^2}\, ds \\ & - \tfrac{1}{2}\sigma^2 \int_0^t 2b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big) \langle g(X(s) + \Phi_{\mathrm{ref}}), T_{\Gamma(s)}\zeta' \rangle_{L^2}\, ds \\ & + \tfrac{1}{2}\sigma^2 \int_0^t b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big)^2 \langle X(s), T_{\Gamma(s)}\zeta'' \rangle_{L^2}\, ds \\ & + \sigma \int_0^t \langle g(X(s) + \Phi_{\mathrm{ref}}), T_{\Gamma(s)}\zeta \rangle_{L^2}\, d\beta_s \\ & - \sigma \int_0^t b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big) \langle X(s), T_{\Gamma(s)}\zeta' \rangle_{L^2}\, d\beta_s \end{aligned} \tag{5.12}$$

holds for all $0 \le t \le T$.

Proof. We note that $\phi_{1;\zeta}$ is C^2-smooth, with derivatives given by

$$D\phi_{1;\zeta}(x,\gamma)[y,\beta] = \langle y, T_\gamma \zeta\rangle_{H^{-1};H^1} - \beta\langle x, T_\gamma\zeta'\rangle_{H^{-1};H^1}, \tag{5.13}$$

together with

$$D^2\phi_{1;\zeta}(x,\gamma)[y,\beta][y,\beta] = -2\beta\langle y, T_\gamma\zeta'\rangle_{H^{-1};H^1} + \beta^2\langle x, T_\gamma\zeta''\rangle_{H^{-1};H^1}. \tag{5.14}$$

Applying a standard Itô formula such as [12, Thm. 1] with $S = I$, the result readily follows. □

Lemma 5.3. *Consider the setting of Proposition 2.1. Then for almost all $\omega \in \Omega$ the identity*

$$\begin{array}{rcl}\phi_{2;\zeta}\big(\Gamma(t)\big) & = & \phi_{2;\zeta}(\Gamma(0)) \\ & & -\int_0^t \big[c + a_\sigma\big(X(s)+\Phi_{\rm ref}, c, T_{\Gamma(s)}\psi_{\rm tw}\big)\big]\langle \Phi_{\rm ref}, T_{\Gamma(s)}\zeta'\rangle_{L^2}\, ds \\ & & +\frac{1}{2}\sigma^2\int_0^t b\big(X(s)+\Phi_{\rm ref}, T_{\Gamma(s)}\psi_{\rm tw}\big)^2\langle\Phi_{\rm ref}, T_{\Gamma(s)}\zeta''\rangle_{L^2}\, ds \\ & & -\sigma\int_0^t b\big(X(s)+\Phi_{\rm ref}, T_{\Gamma(s)}\psi_{\rm tw}\big)\langle\Phi_{\rm ref}, T_{\Gamma(s)}\zeta'\rangle_{L^2}\, d\beta_s\end{array} \tag{5.15}$$

holds for all $0 \le t \le T$.

Proof. We note that $\phi_{2;\zeta}$ is C^2-smooth, with derivatives given by

$$D\phi_{2;\zeta}(\gamma)[\beta] = -\beta\langle\Phi_{\rm ref}, T_\gamma\zeta'\rangle_{L^2}, \tag{5.16}$$

together with

$$D^2\phi_{2;\zeta}(\gamma)[\beta][\beta] = \beta^2\langle\Phi_{\rm ref}, T_\gamma\zeta''\rangle_{L^2}. \tag{5.17}$$

The result again follows from the Itô formula. □

Corollary 5.4. *Consider the setting of Proposition 2.1, suppose that (hPar) is satisfied and pick a test-function $\zeta \in C_c^\infty(\mathbb{R};\mathbb{R}^n)$. Then for almost all $\omega \in \Omega$, the map V defined by (5.1) satisfies the identity*

$$\langle V(t),\zeta\rangle_{L^2} \;=\; \langle V(0),\zeta\rangle_{L^2} + \int_0^t\langle\mathcal{R}_{\sigma;\Phi,c}\big(V(s)\big),\zeta\rangle_{H^{-1};H^1}\, ds + \sigma\int_0^t\langle\mathcal{S}_\Phi\big(V(s)\big),\zeta\rangle_{L^2} d\beta_s \tag{5.18}$$

for all $0 \le t \le T$.

Proof. For any $\gamma \in \mathbb{R}$, we have the identities

$$\alpha_\sigma(u,c,T_\gamma\psi) = \alpha_\sigma(T_{-\gamma}u, c, \psi), \qquad b(u, T_\gamma\psi) = b(T_{-\gamma}u,\psi), \tag{5.19}$$

together with the commutation relations

$$T_\gamma f(u) = f(T_\gamma u), \qquad T_\gamma g(u) = g(T_\gamma u), \qquad T_\gamma A_* u = A_* T_\gamma u. \tag{5.20}$$

By construction, we also have

$$\langle V(t),\zeta\rangle_{L^2} = \phi_{1;\zeta}\big(X(t),\Gamma(t)\big) + \phi_{2;\zeta}\big(\Gamma(t)\big), \tag{5.21}$$

together with

$$T_{-\Gamma(s)}[X(s)+\Phi_{\rm ref}] = \Phi + V(s). \tag{5.22}$$

The derivatives in (5.12) and (5.15) can now be transferred from ζ to yield (5.18).

We emphasise that the identity

$$\frac{1}{2}\sigma^2 b(\Phi+V(s),\psi_{\rm tw})^2[X'' + \Phi_{\rm ref}''] = \frac{1}{2\rho}\sigma^2 b(\Phi+V(s),\psi_{\rm tw})^2 A_*[X(s)+\Phi_{\rm ref}] \tag{5.23}$$

is a crucial ingredient in this computation. This is where we use the requirement in (HA) that all the diffusion coefficients in A_* are equal. □

Proof of Proposition 5.1. Items (i) and (ii) follow immediately from items (i) and (ii) of Proposition 2.1. Turning to (iii), notice first that we have the isometry

$$\|T_\gamma x\|_{H^1} = \|x\|_{H^1} . \tag{5.24}$$

Observe in addition that

$$\|T_\gamma \Phi_{\mathrm{ref}} - \Phi\|_{H^1} \le \|T_\gamma \Phi_{\mathrm{ref}} - \Phi_{\mathrm{ref}}\|_{H^1} + \|\Phi_{\mathrm{ref}} - \Phi\|_{H^1} \le C_1\big[1 + |\gamma|\,\big], \tag{5.25}$$

since Φ_{ref}' and Φ_{ref}'' decay exponentially. In particular, the inclusion (5.6) follows from the corresponding inclusions (2.40) for the pair (X, Γ). The second inclusion (5.7) now follows immediately from the bounds in Lemma's 3.2 and 3.6.

Using Lemma's 3.1, 3.2, 3.6 and 4.2, we obtain the bound

$$\begin{array}{lll} \|\mathcal{R}_{\sigma;\Phi,c}(v)\|_{H^{-1}} & \le & C_2 K_\kappa\big[1 + \|v\|_{H^1}\big] \\ & & +C_2\big[1 + \|v\|_{H^1}^2 \|v\|_{L^2}\big] \\ & & +C_2\sigma^2 K_b\big[1 + \|v\|_{H^1}\big] \\ & & +\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big]\big[1 + \|v\|_{H^1}\big]. \end{array} \tag{5.26}$$

Since items (i) and (iii) imply that

$$\sup_{0\le t\le T} \|V(t,\omega)\|_{L^2} + \int_0^T \|V(t,\omega)\|_{H^1}^2 \, dt < \infty \tag{5.27}$$

for almost all $\omega \in \Omega$, item (iv) follows from the standard bound

$$\int_0^T \|V(t,\omega)\|_{H^1} \, dt \le \sqrt{T}\Big[\int_0^T \|V(t,\omega)\|_{H^1}^2 \, dt\Big]^{1/2}. \tag{5.28}$$

Finally, we note that items (iii) and (iv) imply that the integrals in (5.9) are well-defined. In view of Corollary 5.4, we can apply a standard diagonalisation argument involving the separability of L^2 and the density of test-functions to conclude that (v) holds. □

6 The stochastic time transform

We note that (5.9) can be interpreted as a quasi-linear equation due to the presence of the $\kappa_\sigma A_*$ term. In this section we transform our problem to a semilinear form by rescaling time, using the fact that κ_σ is a scalar. In addition, we investigate the impact of this transformation on the probabilities (2.62).

Recalling the map V defined in Proposition 5.1, we write

$$\tau_\Phi(t,\omega) = \int_0^t \kappa_\sigma\big(\Phi + V(s,\omega), \psi_{\mathrm{tw}}\big)\, ds. \tag{6.1}$$

Using Lemma 3.8 we see that $t \mapsto \tau_\Phi(t)$ is a continuous strictly increasing $(\mathcal{F}_t)$-adapted process that satisfies

$$t \le \tau_\Phi(t) \le K_\kappa t \tag{6.2}$$

for $0 \le t \le T$. In particular, we can define a map

$$t_\Phi : [0,T] \times \Omega \to [0,T] \tag{6.3}$$

for which

$$\tau_\Phi(t_\Phi(\tau,\omega),\omega) = \tau. \tag{6.4}$$

We now introduce the time-transformed map

$$\overline{V} : [0, T] \times \Omega \to L^2 \tag{6.5}$$

that acts as

$$\overline{V}(\tau, \omega) \quad = \quad V\big(t_\Phi(\tau, \omega), \omega\big). \tag{6.6}$$

Before stating our main results, we first investigate the effects of this transformation on the terms appearing in (5.9).

Lemma 6.1. *Consider the setting of Proposition 2.1 and suppose that (hPar) is satisfied. Then the map t_Φ defined in (6.3) satisfies the following properties.*

(i) For every $0 \le \tau \le T$, the random variable $\omega \mapsto t_\Phi(\tau, \omega)$ is an $(\mathcal{F}_t)$-stopping time.

(ii) The map $\tau \mapsto t_\Phi(\tau, \omega)$ is continuous and strictly increasing for all $\omega \in \Omega$.

(iii) For any $0 \le \tau \le T$ and $\omega \in \Omega$ we have the bounds

$$K_\kappa^{-1} \tau \le t_\Phi(\tau, \omega) \le \tau. \tag{6.7}$$

(iv) For every $0 \le t \le T$, the identity

$$t_\Phi(\tau_\Phi(t, \omega), \omega) = t \tag{6.8}$$

holds on the set $\{\omega : \tau_\Phi(t, \omega) \le T\}$.

Proof. On account of the identity

$$\{\omega : t_\Phi(\tau, \omega) \le t\} = \{\omega : \tau_\Phi(t, \omega) \ge \tau\} \tag{6.9}$$

and the fact that the latter set is in $\mathcal{F}_t$, we may conclude that $t_\Phi(\tau)$ is an $(\mathcal{F}_t)$-stopping time. The remaining properties follow directly from (6.2)-(6.4). □

Lemma 6.2. *Consider the setting of Proposition 2.1, recall the maps $(t_\Phi, \overline{V})$ defined by (6.3) and (6.6) and suppose that (hPar) is satisfied. Then there exists a filtration $(\overline{\mathcal{F}}_\tau)_{\tau \ge 0}$ together with a $(\overline{\mathcal{F}}_\tau)$-Brownian motion $(\overline{\beta}_\tau)_{\tau \ge 0}$ so that for any $H \in \mathcal{N}^2([0, T]; (\mathcal{F}_t); L^2)$, the process*

$$\overline{H}(\tau, \omega) = H\big(t_\Phi(\tau, \omega), \omega\big) \tag{6.10}$$

satisfies the following properties.

(i) We have the inclusion

$$\overline{H} \in \mathcal{N}^2([0, T]; (\overline{\mathcal{F}}_\tau); L^2), \tag{6.11}$$

together with the bound

$$E \int_0^T \big\|\overline{H}(\tau)\big\|_{L^2}^2 \, d\tau \le K_\kappa E \int_0^T \|H(t)\|_{L^2}^2 \, dt. \tag{6.12}$$

(ii) For almost all $\omega \in \Omega$, the identity

$$\int_0^{t_\Phi(\tau)} H(s) \, d\beta_s = \int_0^\tau \overline{H}(\tau') \kappa_\sigma(\Phi + \overline{V}(\tau'), \psi_{\mathrm{tw}})^{-1/2} \, d\overline{\beta}_{\tau'} \tag{6.13}$$

holds for all $0 \le \tau \le T$.

Proof. Following [29, §1.2.3], we write

$$\overline{\mathcal{F}}_\tau = \{A \in \cup_{t\geq 0}\mathcal{F}_t : A \cap \{t_\Phi(\tau) \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}. \tag{6.14}$$

The fact that $\overline{H}$ is $(\overline{\mathcal{F}}_\tau)$-progressively measurable can be established following the proof of [28, Lemma 10.8(c)]. In addition, we note that for almost all $\omega \in \Omega$ the path

$$t \mapsto \|V(t,\omega)\|_{L^2}^2 \tag{6.15}$$

is in $L^1([0,T])$, which allows us to apply the deterministic substitution rule to obtain

$$\int_0^{t_\Phi(\tau)} \|V(s)\|_{L^2}^2 \, ds = \int_0^\tau \big\|V\big(t_\Phi(\tau')\big)\big\|_{L^2}^2 \, \partial_\tau t_\Phi(\tau')\, d\tau'. \tag{6.16}$$

We now note that

$$\begin{array}{lcl} \partial_\tau t_\Phi(\tau') & = & \big[\partial_t \tau_\Phi\big(t_\Phi(\tau')\big)\big]^{-1} \\ & = & \kappa_\sigma\big(\Phi + V\big(t_\Phi(\tau')\big), \psi_{\mathrm{tw}}\big)^{-1} \\ & = & \kappa_\sigma\big(\Phi + \overline{V}(\tau'), \psi_{\mathrm{tw}}\big)^{-1}. \end{array} \tag{6.17}$$

In particular, we see that

$$|\partial_\tau t_\Phi(\tau')| \geq K_\kappa^{-1} \tag{6.18}$$

and hence

$$\int_0^\tau \big\|V\big(t_\Phi(\tau')\big)\big\|_{L^2}^2 \, d\tau' \leq K_\kappa \int_0^{t_\Phi(\tau)} \|V(s)\|_{L^2}^2 \, ds. \tag{6.19}$$

The bound (6.12) now follows from $t_\Phi(T,\omega) \leq T$.

To obtain (ii), we introduce the Brownian-motion $(\overline{\beta}_\tau)_{\tau\geq 0}$ that is given by

$$\overline{\beta}_\tau = \int_0^\tau \frac{1}{\sqrt{\partial_\tau t_\Phi(\tau')}}\, d\beta_{t_\Phi(\tau')}. \tag{6.20}$$

For any test-function $\zeta \in C_c^\infty(\mathbb{R};\mathbb{R}^n)$ and $0 \leq t \leq T$, the proof of [29, Lem. 5.1.3.5] implies that for almost all $\omega \in \Omega$ the identity

$$\begin{array}{lcl} \int_0^{t_\Phi(\tau)} \langle H(s), \zeta\rangle_{L^2} d\beta_s & = & \int_0^\tau \langle H\big(t_\Phi(\tau')\big), \zeta\rangle_{L^2} \sqrt{\partial_\tau t_\Phi(\tau')}\, d\overline{\beta}_{\tau'} \\ & = & \int_0^\tau \langle \overline{H}(\tau'), \zeta\rangle_{L^2} \kappa_\sigma\big(\Phi + \overline{V}(\tau'), \psi_{\mathrm{tw}}\big)^{-1/2} \, d\overline{\beta}_{\tau'} \end{array} \tag{6.21}$$

holds for all $0 \leq \tau \leq T$. Since (i) and (ii) together imply that the right-hand side of (6.13) is well-defined as a stochastic-integral, a standard diagonalisation argument involving the separability of L^2 shows that both sides must be equal for almost all $\omega \in \Omega$. □

In order to formulate the time-transformed SPDE, we introduce the nonlinearity

$$\begin{array}{lcl} \overline{\mathcal{R}}_{\sigma;\Phi,c}(v) & = & \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})^{-1}\mathcal{R}_{\sigma;\Phi,c}(v) - \mathcal{L}_{\mathrm{tw}} v \\ & = & A_*[\Phi + v] + \mathcal{J}_\sigma(\Phi + v, c, \psi_{\mathrm{tw}}) + \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})^{-1} a(\Phi + v, c, \psi_{\mathrm{tw}})[\Phi' + v'] \\ & & \quad -\mathcal{L}_{\mathrm{tw}} v, \end{array} \tag{6.22}$$

together with

$$\begin{array}{lcl} \overline{\mathcal{S}}_{\sigma;\Phi}(v) & = & \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})^{-1/2} \mathcal{S}_\Phi(v) \\ & = & \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})^{-1/2} \Big[g(\Phi + v) + b(\Phi + v, \psi_{\mathrm{tw}})[\Phi' + v']\Big]. \end{array} \tag{6.23}$$

Proposition 6.3. *Consider the setting of Proposition 2.1 and suppose that (hPar) is satisfied. Then the map*

$$\overline{V} : [0,T] \times \Omega \to L^2 \tag{6.24}$$

defined by the transformations (5.1) and (6.6) satisfies the following properties.

(i) For almost all $\omega \in \Omega$*, the map* $\tau \mapsto \overline{V}(\tau;\omega)$ *is of class* $C\big([0,T];L^2\big)$.

(ii) For all $\tau \in [0,T]$*, the map* $\omega \mapsto \overline{V}(\tau,\omega)$ *is* $(\overline{\mathcal{F}}_\tau)$*-measurable.*

(iii) We have the inclusion

$$\overline{V} \in \mathcal{N}^2\big([0,T];(\overline{\mathcal{F}})_\tau;H^1\big), \tag{6.25}$$

together with

$$\overline{\mathcal{S}}_{\sigma;\Phi}(\overline{V}) \in \mathcal{N}^2\big([0,T];(\overline{\mathcal{F}})_\tau;L^2\big). \tag{6.26}$$

(iv) For almost all $\omega \in \Omega$*, we have the inclusion*

$$\overline{\mathcal{R}}_{\sigma;\Phi,c}\big(\overline{V}(\cdot,\omega)\big) \in L^1([0,T];L^2). \tag{6.27}$$

(v) For almost all $\omega \in \Omega$*, the identity*

$$\begin{aligned} \overline{V}(\tau) &= \overline{V}(0) + \int_0^\tau \Big[\mathcal{L}_{\mathrm{tw}}\overline{V}(\tau') + \overline{\mathcal{R}}_{\sigma;\Phi,c}\big(\overline{V}(\tau')\big)\Big]\,d\tau' \\ &\quad + \sigma \int_0^\tau \overline{\mathcal{S}}_{\sigma;\Phi}\big(\overline{V}(\tau')\big)\,d\beta_{\tau'} \end{aligned} \tag{6.28}$$

holds for all $0 \le t \le T$.

(vi) For almost all $\omega \in \Omega$*, the identity*

$$\begin{aligned} \overline{V}(\tau) &= S(\tau)\overline{V}(0) + \int_0^\tau S(\tau-\tau')\overline{\mathcal{R}}_{\sigma;\Phi,c}\big(\overline{V}(\tau')\big)\,d\tau' \\ &\quad + \sigma \int_0^\tau S(\tau-\tau')\overline{\mathcal{S}}_{\sigma;\Phi}\big(\overline{V}(\tau')\big)\,d\beta_{\tau'} \end{aligned} \tag{6.29}$$

holds for all $\tau \in [0,T]$*, in which*

$$S : [0,\infty) \to \mathcal{L}(L^2;L^2) \tag{6.30}$$

denotes the analytic semigroup generated by $\mathcal{L}_{\mathrm{tw}}$.

Proof. Items (i)-(iii) follow by applying (i) of Lemma 6.2 to the maps V, $\partial_\xi V$ and using the definition (6.23). Item (iv) can be obtained from the computation (5.26), noting that the $A_* v$ contribution is no longer present.

Item (v) can be obtained by applying the stochastic time-transform (6.13) and the deterministic time-transform

$$\int_0^{t_\Phi(\tau)} \mathcal{R}_{\sigma;\Phi,c}\big(V(s)\big)\,ds = \int_0^\tau \overline{\mathcal{R}}_{\sigma;\Phi,c}\big(\overline{V}(\tau')\big)\big[\kappa_\sigma\big(\Phi + \overline{V}(\tau'),\psi_{\mathrm{tw}}\big)\big]^{-1}\,d\tau' \tag{6.31}$$

to the integral equation (5.9).

Turning to (vi), we note that A_* generates a standard diagonal heat-semigroup, which is obviously analytic. Noting that

$$\mathcal{L}_{\mathrm{tw}} - A_* \in \mathcal{L}(H^1;L^2) \tag{6.32}$$

and recalling the interpolation estimate

$$\|v\|_{H^1} \le C_1 \|v\|_{H^2}^{1/2} \|v\|_{L^2}^{1/2}, \tag{6.33}$$

we may apply [38, Prop 3.2.2(iii)] to conclude that also $\mathcal{L}_{\mathrm{tw}}$ generates an analytic semigroup. We may now apply [41, Prop 6.3] and the computation in the proof of [38, Prop 4.1.4] to conclude the integral identity (6.29). □

We now introduce the scalar functions

$$\begin{aligned} N_{\varepsilon,\alpha}(t) &= e^{\alpha t}\left\|V(t)\right\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)}e^{\alpha s}\left\|V(s)\right\|_{H^1}^2\,ds, \\ \overline{N}_{\varepsilon,\alpha}(\tau) &= e^{\alpha\tau}\left\|\overline{V}(\tau)\right\|_{L^2}^2 + \int_0^\tau e^{-\varepsilon(\tau-\tau')}e^{\alpha\tau'}\left\|\overline{V}(\tau')\right\|_{H^1}^2\,d\tau', \end{aligned} \tag{6.34}$$

together with the associated probabilities

$$\begin{aligned} p_{\varepsilon,\alpha}(T,\eta) &= P\Big(\sup_{0\le t\le T} N_{\varepsilon,\alpha}(t) > \eta\Big), \\ \overline{p}_{\varepsilon,\alpha}(T,\eta) &= P\Big(\sup_{0\le \tau\le T} \overline{N}_{\varepsilon,\alpha}(\tau) > \eta\Big). \end{aligned} \tag{6.35}$$

Our second main result shows that these two sets of probabilities can be effectively compared with each other.

Proposition 6.4. *Consider the setting of Proposition 2.1 and recall the maps V and $\overline{V}$ defined by (5.1) and (6.6). Then we have the bound*

$$p_{\varepsilon,\alpha}(T,\eta) \le \overline{p}_{K_\kappa^{-1}\varepsilon,\alpha}\big(K_\kappa T, K_\kappa^{-1}\eta\big). \tag{6.36}$$

Proof. We note that

$$e^{\alpha t}\left\|V(t)\right\|_{L^2} = e^{\alpha t}\left\|\overline{V}\big(\tau_\Phi(t)\big)\right\|_{L^2} \le e^{\alpha\tau_\Phi(t)}\left\|\overline{V}\big(\tau_\Phi(t)\big)\right\|_{L^2}, \tag{6.37}$$

which implies that

$$\sup_{0\le t\le T} e^{\alpha t}\left\|V(t)\right\|_{L^2}^2 \le \sup_{0\le\tau\le K_\kappa T} e^{\alpha\tau}\left\|\overline{V}(\tau)\right\|_{L^2}^2. \tag{6.38}$$

In addition, we compute

$$\int_0^t e^{-\varepsilon(t-s)}e^{\alpha s}\left\|V(s)\right\|_{H^1}^2\,ds = \int_0^{\tau_\Phi(t)} e^{-\varepsilon(t-t_\Phi(\tau'))}e^{\alpha t_\Phi(\tau')}\left\|\overline{V}(\tau')\right\|_{H^1}^2 \kappa_\sigma\big(\Phi+\overline{V}(\tau'),\psi_{\mathrm{tw}}\big)^{-1}\,d\tau'. \tag{6.39}$$

Using (6.18) we obtain the estimate

$$t - t_\Phi(\tau') = t_\Phi\big(\tau_\Phi(t)\big) - t_\Phi(\tau') = \int_{\tau'}^{\tau_\Phi(t)} \partial_\tau t_\Phi(\tau'')\,d\tau'' \ge K_\kappa^{-1}\left|\tau_\Phi(t)-\tau'\right|, \tag{6.40}$$

which yields

$$\int_0^t e^{-\varepsilon(t-s)}e^{\alpha s}\left\|V(s)\right\|_{H^1}^2\,ds \le K_\kappa\int_0^{\tau_\Phi(t)} e^{-K_\kappa^{-1}\varepsilon(\tau_\Phi(t)-\tau')}e^{\alpha\tau'}\left\|\overline{V}(\tau')\right\|_{H^1}^2\,d\tau'. \tag{6.41}$$

In particular, we conclude that

$$\sup_{0\le t\le T}\int_0^t e^{-\varepsilon(t-s)}e^{\alpha s}\left\|V(s)\right\|_{H^1}^2\,ds \le \sup_{0\le\tau\le K_\kappa T} K_\kappa\int_0^\tau e^{-K_\kappa^{-1}\varepsilon(\tau-\tau')}e^{\alpha\tau'}\left\|\overline{V}(\tau')\right\|_{H^1}^2\,d\tau'. \tag{6.42}$$

This yields the implication

$$\sup_{0\le\tau\le K_\kappa T}\overline{N}_{K_\kappa^{-1}\varepsilon,\alpha}(\tau) \le K_\kappa^{-1}\eta \Rightarrow \sup_{0\le t\le T} N_{\varepsilon,\alpha}(t) \le \eta, \tag{6.43}$$

from which the desired inequality immediately follows. □

7 The stochastic wave

In this section we set out to construct the branch of modified waves (Φ_σ, c_σ) and analyse the phase condition

$$\langle T_{-\gamma_0}[u_0] - \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} = 0 \tag{7.1}$$

for $u_0 \approx \Phi_\sigma$. In particular, we establish Propositions 2.2, 2.3 and 2.6.

A key role in our analysis is reserved for the function

$$\begin{aligned} \mathcal{M}_{\sigma;\Phi,c}(v,d) &= \mathcal{J}_\sigma(\Phi + v, c + d, \psi_{\mathrm{tw}}) - \mathcal{J}_0(\Phi, c) \\ &\quad - d\Phi_0' + [A_* - \mathcal{L}_{\mathrm{tw}}]v, \end{aligned} \tag{7.2}$$

defined for $(\Phi, c) \in \mathcal{U}_{H^1} \times \mathbb{R}$ and $(v, d) \in H^1 \times \mathbb{R}$. Indeed, we will construct a solution to

$$A_*\Phi_\sigma + \mathcal{J}_\sigma(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}) = 0 \tag{7.3}$$

by writing

$$\Phi_\sigma = \Phi_0 + v, \qquad c_\sigma = c_0 + d. \tag{7.4}$$

Using the fact that the pair (Φ_0, c_0) is a solution to (7.3) for $\sigma = 0$, one readily verifies that the pair $(v, d) \in H^2 \times \mathbb{R}$ must satisfy the system

$$d\Phi_0' + \mathcal{L}_{\mathrm{tw}} v = -\mathcal{M}_{\sigma;\Phi_0,c_0}(v, d). \tag{7.5}$$

In addition, the function $\mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}$ will be used in §8 to obtain bounds on the nonlinearity $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$.

In §7.1 we obtain global and Lipschitz bounds on $\mathcal{M}_{\sigma;\Phi,c}$. These bounds are subsequently used in §7.2 to setup two fixed-point constructions that provide solutions to (7.1) and (7.3).

7.1 Bounds for $\mathcal{M}_\sigma$

In order to streamline our estimates, it is convenient to decompose the function $\mathcal{J}_\sigma$ as

$$\begin{aligned} \mathcal{J}_\sigma(u, \bar{c}, \psi_{\mathrm{tw}}) &= \kappa_\sigma(u, \psi_{\mathrm{tw}})^{-1}\Big[f(u) + \bar{c}u' + \sigma^2 b(u, \psi_{\mathrm{tw}})\partial_\xi[g(u)]\Big] \\ &= \mathcal{J}_0(u, \bar{c}) + \mathcal{E}_{\sigma;I}(u, \bar{c}) + \mathcal{E}_{\sigma;II}(u). \end{aligned} \tag{7.6}$$

Here we have introduced the function

$$\begin{aligned} \mathcal{E}_{\sigma;I}(u, \bar{c}) &= \nu_\sigma^{(-1)}(u, \psi_{\mathrm{tw}})[f(u) + \bar{c}u'] \\ &= \nu_\sigma^{(-1)}(u, \psi_{\mathrm{tw}})\mathcal{J}_0(u, \bar{c}), \end{aligned} \tag{7.7}$$

together with

$$\mathcal{E}_{\sigma;II}(u) = \sigma^2\kappa_\sigma(u, \psi_{\mathrm{tw}})^{-1} b(u, \psi_{\mathrm{tw}})\partial_\xi[g(u)] \tag{7.8}$$

where ν_σ^{-1} is as defined in (3.55).

This decomposition allows us to rewrite (7.2) in the intermediate form

$$\mathcal{M}_{\sigma;\Phi,c}(v, d) = \mathcal{M}_{0;\Phi,c}(v, d) + \mathcal{E}_{\sigma;I}(\Phi + v, c + d) + \mathcal{E}_{\sigma;II}(\Phi + v). \tag{7.9}$$

We now make a final splitting

$$\begin{aligned} \mathcal{M}_{0;\Phi,c}(v, d) &= \mathcal{J}_0(\Phi + v, c + d) - \mathcal{J}_0(\Phi, c) - Df(\Phi_0)v - c_0 v' - d\Phi_0' \\ &= \mathcal{N}_{I;f,\Phi}(v) + \mathcal{N}_{II;\Phi,c}(v, d), \end{aligned} \tag{7.10}$$

in which we have introduced the function

$$\mathcal{N}_{I;f,\Phi}(v) = f(\Phi + v) - f(\Phi) - Df(\Phi)v, \tag{7.11}$$

together with

$$\mathcal{N}_{II;\Phi,c}(v,d) \quad = \quad dv' + \big[Df(\Phi) - Df(\Phi_0)\big]v + (c-c_0)v' + d[\Phi' - \Phi_0']. \tag{7.12}$$

We hence arrive at the convenient final expression

$$\mathcal{M}_{\sigma;\Phi,c}(v,d) \quad = \quad \mathcal{N}_{I;f,\Phi}(v) + \mathcal{N}_{II;\Phi,c}(v,d) + \mathcal{E}_{\sigma;I}(\Phi+v, c+d) + \mathcal{E}_{\sigma;II}(\Phi+v) \tag{7.13}$$

and set out to analyse each of these terms separately.

Lemma 7.1. *Suppose that (Hf) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $v \in H^1$ we have the bound*

$$\|\mathcal{N}_{I;f,\Phi}(v)\|_{L^2} \quad \leq \quad K\big[1 + \|v\|_{H^1}\big]\,\|v\|_{H^1}\,\|v\|_{L^2}\,, \tag{7.14}$$

while for any pair $(v_A, v_B) \in H^1 \times H^1$ we have the estimates

$$\begin{array}{rcl} \|\mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B)\|_{L^2} & \leq & K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\,\|v_A\|_{H^1} + \|v_B\|_{H^1}\big] \\ & & \times\,\|v_A - v_B\|_{L^2}\,, \\ |\langle \mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B), \psi_{\mathrm{tw}}\rangle_{L^2}| & \leq & K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\,\|v_A\|_{L^2} + \|v_B\|_{L^2}\big] \\ & & \times\,\|v_A - v_B\|_{L^2}\,. \end{array} \tag{7.15}$$

Proof. Using (3.6) and (hPar) we obtain the pointwise bound

$$|\mathcal{N}_{I;f,\Phi}(v)| \leq C_1[1 + |v|]\,|v|^2\,, \tag{7.16}$$

from which (7.14) easily follows. In addition, we may compute

$$\begin{array}{rcl} \mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B) & = & f(\Phi + v_A) - f(\Phi + v_B) - Df(\Phi + v_B)(v_A - v_B) \\ & & +\big(Df(\Phi + v_B) - Df(\Phi)\big)(v_A - v_B) \\ & = & \mathcal{N}_{I;f,\Phi+v_B}(v_A - v_B) + \big(Df(\Phi + v_B) - Df(\Phi)\big)(v_A - v_B). \end{array} \tag{7.17}$$

Applying (3.6) and (hPar) a second time, we obtain the pointwise bound

$$\begin{array}{rcl} |\mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B)| & \leq & C_2[1 + |v_A| + |v_B|\,]\,|v_A - v_B|^2 \\ & & +C_2[1 + |v_B|]\,|v_B|\,|v_A - v_B| \\ & \leq & C_3[1 + |v_A| + |v_B|\,]\big[\,|v_A| + |v_B|\,\big]\,|v_A - v_B|\,, \end{array} \tag{7.18}$$

from which the estimates in (7.15) can be readily obtained. □

Lemma 7.2. *Suppose that (Hf) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $(v,d) \in H^1 \times \mathbb{R}$ we have the bound*

$$\|\mathcal{N}_{II;\Phi,c}(v,d)\|_{L^2} \quad \leq \quad K\big[\,|c - c_0| + \|\Phi - \Phi_0\|_{H^1} + |d|\,\big]\big[\,\|v\|_{H^1} + |d|\,\big], \tag{7.19}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$ the expression

$$\Delta_{AB}\mathcal{N}_{II;\Phi,c} = \mathcal{N}_{II;\Phi,c}(v_A, d_A) - \mathcal{N}_{II;\Phi,c}(v_B, d_B) \tag{7.20}$$

satisfies the estimates

$$\begin{array}{rcl} \|\Delta_{AB}\mathcal{N}_{II;\Phi,c}\|_{L^2} & \leq & K\big[\,\|v_A\|_{H^1} + |d_B| + \|\Phi - \Phi_0\|_{H^1} + |c - c_0|\,\big] \\ & & \times\big[\,\|v_A - v_B\|_{H^1} + |d_A - d_B|\,\big], \\ |\langle \Delta_{AB}\mathcal{N}_{II;\Phi,c}, \psi_{\mathrm{tw}}\rangle_{L^2}| & \leq & K\big[\,\|v_A\|_{L^2} + |d_B| + \|\Phi - \Phi_0\|_{L^2} + |c - c_0|\,\big] \\ & & \times\big[\,\|v_A - v_B\|_{L^2} + |d_A - d_B|\,\big]. \end{array} \tag{7.21}$$

Proof. In view of (hPar), we obtain the pointwise bound

$$|\mathcal{N}_{II;\Phi,c}(v,d)| \le \big[\, |d| + |c - c_0| \,\big]\, |v'| + C_1 \, |\Phi - \Phi_0| \, |v| + |\Phi' - \Phi_0'| \, |d| \,, \tag{7.22}$$

from which (7.19) follows. In addition, we obtain the pointwise bound

$$\begin{array}{lcl} |\Delta_{AB}\mathcal{N}_{II;\Phi,c}| & \le & |d_A - d_B| \, |v_A'| + \big[\, |d_B| + |c - c_0| \,\big] \, |v_A' - v_B'| \\ & & \quad + K \, |\Phi - \Phi_0| \, \big[\, |v_A - v_B| \,\big] + |\Phi' - \Phi_0'| \, |d_A - d_B| \end{array} \tag{7.23}$$

from which (7.21) follows. □

Lemma 7.3. *Suppose that (Hf), (Hg) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $0 \le \sigma \le 1$ and $(v,d) \in H^1 \times \mathbb{R}$, we have the bound*

$$\|\mathcal{E}_{\sigma;I}(\Phi + v, c + d)\|_{L^2} \quad \le \quad K\sigma^2(1 + |d|)\big[1 + \|v\|_{H^1} + \|v\|_{H^1}^2 \, \|v\|_{L^2} \,\big], \tag{7.24}$$

while for any $0 \le \sigma \le 1$ and any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$, the expression

$$\Delta_{AB}\mathcal{E}_{\sigma;I} = \mathcal{E}_{\sigma;I}(\Phi + v_A, c + d_A) - \mathcal{E}_{\sigma;I}(\Phi + v_B, c + d_B) \tag{7.25}$$

satisfies the estimates

$$\begin{array}{lcl} \|\Delta_{AB}\mathcal{E}_{\sigma;I}\|_{L^2} & \le & K\sigma^2(1 + |d_A|)\big[1 + \|v_A\|_{H^1} + \|v_A\|_{H^1}^2 \, \|v_A\|_{L^2} \,\big] \, \|v_A - v_B\|_{L^2} \\ & & \quad + K\sigma^2\big[1 + |d_B| + \|v_A\|_{H^1} \, \|v_A\|_{L^2} + \|v_B\|_{H^1} \, \|v_B\|_{L^2} \,\big] \, \|v_A - v_B\|_{H^1} \\ & & \quad + K\sigma^2\big[1 + \|v_A\|_{H^1} \,\big] \, |d_A - d_B| \,, \\ |\langle \Delta_{AB}\mathcal{E}_{\sigma;I}, \psi_{\mathrm{tw}} \rangle_{L^2}| & \le & K\sigma^2(1 + |d_A| + |d_B|)\big[1 + \|v_A\|_{H^1} \, \|v_A\|_{L^2}^2 \,\big] \, \|v_A - v_B\|_{L^2} \\ & & \quad + K\sigma^2\big[\, \|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2 \,\big] \, \|v_A - v_B\|_{H^1} \\ & & \quad + K\sigma^2[1 + \|v_A\|_{L^2}] \, |d_A - d_B| \,. \end{array} \tag{7.26}$$

Proof. The bound (7.24) follows directly from Lemma's 3.4 and 3.9. In addition, these results allow us to compute

$$\begin{array}{lcl} \|\Delta_{AB}\mathcal{E}_{\sigma;I}\|_{L^2} & \le & \Big|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\mathrm{tw}}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big| \, \|\mathcal{J}_0(\Phi + v_A, c + d_A)\|_{L^2} \\ & & \quad + \Big|\nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big| \, \|\mathcal{J}_0(\Phi + v_A, c + d_A) - \mathcal{J}_0(\Phi + v_B, c + d_B)\|_{L^2} \\ & \le & C_1\sigma^2 \, \|v_A - v_B\|_{L^2} \, (1 + |d_A|)\big[1 + \|v_A\|_{H^1} + \|v_A\|_{H^1}^2 \, \|v\|_{L^2} \,\big] \\ & & \quad + C_1\sigma^2\big[\, \|v_A\|_{H^1} \, \|v_A\|_{L^2} + \|v_B\|_{H^1} \, \|v_B\|_{L^2} \,\big] \, \|v_A - v_B\|_{H^1} \\ & & \quad + C_1\sigma^2[1 + \|v_A\|_{H^1}] \, |d_A - d_B| \\ & & \quad + C_1\sigma^2(1 + |d_B|) \, \|v_A - v_B\|_{H^1} \,, \end{array} \tag{7.27}$$

together with

$$
\begin{aligned}
|\langle \Delta_{AB}\mathcal{E}_{\sigma;I}, \psi_{\rm tw}\rangle_{L^2}| \quad &\leq \quad \left|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\rm tw}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\rm tw})\right| |\langle \mathcal{J}_0(\Phi + v_A, c + d_A), \psi_{\rm tw}\rangle_{L^2}| \\
&\qquad + \left|\nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\rm tw})\right| |\langle \mathcal{J}_0(\Phi + v_A, c + d_A) - \mathcal{J}_0(\Phi + v_B, c + d_B), \psi_{\rm tw}\rangle_{L^2}| \\
&\leq \quad C_2\sigma^2 \|v - w\|_{L^2} (1 + |d_A|)\big[1 + \|v_A\|_{H^1} \|v_A\|_{L^2}^2\big] \\
&\qquad + C_2\sigma^2[\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2] \|v_A - v_B\|_{H^1} \\
&\qquad + C_2\sigma^2[1 + \|v_A\|_{L^2}] |d_A - d_B| \\
&\qquad + C_2\sigma^2(1 + |d_B|) \|v_A - v_B\|_{L^2} .
\end{aligned}
\tag{7.28}
$$

These terms can all be absorbed by the expressions in (7.26). □

Lemma 7.4. *Suppose that (Hg) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $0 \leq \sigma \leq 1$ and $v \in H^1$ we have the bound*

$$
\|\mathcal{E}_{\sigma;II}(\Phi + v)\|_{L^2} \quad \leq \quad K\sigma^2\big[1 + \|v\|_{H^1}\big],
\tag{7.29}
$$

while for any $0 \leq \sigma \leq 1$ and any pair $(v_A, v_B) \in H^1 \times H^1$ the expression

$$
\Delta_{AB}\mathcal{E}_{\sigma;II} = \mathcal{E}_{\sigma;II}(\Phi + v_A) - \mathcal{E}_{\sigma;II}(\Phi + v_B)
\tag{7.30}
$$

satisfies the estimates

$$
\begin{aligned}
\|\Delta_{AB}\mathcal{E}_{\sigma;II}\|_{L^2} \quad &\leq \quad K\sigma^2\big[1 + \|v_A\|_{H^1}\big] \|v_A - v_B\|_{H^1} , \\
|\langle \Delta_{AB}\mathcal{E}_{\sigma;II}, \psi_{\rm tw}\rangle_{L^2}| \quad &\leq \quad K\sigma^2\big[1 + \|v_A\|_{L^2}\big] \|v_A - v_B\|_{L^2} .
\end{aligned}
\tag{7.31}
$$

Proof. The bound (7.29) follows directly from Lemma's 3.2, 3.6 and 3.8. In addition, we may compute

$$
\begin{aligned}
\|\Delta_{AB}\mathcal{E}_{\sigma;II}\|_{L^2} \quad &\leq \quad \sigma^2 \left|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\rm tw}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\rm tw})\right| K_b \|\partial_\xi[g(\Phi + v)]\|_{L^2} \\
&\qquad + \sigma^2 K_\nu |b(\Phi + v_A, \psi_{\rm tw}) - b(\Phi + v_B, \psi_{\rm tw})| \|\partial_\xi[g(\Phi + v_A)]\|_{L^2} \\
&\qquad + \sigma^2 K_\kappa K_b \|\partial_\xi[g(\Phi + v_A) - g(\Phi + v_B)]\|_{L^2} \\
&\leq \quad C_1\sigma^2 \|v_A - v_B\|_{L^2} \big[1 + \|v_A\|_{H^1}\big] \\
&\qquad + C_1\sigma^2 \|v_A - v_B\|_{L^2} \big[1 + \|v_A\|_{H^1}\big] \\
&\qquad + C_1\sigma^2\big[1 + \|v_A\|_{H^1}\big] \|v_A - v_B\|_{H^1} ,
\end{aligned}
\tag{7.32}
$$

together with

$$
\begin{aligned}
|\langle \Delta_{AB}\mathcal{E}_{\sigma;II}, \psi_{\rm tw}\rangle_{L^2}| \quad &\leq \quad \sigma^2 \left|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\rm tw}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\rm tw})\right| K_b |\langle \partial_\xi[g(\Phi + v)], \psi_{\rm tw}\rangle_{L^2}| \\
&\qquad + \sigma^2 K_\nu |b(\Phi + v_A, \psi_{\rm tw}) - b(\Phi + v_B, \psi_{\rm tw})| |\langle \partial_\xi[g(\Phi + v)], \psi_{\rm tw}\rangle_{L^2}| \\
&\qquad + \sigma^2 K_\nu K_b |\langle \partial_\xi[g(\Phi + v_A) - g(\Phi + v_B)], \psi_{\rm tw}\rangle_{L^2}| \\
&\leq \quad C_2\sigma^2 \|v_A - v_B\|_{L^2} \big[1 + \|v_A\|_{L^2}\big] \\
&\qquad + C_2\sigma^2 \|v_A - v_B\|_{L^2} \big[1 + \|v_A\|_{L^2}\big] \\
&\qquad + C_2\sigma^2 \|v_A - v_B\|_{L^2} .
\end{aligned}
\tag{7.33}
$$

These expressions can be absorbed into the bounds (7.31). □

Corollary 7.5. *Suppose that (Hf), (Hg) and (hPar) are satisfied. Then there exists $K > 0$ so that the following holds true. For any $0 \le \sigma \le 1$ and any $(v, d) \in H^1 \times \mathbb{R}$ that has $|d| \le 1$, we have the estimate*

$$\begin{aligned} \|\mathcal{M}_{\sigma;\Phi,c}(v,d)\|_{L^2} \quad \le \quad & K\big[1 + \|v\|_{H^1}\big] \|v\|_{H^1} \|v\|_{L^2} \\ & + K\big[\,|c - c_0| + \|\Phi - \Phi_0\|_{H^1} + |d|\,\big]\big[\,\|v\|_{H^1} + |d|\,\big] \\ & + K\sigma^2\big[1 + \|v\|_{H^1}\big]. \end{aligned} \tag{7.34}$$

In addition, for any $0 \le \sigma \le 1$ and any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$ for which $|d_A| \le 1$ and $|d_B| \le 1$, the expression

$$\Delta_{AB}\mathcal{M}_{\sigma;\Phi,c} = \mathcal{M}_{\sigma;\Phi,c}(v_A, d_A) - \mathcal{M}_{\sigma;\Phi,c}(v_B, d_B) \tag{7.35}$$

satisfies the estimates

$$\begin{aligned} \|\Delta_{AB}\mathcal{M}_{\sigma;\Phi,c}\|_{L^2} \quad \le \quad & K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\,\|v_A\|_{H^1} + \|v_B\|_{H^1}\,\big] \|v_A - v_B\|_{L^2} \\ & + K\big[\sigma^2 + \|v_A\|_{H^1} + |d_B| + \|\Phi - \Phi_0\|_{H^1} + |c - c_0|\,\big] \\ & \quad \times\big[\,\|v_A - v_B\|_{H^1} + |d_A - d_B|\,\big] \\ & + K\sigma^2 \|v_A\|_{H^1}^2 \|v_A\|_{L^2} \|v_A - v_B\|_{L^2} \\ & + K\sigma^2\Big[\,\|v_A\|_{H^1} \|v_A\|_{L^2} + \|v_B\|_{H^1} \|v_B\|_{L^2}\,\Big] \|v_A - v_B\|_{H^1}, \\ |\langle \Delta_{AB}\mathcal{M}_{\sigma;\Phi,c}, \psi_{\mathrm{tw}}\rangle_{L^2}| \quad \le \quad & K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\,\|v_A\|_{L^2} + \|v_B\|_{L^2}\,\big] \|v_A - v_B\|_{L^2} \\ & + K\big[\sigma^2 + \|v_A\|_{L^2} + |d_B| + \|\Phi - \Phi_0\|_{L^2} + |c - c_0|\,\big] \\ & \quad \times\big[\,\|v_A - v_B\|_{L^2} + |d_A - d_B|\,\big] \\ & + K\sigma^2 \|v_A\|_{H^1} \|v_A\|_{L^2}^2 \|v_A - v_B\|_{L^2} \\ & + K\sigma^2\Big[\,\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2\,\Big] \|v_A - v_B\|_{H^1}. \end{aligned} \tag{7.36}$$

Proof. In view of the identity (7.13) it suffices to note that the terms (7.14), (7.19), (7.24) and (7.29) can be absorbed in (7.34), while the expressions (7.15), (7.21), (7.26) and (7.31) can be absorbed in (7.36). □

Corollary 7.6. *Suppose that (Hf) and (Hg) are satisfied. Then there exists $K > 0$ so that the following holds true. For any $0 \le \sigma \le 1$ and any $(v, d) \in H^1 \times \mathbb{R}$ that has $\|v\|_{H^1} \le 1$ together with $|d| \le 1$, we have the estimate*

$$\|\mathcal{M}_{\sigma;\Phi_0,c_0}(v,d)\|_{L^2} \quad \le \quad K\big[\,\|v\|_{L^2} + |d|\,\big]\big[\,\|v\|_{H^1} + |d|\,\big] + K\sigma^2. \tag{7.37}$$

In addition, for any $0 \le \sigma \le 1$ and any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$ for which the bounds

$$\|v_A\|_{H^1} \le 1, \qquad |d_A| \le 1, \qquad \|v_B\|_{H^1} \le 1, \qquad |d_B| \le 1 \tag{7.38}$$

hold, the expression

$$\Delta_{AB}\mathcal{M}_{\sigma;\Phi_0,c_0} = \mathcal{M}_{\sigma;\Phi_0,c_0}(v_A, d_A) - \mathcal{M}_{\sigma;\Phi_0,c_0}(v_B, d_B) \tag{7.39}$$

satisfies the estimate

$$\|\Delta_{AB}\mathcal{M}_{\sigma;\Phi_0,c_0}\|_{L^2} \quad \le \quad K\big[\sigma^2 + \|v_A\|_{H^1} + \|v_B\|_{H^1} + |d_B|\,\big]\big[\,\|v_A - v_B\|_{H^1} + |d_A - d_B|\,\big]. \tag{7.40}$$

Proof. These bounds can easily be obtained by simplifying the corresponding expressions from Corollary 7.5. □

7.2 Fixed-point constructions

As a final preparation before setting up our fixed-point problems, we need to control the higher order effects that arise when translating the adjoint eigenfunction ψ_{tw}. In particular, for any $\gamma \in \mathbb{R}$ we introduce the function

$$\mathcal{N}_{\mathrm{tw}}(\gamma) = T_\gamma \psi_{\mathrm{tw}} - \psi_{\mathrm{tw}} + \gamma \psi'_{\mathrm{tw}} \tag{7.41}$$

and obtain the following bounds.

Lemma 7.7. *Suppose that (HTw) and (HS) hold. Then there exists $K > 0$ so that for any $\gamma \in \mathbb{R}$ we have the bound*

$$\|\mathcal{N}_{\mathrm{tw}}(\gamma)\|_{L^2} \le K\gamma^2, \tag{7.42}$$

while for any pair $(\gamma_A, \gamma_B) \in \mathbb{R}^2$ we have the estimate

$$\|\mathcal{N}_{\mathrm{tw}}(\gamma_A) - \mathcal{N}_{\mathrm{tw}}(\gamma_B)\|_{L^2} \le K\big[\, |\gamma_A| + |\gamma_B| \,\big]\, |\gamma_A - \gamma_B| \,. \tag{7.43}$$

Proof. In view of (4.31), we have the a-priori bound

$$\|\mathcal{N}_{\mathrm{tw}}(\gamma)\|_{L^2} \le C_1\big[1 + |\gamma|\,\big], \tag{7.44}$$

together with

$$\|\mathcal{N}_{\mathrm{tw}}(\gamma_A) - \mathcal{N}_{\mathrm{tw}}(\gamma_B)\|_{L^2} \le C_1\, |\gamma_A - \gamma_B| \,. \tag{7.45}$$

In particular, we can restrict our attention to the situation where $|\gamma| \le 1$ and $|\gamma_A| + |\gamma_B| \le 1$. In this case we obtain the pointwise bounds

$$|\mathcal{N}_{\mathrm{tw}}(\gamma)(\xi)| \le \frac{1}{2}\gamma^2 \sup_{\xi - 1 \le \xi' \le \xi + 1} |\psi''_{\mathrm{tw}}(\xi')| \tag{7.46}$$

together with

$$|\mathcal{N}_{\mathrm{tw}}(\gamma_A)(\xi) - \mathcal{N}_{\mathrm{tw}}(\gamma_B)(\xi)| \le \big[\sup_{\xi - 1 \le \xi' \le \xi + 1} |\psi''_{\mathrm{tw}}(\xi')|\,\big]\big[\frac{1}{2}(\gamma_A - \gamma_B)^2 + |\gamma_B|\, |\gamma_A - \gamma_B|\,\big]. \tag{7.47}$$

The desired bounds now follow from the exponential decay of ψ''_{tw}. □

Proof of Proposition 2.2. As a consequence of (HS), there exists a bounded linear map

$$\mathcal{L}_{\mathrm{inv}} : L^2 \to H^2 \times \mathbb{R} \tag{7.48}$$

so that for any $h \in L^2$, the pair $(v, d) = \mathcal{L}_{\mathrm{inv}} h$ is the unique solution in $H^2 \times \mathbb{R}$ to the problem

$$\mathcal{L}_{\mathrm{tw}} v = h - \Phi'_0 d. \tag{7.49}$$

Indeed, we take $d = \langle h, \psi_{\mathrm{tw}} \rangle_{L^2}$, which in view of the normalisation (2.14) ensures that the right-hand side of (7.49) is in the range of $\mathcal{L}_{\mathrm{tw}}$.

It now suffices to find a solution to the fixed-point problem

$$(v, d) = -\mathcal{L}_{\mathrm{inv}} \mathcal{M}_{\sigma;\Phi_0,c_0}(v, d). \tag{7.50}$$

Upon introducing the set

$$\mathcal{Z}_\Theta = \big\{(v, d) \in H^2 \times \mathbb{R} : \|v\|_{H^2} + |d| \le \min\{1, \Theta\sigma^2\}\big\} \subset H^2 \times \mathbb{R} \tag{7.51}$$

and applying Corollary 7.6, we see that for any $(v, d) \in \mathcal{Z}_\Theta$ we have

$$\|\mathcal{M}_{\sigma;\Phi_0,c_0}(v, d)\|_{L^2} \le K\big(\Theta^4\sigma^4 + \sigma^2\big) = K\sigma^2\big(\Theta^2\sigma^2 + 1\big), \tag{7.52}$$

while for any two pairs $(v_A, d_A) \in \mathcal{Z}_\Theta$ and $(v_B, d_B) \in \mathcal{Z}_\Theta$ we have

$$\|\mathcal{M}_{\sigma;\Phi_0,c_0}(v_A, d_A) - \mathcal{M}_{\sigma;\Phi_0,c_0}(v_B, d_B)\|_{L^2} \le K\sigma^2\big[1 + 2\Theta\big]\big[\, \|v_A - v_B\|_{H^1} + |d_A - d_B|\,\big]. \tag{7.53}$$

In particular, choosing Θ to be sufficiently large and $\delta_\sigma > 0$ to be sufficiently small, we see that the map $-\mathcal{L}_{\mathrm{inv}}\mathcal{M}_{\sigma;\Phi_0,c_0}$ is a contraction on $\mathcal{Z}_\Theta$ for all $0 \le \sigma \le \delta_\sigma$. □

Proof of Proposition 2.3. We first recall that

$$\langle \Phi_{\mathrm{ref}}, \psi'_{\mathrm{tw}} \rangle_{L^2} = -\langle \Phi'_{\mathrm{ref}}, \psi_{\mathrm{tw}} \rangle_{L^2} = -\langle \Phi'_0, \psi_{\mathrm{tw}} \rangle_{L^2} = -1. \tag{7.54}$$

Writing $u_0 = x_0 + \Phi_{\mathrm{ref}}$, this allows us to compute

$$\begin{array}{rcl} \langle v_\gamma, \psi_{\mathrm{tw}} \rangle_{L^2} & = & \langle x_0 + \Phi_{\mathrm{ref}}, T_\gamma \psi_{\mathrm{tw}} \rangle_{L^2} - \langle \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} \\ & = & \langle x_0 + \Phi_{\mathrm{ref}}, \psi_{\mathrm{tw}} - \gamma \psi'_{\mathrm{tw}} + \mathcal{N}_{\mathrm{tw}}(\gamma) \rangle_{L^2} - \langle \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} \\ & = & \gamma + \langle x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} \\ & & \quad + \mathcal{E}_\sigma(x_0, \gamma), \end{array} \tag{7.55}$$

in which we have introduced the expression

$$\mathcal{E}_\sigma(x_0, \gamma) = -\gamma \langle x_0, \psi'_{\mathrm{tw}} \rangle_{L^2} + \langle x_0 + \Phi_{\mathrm{ref}}, \mathcal{N}_{\mathrm{tw}}(\gamma) \rangle_{L^2}. \tag{7.56}$$

Using Lemma 7.7, we obtain the estimate

$$|\mathcal{E}_\sigma(x_0, \gamma)| \le C_1 \, \|x_0\|_{L^2} \, |\gamma| + C_1 \big[1 + \|x_0\|_{L^2} \big] \gamma^2, \tag{7.57}$$

together with the Lipschitz bound

$$\begin{array}{rcl} \|\mathcal{E}_\sigma(x_0, \gamma_A) - \mathcal{E}_\sigma(x_0, \gamma_B)\|_{L^2} & \le & C_2 \, \|x_0\|_{L^2} \, |\gamma_A - \gamma_B| \\ & & \quad + C_2 \big[1 + \|x_0\|_{L^2} \big] \big[\, |\gamma_A| + |\gamma_B| \, \big] \, |\gamma_A - \gamma_B| \, . \end{array} \tag{7.58}$$

In particular, upon choosing $\delta_{\mathrm{fix}} > 0$ to be sufficiently small and imposing the restriction

$$\|x_0\|_{L^2} + \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} < \delta_{\mathrm{fix}}, \tag{7.59}$$

we can define γ_0 as the unique solution to the fixed-point problem

$$-\gamma = \langle x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} + \mathcal{E}_\sigma(x_0, \gamma) \tag{7.60}$$

on the set

$$\Sigma_{x_0} = \{ \gamma : |\gamma| \le 2 \, \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} \, \|\psi_{\mathrm{tw}}\|_{L^2} \}. \tag{7.61}$$

By choosing $\delta_\sigma > 0$ and $\delta_0 > 0$ to be sufficiently small, the bound (2.49) allows us to conclude that (7.59) is satisfied whenever (2.50) holds.

For any $\gamma \in \mathbb{R}$ we can compute

$$\begin{array}{rcl} \|T_{-\gamma} \Phi_\sigma - \Phi_\sigma\|^2_{L^2} & = & \int \big(\Phi_\sigma(\xi + \gamma) - \Phi_\sigma(\xi) \big)^2 \, d\xi \\ & = & \int \big[\int_0^\gamma \Phi'_\sigma(\xi + s) \, ds \big]^2 \, d\xi \\ & \le & \int |\gamma| \int_0^\gamma \Phi'_\sigma(\xi + s)^2 \, ds \, d\xi \\ & = & |\gamma|^2 \int \Phi'_\sigma(\xi)^2 \, d\xi \\ & = & |\gamma|^2 \, \|\Phi'_\sigma\|^2_{L^2} \, . \end{array} \tag{7.62}$$

In particular, we obtain the bound

$$\begin{array}{rcl} \|v_{\gamma_0}\|_{L^2} & = & \|x_0 + \Phi_{\mathrm{ref}} - T_{\gamma_0} \Phi_\sigma\|_{L^2} \\ & \le & \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} + \|T_{\gamma_0} \Phi_\sigma - \Phi_\sigma\|_{L^2} \\ & \le & \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} + C_3 \, |\gamma_0| \, . \end{array} \tag{7.63}$$

The desired estimate (2.53) hence follows from $\gamma_0 \in \Sigma_{x_0}$. The final estimate (2.54) follows in a similar fashion, exploiting $\Phi''_\sigma \in L^2$. □

Proof of Proposition 2.6. For convenience, we introduce the notation

$$\alpha_\sigma = \big[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{1/2}. \tag{7.64}$$

Using the definitions (2.68) one easily verifies the identities

$$\Phi_\sigma'(\xi) = \alpha_\sigma \Phi_0'(\alpha_\sigma \xi), \qquad \Phi_\sigma''(\xi) = \alpha_\sigma^2 \Phi_0''(\alpha_\sigma \xi), \tag{7.65}$$

which yields

$$g\big(\Phi_\sigma(\xi)\big) = g\big(\Phi_0(\alpha_\sigma\xi)\big) = \vartheta_0 \Phi_0'\big(\alpha_\sigma\xi\big) = \vartheta_0 \alpha_\sigma^{-1}\Phi_\sigma'(\xi), \tag{7.66}$$

together with

$$f(\Phi_\sigma) + c_\sigma \Phi_\sigma' = -\alpha_\sigma^{-2} A_* \Phi_\sigma. \tag{7.67}$$

Since the cut-off functions in the definition of b act as the identity for small $\sigma \geq 0$, we obtain

$$\begin{array}{rcl} b(\Phi_\sigma, \psi_{\rm tw}) & = & -\vartheta_0 \alpha_\sigma^{-1}, \\ \kappa_\sigma(\Phi_\sigma, \psi_{\rm tw}) & = & 1 + \frac{1}{2\rho}\vartheta_0^2 \alpha_\sigma^{-2}, \end{array} \tag{7.68}$$

which implies

$$\begin{array}{rcl} \mathcal{J}_\sigma(\Phi_\sigma, c_\sigma, \psi_{\rm tw}) & = & [1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\alpha_\sigma^{-2}]^{-1}\big[f(\Phi_\sigma) + c_\sigma\Phi_\sigma' - \sigma^2\vartheta_0^2\alpha_\sigma^{-2}\Phi_\sigma''\big] \\ & = & -[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\alpha_\sigma^{-2}]^{-1}\big[\alpha_\sigma^{-2}A_*\Phi_\sigma + \frac{\sigma^2}{\rho}\vartheta_0^2\alpha_\sigma^{-2}A_*\Phi_\sigma\big] \\ & = & -[\alpha_\sigma^2 + \frac{1}{2\rho}\sigma^2\vartheta_0^2]^{-1}\big[1 + \frac{\sigma^2}{\rho}\vartheta_0^2\big]A_*\Phi_\sigma \\ & = & -A_*\Phi_\sigma. \end{array} \tag{7.69}$$

The claims now follow from the uniqueness statement in Proposition 2.2. □

8 Bounds on mild nonlinearities

In this section we set out to obtain bounds on the nonlinearities $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$ and $\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}$ defined in (6.22)-(6.23). In addition, we show that our choices (2.27) and (2.29) for a_σ and b prevent these nonlinearities from having a component in the subspace of L^2 on which the semigroup $S(t)$ does not decay, provided the cut-offs are not hit.

Our main result below shows that the construction of Φ_σ has eliminated all $\mathcal{O}(1)$-terms from the deterministic nonlinearity $\overline{\mathcal{R}}$, leaving only a small linear contribution together with the expected higher order terms. It is important to note here that these higher order terms depend at most quadratically on $\|v\|_{H^1}$, besides powers of $\|v\|_{L^2}$.

In general, the stochastic nonlinearity $\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}$ will have an $\mathcal{O}(1)$-term, but we have an explicit expression for this contribution so we also discuss the case when this contribution disappears. In both cases, the higher order terms depend at most linearly on $\|v\|_{H^1}$.

Proposition 8.1. *Consider the setting of Proposition 2.2 and recall the definitions (6.22) and (6.23). Then there exists $K > 0$ so that for any $0 \leq \sigma \leq \delta_\sigma$ and any $v \in H^1$, the following properties hold true.*

(i) We have the bound

$$\big\|\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v)\big\|_{L^2} \quad \leq \quad K\sigma^2\|v\|_{H^1} + K\|v\|_{H^1}^2\big[1 + \|v\|_{L^2}^2 + \sigma^2\|v\|_{L^2}^3\big]. \tag{8.1}$$

(ii) We have the estimate

$$\big\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)\big\|_{L^2} \leq K\big[1 + \|v\|_{H^1}\big]. \tag{8.2}$$

(iii) If the inequality

$$\|v\|_{L^2} \le \min\{1, [4\,\|\psi_{\rm tw}\|_{H^1}]^{-1}\} \tag{8.3}$$

holds, then we have the identities

$$\langle \overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v), \psi_{\rm tw}\rangle_{L^2} = \langle \overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v), \psi_{\rm tw}\rangle_{L^2} = 0. \tag{8.4}$$

(iv) If the identity

$$g(\Phi_\sigma) = -b(\Phi_\sigma, \psi_{\rm tw})\Phi_\sigma' \tag{8.5}$$

holds, then we have the bound

$$\big\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)\big\|_{L^2} \le K\,\|v\|_{H^1}\,. \tag{8.6}$$

In order to derive a compact expression for $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$, it is convenient to recall the definition (7.2) and introduce the function

$$\overline{\mathcal{R}}_{\sigma;I}(v) = \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(v,0) - \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(0,0). \tag{8.7}$$

We note that the bounds in Corollary 7.5 are directly applicable to this function.

Lemma 8.2. *Consider the setting of Proposition 2.2. Then for any $0 \le \sigma \le \delta_\sigma$ and $v \in H^1$, we have the identity*

$$\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v) = \overline{\mathcal{R}}_{\sigma;I}(v) - \Big[\chi_{\rm low}\big(\langle \partial_\xi[\Phi_\sigma + v], \psi_{\rm tw}\rangle_{L^2}\big)\Big]^{-1}\langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\rm tw}\rangle_{L^2}[\Phi_\sigma' + v']. \tag{8.8}$$

Proof. Inspecting (7.2) and using the defining property (2.48) for (Φ_σ, c_σ), we see that

$$-\mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(0,0) = A_*\Phi_\sigma + \mathcal{J}_0(\Phi_\sigma, c_\sigma). \tag{8.9}$$

Applying (7.2) once more, we hence find

$$\begin{array}{rcl} \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\rm tw}) & = & \mathcal{J}_0(\Phi_\sigma, c_\sigma) + [\mathcal{L}_{\rm tw} - A_*]v + \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(v,0) \\ & = & [\mathcal{L}_{\rm tw} - A_*]v - A_*\Phi_\sigma + \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(v,0) - \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(0,0) \\ & = & [\mathcal{L}_{\rm tw} - A_*]v - A_*\Phi_\sigma + \overline{\mathcal{R}}_{\sigma;I}(v). \end{array} \tag{8.10}$$

Writing

$$\mathcal{I}_\sigma(v) \quad = \quad \langle \Phi_\sigma + v, A_*\psi_{\rm tw}\rangle_{L^2} + \langle \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\rm tw}), \psi_{\rm tw}\rangle_{L^2} \tag{8.11}$$

and using $\mathcal{L}_{\rm tw}^{\rm adj}\psi_{\rm tw} = 0$, we may compute

$$\begin{array}{rcl} \mathcal{I}_\sigma(v) & = & \langle \Phi_\sigma, A_*\psi_{\rm tw}\rangle_{L^2} + \langle v, [A_* - \mathcal{L}_{\rm tw}^{\rm adj}]\psi_{\rm tw}\rangle_{L^2} \\ & & \quad + \langle \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\rm tw}), \psi_{\rm tw}\rangle_{L^2} \\ & = & \langle A_*\Phi_\sigma, \psi_{\rm tw}\rangle_{L^2} + \langle [A_* - \mathcal{L}_{\rm tw}]v, \psi_{\rm tw}\rangle_{L^2} \\ & & \quad + \langle \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\rm tw}), \psi_{\rm tw}\rangle_{L^2} \\ & = & \langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\rm tw}\rangle_{L^2}. \end{array} \tag{8.12}$$

In view of the definition (2.29) for a_σ, we now obtain

$$\begin{array}{rcl} \kappa_\sigma(\Phi_\sigma + v, \psi_{\rm tw})^{-1} a_\sigma\big(\Phi_\sigma + v, c_\sigma, \psi_{\rm tw}\big) & = & -\Big[\chi_{\rm low}\big(\langle \partial_\xi[\Phi_\sigma + v], \psi_{\rm tw}\rangle_{L^2}\big)\Big]^{-1}\mathcal{I}_\sigma(v) \\ & = & -\Big[\chi_{\rm low}\big(\langle \partial_\xi[\Phi_\sigma + v], \psi_{\rm tw}\rangle_{L^2}\big)\Big]^{-1}\langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\rm tw}\rangle_{L^2}. \end{array} \tag{8.13}$$

In particular, the desired identity (8.8) follows directly from the definition (6.22). □

Lemma 8.3. *Consider the setting of Proposition 2.2. Then there exists $K > 0$ so that for any $v \in H^1$ and $0 \le \sigma \le \delta_\sigma$ we have the bound*

$$\left\|\overline{\mathcal{R}}_{\sigma;I}(v)\right\|_{L^2} \quad \le \quad K\sigma^2 \left\|v\right\|_{H^1} + K \left\|v\right\|_{H^1}^2 \left[1 + \left\|v\right\|_{L^2} + \sigma^2 \left\|v\right\|_{L^2}^2\right], \tag{8.14}$$

together with

$$\left|\langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\mathrm{tw}}\rangle_{L^2}\right| \quad \le \quad K \left\|v\right\|_{L^2} \left[\sigma^2 + \left\|v\right\|_{L^2}\right] + K \left\|v\right\|_{H^1} \left[\, \left\|v\right\|_{L^2}^2 + \sigma^2 \left\|v\right\|_{L^2}^3\right]. \tag{8.15}$$

Proof. Applying Corollary 7.5, we find

$$\begin{aligned}\left\|\overline{\mathcal{R}}_{\sigma;I}(v)\right\|_{L^2} \quad \le \quad & C_1\left[1 + \left\|v\right\|_{H^1}\right] \left\|v\right\|_{H^1} \left\|v\right\|_{L^2} \\ & +C_1\left[\sigma^2 + \left\|v\right\|_{H^1}\right] \left\|v\right\|_{H^1} \\ & +C_1\sigma^2 \left\|v\right\|_{H^1}^2 \left\|v\right\|_{L^2} \left\|v\right\|_{L^2} \\ & +C_1\sigma^2 \left\|v\right\|_{H^1} \left\|v\right\|_{L^2} \left\|v\right\|_{H^1},\end{aligned} \tag{8.16}$$

together with

$$\begin{aligned}\left|\langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\mathrm{tw}}\rangle_{L^2}\right| \quad \le \quad & C_2\left[1 + \left\|v\right\|_{H^1}\right] \left\|v\right\|_{L^2} \left\|v\right\|_{L^2} \\ & +C_2\left[\sigma^2 + \left\|v\right\|_{L^2}\right] \left\|v\right\|_{L^2} \\ & +C_2\sigma^2 \left\|v\right\|_{H^1} \left\|v\right\|_{L^2}^2 \left\|v\right\|_{L^2} \\ & +C_2\sigma^2 \left\|v\right\|_{L^2}^2 \left\|v\right\|_{H^1}.\end{aligned} \tag{8.17}$$

These expressions can be absorbed into (8.14) and (8.15). □

Lemma 8.4. *Consider the setting of Proposition 2.2. Then there exists $K > 0$ so that for any $0 \le \sigma \le \delta_\sigma$ and any $v \in H^1$ we have the bound*

$$\left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v) - \overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\right\|_{L^2} \le K \left\|v\right\|_{H^1}. \tag{8.18}$$

Proof. Writing

$$\begin{aligned}\mathcal{I} \quad = \quad & \overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v) - \overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0) \\ = \quad & \kappa_\sigma(\Phi_\sigma + v, \psi_{\mathrm{tw}})^{-1/2}\Big[g(\Phi_\sigma + v) + b(\Phi_\sigma + v, \psi_{\mathrm{tw}})\partial_\xi[\Phi_\sigma + v]\Big] \\ & -\kappa_\sigma(\Phi_\sigma, \psi_{\mathrm{tw}})^{-1/2}\Big[g(\Phi_\sigma) + b(\Phi_\sigma, \psi_{\mathrm{tw}})\partial_\xi[\Phi_\sigma]\Big]\end{aligned} \tag{8.19}$$

and using Lemma's 3.2, 3.6, 3.8 and 3.9, we compute

$$\begin{aligned}\left\|\mathcal{I}\right\|_{L^2} \quad \le \quad & \Big|\nu_\sigma^{(-1/2)}(\Phi_\sigma + v, \psi_{\mathrm{tw}}) - \nu_\sigma^{(-1/2)}(\Phi_\sigma, \psi_{\mathrm{tw}})\Big| \left[\, \left\|g(\Phi_\sigma)\right\|_{L^2} + K_b \left\|\Phi_\sigma'\right\|_{L^2}\right] \\ & +K_\kappa \left\|g(\Phi_\sigma + v) - g(\Phi_\sigma)\right\|_{L^2} \\ & +K_\kappa \left|b(\Phi_\sigma + v, \psi_{\mathrm{tw}}) - b(\Phi_\sigma, \psi_{\mathrm{tw}})\right| \left\|\Phi_\sigma'\right\|_{L^2} \\ & +K_\kappa K_b \left\|v'\right\|_{L^2}.\end{aligned} \tag{8.20}$$

Applying these results once more, we find

$$\begin{aligned}\left\|\mathcal{I}\right\|_{L^2} \quad \le \quad & C_1\sigma^2 \left\|v\right\|_{L^2} + C_1 \left\|v\right\|_{L^2} + C_1 \left\|v\right\|_{L^2} + C_1 \left\|v\right\|_{H^1} \\ \le \quad & C_2 \left\|v\right\|_{H^1},\end{aligned} \tag{8.21}$$

as desired. □

Proof of Proposition 8.1. To obtain (i), we use (8.8) together with Lemma 8.3 to compute

$$\begin{array}{rcl} \left\|\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v)\right\|_{L^2} & \leq & \left\|\overline{\mathcal{R}}_{\sigma;I}(v)\right\|_{L^2} + C_1 \left|\langle\overline{\mathcal{R}}_{\sigma;I}(v),\psi_{\mathrm{tw}}\rangle_{L^2}\right| \left[1+\|v\|_{H^1}\right] \\ & \leq & C_2\sigma^2\|v\|_{H^1} + C_2\|v\|_{H^1}^2\left[1+\|v\|_{L^2}+\sigma^2\|v\|_{L^2}^2\right] \\ & & +C_2\|v\|_{L^2}\left[\sigma^2+\|v\|_{L^2}\right]\left[1+\|v\|_{H^1}\right] \\ & & +C_2\|v\|_{H^1}\left[\|v\|_{L^2}^2+\sigma^2\|v\|_{L^2}^3\right]\left[1+\|v\|_{H^1}\right]. \end{array} \tag{8.22}$$

These terms can all be absorbed into (8.1).

The bound (ii) follows directly from Lemma 8.4, using the estimate

$$\left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)\right\|_{L^2} \leq \left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\right\|_{L^2} + \left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)-\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\right\|_{L^2} \tag{8.23}$$

and the a-priori bound

$$\left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\right\|_{L^2} \leq C_3. \tag{8.24}$$

The bound (iv) follows in the same fashion, since the condition (8.5) implies that

$$\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0) = 0. \tag{8.25}$$

Finally, (iii) follows from the identities (8.8) and (3.49), using the proof of Lemma 3.7 to show that the cut-off function χ_{low} in (8.8) acts as the identity. □

9 Nonlinear stability of mild solutions

In this section we prove Theorems 2.4 and 2.5, providing a orbital and an exponential stability result for the stochastic waves (Φ_σ, c_σ) on timescales of order σ^{-2}. Recalling the function (6.34), our key statement is that $E\overline{N}_{\varepsilon,\alpha}$ can be bounded in terms of itself, the noise-strength σ and the initial condition $\left\|\overline{V}(0)\right\|_{H^1}^2$. This requires a number of technical regularity estimates, which we obtain in §9.2-9.3.

In order to prevent cumbersome notation and to highlight the broad applicability of our techniques here, we do not refer to the specific functions $\overline{V}$ and the specific nonlinearities $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$ here. Instead, we assume the following general condition concerning the form of our nonlinearities.

(hFB) We have $\|B_{\mathrm{cn}}\|_{L^2} = K_{B;\mathrm{cn}} < \infty$ and the maps

$$F_{\mathrm{lin}} : H^1 \to L^2, \qquad F_{\mathrm{nl}} : H^1 \to L^2, \qquad B_{\mathrm{lin}} : H^1 \to L^2 \tag{9.1}$$

satisfy the bounds

$$\begin{array}{rcl} \|F_{\mathrm{lin}}(v)\|_{L^2} & \leq & K_{F;\mathrm{lin}}\|v\|_{H^1}, \\ \|F_{\mathrm{nl}}(v)\|_{L^2} & \leq & K_{F;\mathrm{nl}}\|v\|_{H^1}^2(1+\|v\|_{L^2}^m), \\ \|B_{\mathrm{lin}}(v)\|_{L^2} & \leq & K_{B;\mathrm{lin}}\|v\|_{H^1} \end{array} \tag{9.2}$$

for some $m > 0$. In addition, there exists $\eta_0 > 0$ so that

$$\langle\sigma^2 F_{\mathrm{lin}}(v) + F_{\mathrm{nl}}(v), \psi_{\mathrm{tw}}\rangle_{L^2} = 0, \qquad \langle B_{\mathrm{cn}} + B_{\mathrm{lin}}(v), \psi_{\mathrm{tw}}\rangle_{L^2} = 0 \tag{9.3}$$

whenever $\|v\|_{L^2} \leq \eta_0$.

Using the nonlinearities above, we can discuss the mild formulation of the SPDE that we are interested in. At present, we simply assume that a solution is a-priori available, but one can also set out to construct such a solution directly.

(hSol) For any $T > 0$, there exists a continuous $(\mathcal{F}_t)$-adapted process $V : \Omega \times [0,T] \to L^2$ for which we have the inclusions

$$V \in \mathcal{N}^2\big([0,T];(\mathcal{F}_t);H^1\big), \qquad B_{\mathrm{lin}}(V) \in \mathcal{N}^2\big([0,T];(\mathcal{F}_t);L^2\big). \tag{9.4}$$

In addition, for almost all $\omega \in \Omega$ we have the inclusions

$$F_{\mathrm{lin}}\big(V(\cdot,\omega)\big) \in L^1([0,T];L^2), \qquad F_{\mathrm{nl}}\big(V(\cdot,\omega)\big) \in L^1([0,T];L^2) \tag{9.5}$$

together with the identity

$$\begin{aligned} V(t) \quad = \quad & S(t)V(0) + \sigma^2 \int_0^t S(t-s)F_{\mathrm{lin}}\big(V(s)\big)\,ds + \int_0^t S(t-s)F_{\mathrm{nl}}\big(V(s)\big)\,ds \\ & +\sigma \int_0^t S(t-s)B_{\mathrm{cn}}\,d\beta_s + \sigma \int_0^t S(t-s)B_{\mathrm{lin}}\big(V(s)\big)\,d\beta_s, \end{aligned} \tag{9.6}$$

which holds for all $t \in [0,T]$. Finally, we have $\langle V(0), \psi_{\mathrm{tw}}\rangle_{L^2} = 0$.

For any $\varepsilon > 0$ and $\alpha \geq 0$, we recall the notation

$$N_{\varepsilon,\alpha}(t) = e^{\alpha t}\|V(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)}e^{\alpha s}\|V(s)\|_{H^1}^2\,ds. \tag{9.7}$$

For any $T > 0$ and $\eta > 0$, we introduce the $(\mathcal{F}_t)$-stopping time

$$\tau_{\varepsilon,\alpha}(T,\eta) = \inf\Big\{0 \leq t < T : N_{\varepsilon,\alpha}(t) > \eta\Big\}, \tag{9.8}$$

writing $\tau_{\varepsilon,\alpha}(T,\eta) = T$ if the set is empty. Our two main results here, which we establish in §9.3 provide bounds on the expectation of $\sup_{0\leq t\leq \tau_{\varepsilon,\alpha}(T,\eta)} N_{\varepsilon,\alpha}(t)$.

Proposition 9.1. *Assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) are satisfied. Pick a constant $0 < \varepsilon < \beta$, together with two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any $T > 1$, any $0 < \eta \leq \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma T^{-1/2}$ we have the bound*

$$E \sup_{0\leq t\leq \tau_{\varepsilon,0}(T,\eta)} N_{\varepsilon,0}(t) \quad \leq \quad K\Big[\|V(0)\|_{H^1}^2 + \sigma^2 T\Big]. \tag{9.9}$$

Proposition 9.2. *Assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) are satisfied and that $B_{\mathrm{cn}} = 0$. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \frac{\alpha}{2} < \beta$, together with two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any $T > 1$, any $0 < \eta \leq \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma T^{-1/2}$ we have the bound*

$$E \sup_{0\leq t\leq \tau_{\varepsilon,\alpha}(T,\eta)} N_{\varepsilon,\alpha}\big(t\big) \quad \leq \quad K\|V(0)\|_{H^1}^2. \tag{9.10}$$

Exploiting the technique used in Stannat [49], these bounds can be turned into estimates concerning the probabilities

$$p_{\varepsilon,\alpha}(T,\eta) = P\Big(\sup_{0\leq t\leq T}\big[N_{\varepsilon,\alpha}(t)\big] > \eta\Big). \tag{9.11}$$

This allows our main stability theorems to be established.

Corollary 9.3. *Consider the setting of Proposition 9.1. Then there exists a constant $K > 0$ so that for any $T > 1$, any $0 < \eta \leq \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma T^{-1/2}$, we have the bound*

$$p_{\varepsilon,0}(T,\eta) \leq \eta^{-1}K\big[\|V(0)\|_{H^1}^2 + \sigma^2 T\big]. \tag{9.12}$$

Proof. Upon computing

$$\begin{aligned} \eta p_{\varepsilon,0}(T,\eta) \quad & = \quad \eta P\big(\tau_{\varepsilon,0}(T,\eta) < T\big) \\ & = \quad E\Big[\mathbf{1}_{\tau_{\varepsilon,0}(T,\eta)<T} N_{\varepsilon,0}\big(\tau_{\varepsilon,0}(T,\eta)\big)\Big] \\ & \leq \quad E N_{\varepsilon,0}\big(\tau_{\varepsilon,0}(T,\eta)\big) \\ & \leq \quad E \sup_{0\leq t\leq \tau_{\varepsilon,0}(T,\eta)} N_{\varepsilon,0}(t), \end{aligned} \tag{9.13}$$

the result follows from (9.9). □

Corollary 9.4. *Consider the setting of Proposition 9.2. Then there exists a constant $K > 0$ so that for any $T > 1$, any $0 < \eta \leq \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma T^{-1/2}$ we have the bound*

$$p_{\varepsilon,\alpha}(T, \eta) \leq \eta^{-1} K \|V(0)\|_{H^1}^2. \tag{9.14}$$

Proof. Upon computing

$$\begin{aligned} \eta p_{\varepsilon,\alpha}(T,\eta) &= \eta P\big(\tau_{\varepsilon,\alpha}(T,\eta) < T\big) \\ &= E\Big[\mathbf{1}_{\tau_{\varepsilon,\alpha}(T,\eta)<T} N_{\varepsilon,\alpha}\big(\tau_{\varepsilon,\alpha}(T,\eta)\big)\Big] \\ &\leq E N_{\varepsilon,\alpha}\big(\tau_{\varepsilon,\alpha}(T,\eta)\big) \\ &\leq E \sup_{0\leq t\leq \tau_{\varepsilon,\alpha}(T,\eta)} N_{\varepsilon,\alpha}\big(t\big), \end{aligned} \tag{9.15}$$

the result follows from (9.10). □

Proof of Theorems 2.4 and 2.5. On account of Propositions 2.3 and 6.3, the map $\overline{V}$ defined in (6.6) satisfies the conditions of (hSol) with $\big(\overline{\beta}_\tau, \overline{\mathcal{F}}_\tau\big)_{\tau\geq 0}$ as the relevant Brownian motion. In addition, Proposition 8.1 guarantees that (hFB) is satisfied. The desired estimates now follow from Corollaries 9.3 and 9.4, using Proposition 6.4 to reverse the time-transform. □

9.1 Setup

In order to establish Propositions 9.1-9.2 we need to estimate each of the terms featuring in the identity (9.6). The regularity structure of the semigroup $S(t)$ is crucial for our purposes here, so we discuss this in some detail using the terminology used in [25, §10].

In particular, for any $0 < \varphi < \pi$ we introduce the sector

$$\Sigma_\varphi = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \varphi\}, \tag{9.16}$$

in which we take $\arg(z) \in (-\pi, \pi)$. We recall that a linear operator $\mathcal{L} : D(\mathcal{L}) \subset X \to X$ on a Banach space X is called sectorial if the spectrum of $\mathcal{L}$ is contained in $\overline{\Sigma}_\omega$ for some $0 < \omega(\mathcal{L}) < \frac{\pi}{2}$, while the resolvent operators $R(z, \mathcal{L}) = (z - \mathcal{L})^{-1}$ satisfy the bound

$$\sup_{z\in\mathbb{C}\setminus\overline{\Sigma}_{\omega(\mathcal{L})}} \|zR(z,\mathcal{L})\|_{\mathcal{L}(X,X)} < \infty. \tag{9.17}$$

Our spectral assumptions (HS) combined with the fact that $\mathcal{L}_{\mathrm{tw}}$ is a lower-order perturbation to the diffusion operator A_* guarantee that $-\mathcal{L}_{\mathrm{tw}}$ is sectorial. This means that $\mathcal{L}_{\mathrm{tw}}$ generates an analytic semigroup. In order to isolate the behaviour caused by the neutral eigenmode, we introduce the map $Q : L^2 \to L^2$ that acts as

$$Qv = v - \langle v, \psi_{\mathrm{tw}}\rangle_{L^2} \Phi_0'. \tag{9.18}$$

This projection allows us to formulate several important estimates.

Lemma 9.5 (see [38])**.** *Assume that (HTw) and (HS) hold and consider the analytic semigroup $S(t)$ generated by $\mathcal{L}_{\mathrm{tw}}$. Then there is a constant $M \geq 1$ for which we have the bounds*

$$\begin{aligned} \|S(t)Q\|_{\mathcal{L}(L^2,L^2)} &\leq Me^{-\beta t}, && 0 < t < \infty, \\ \|S(t)Q\|_{\mathcal{L}(L^2,H^1)} &\leq Mt^{-\frac{1}{2}}, && 0 < t \leq 2, \\ \|S(t)Q\|_{\mathcal{L}(L^2,H^1)} &\leq Me^{-\beta t}, && t \geq 1, \\ \|[\mathcal{L}_{\mathrm{tw}} - A_*]S(t)Q\|_{\mathcal{L}(L^2,L^2)} &\leq Mt^{-\frac{1}{2}}, && 0 < t \leq 2, \\ \Big\|[\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} - A_*]S(t)Q\Big\|_{\mathcal{L}(L^2,L^2)} &\leq Mt^{-\frac{1}{2}}, && 0 < t \leq 2. \end{aligned} \tag{9.19}$$

In order to understand the combination $S(t)Q$ as an independent semigroup, we introduce the spaces

$$L^2_Q = \{v \in L^2 : (I - Q)v = 0\}, \qquad H^2_Q = \{v \in H^2 : (I - Q)v = 0\} \tag{9.20}$$

and consider the operator $\mathcal{L}^Q_{\mathrm{tw}} : H^2_Q \to L^2_Q$ that arises upon restricting $\mathcal{L}_{\mathrm{tw}}$ to act on H^2_Q. Note that this is well-defined since $\mathrm{Range}\big(\mathcal{L}_{\mathrm{tw}}\big) = L^2_Q$. For any $\theta \in \mathbb{R}$, we now introduce the linear operators

$$B_\theta = -\big[\mathcal{L}_{\mathrm{tw}} + \theta\big], \qquad B^Q_\theta = -\big[\mathcal{L}^Q_{\mathrm{tw}} + \theta\big]. \tag{9.21}$$

Lemma 9.6. *Assume that (HTw) and (HS) hold and pick any $0 \le \theta \le \beta$. Then the operator B^Q_θ is sectorial on L^2_Q and the semigroup generated by $-B^Q_\theta$ corresponds with the restriction of $e^{\theta t} S(t)$ to L^2_Q.*

Proof. Note first that $\mathcal{L}^Q_{\mathrm{tw}}$ is bijective since we have projected out the one-dimensional kernel. For any $v \in L^2_Q$ and λ in the resolvent set of $\mathcal{L}_{\mathrm{tw}}$, we may compute

$$\begin{array}{lcl} 0 & = & (I - Q)\mathcal{L}_{\mathrm{tw}} R(\lambda, \mathcal{L}_{\mathrm{tw}})v \\ & = & (I - Q)\big[-v + \lambda R(\lambda, \mathcal{L}_{\mathrm{tw}})v\big] \\ & = & \lambda (I - Q) R(\lambda, \mathcal{L}_{\mathrm{tw}})v. \end{array} \tag{9.22}$$

which implies that $R(\lambda, \mathcal{L}_{\mathrm{tw}})v \in L^2_Q$. In particular, the resolvent set of $\mathcal{L}_{\mathrm{tw}}$ is contained in the resolvent set of $\mathcal{L}^Q_{\mathrm{tw}}$. The stated properties now follow in a standard fashion; see for example [38, Prop 3.1.5]. □

In order to define our final regularity concept, we need to introduce the Hardy spaces

$$\begin{array}{lcl} H^1(\Sigma_\varphi) & = & \{f : \Sigma_\varphi \to \mathbb{C} \text{ holomorphic for which} \\ & & \qquad \|f\|_{H^1(\Sigma_\varphi)} := \sup_{|\nu| < \varphi} \int_0^\infty t^{-1} f(e^{i\nu} t)\, dt < \infty\}, \\ H^\infty(\Sigma_\varphi) & = & \{f : \Sigma_\varphi \to \mathbb{C} \text{ holomorphic for which} \\ & & \qquad \|f\|_{H^\infty(\Sigma_\varphi)} := \sup_{z \in \Sigma_\varphi} |f(z)| < \infty\}. \end{array} \tag{9.23}$$

If $\mathcal{L}$ is sectorial on a Banach space X, then for any $\omega(\mathcal{L}) < \varphi < \pi$ and any $h \in H^1(\Sigma_\varphi)$ one can define

$$h(\mathcal{L}) = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} R(z, \mathcal{L}) h(z)\, dz \in \mathcal{L}(X, X) \tag{9.24}$$

by picking an arbitrary $\nu \in (\omega(\mathcal{L}), \varphi)$ and traversing the boundary in a downward fashion, keeping the spectrum of $\mathcal{L}$ on the left. It is however unclear if this integral converges if we take $h \in H^\infty(\Sigma_\varphi)$. The following result states that this is indeed the case for the sectorial operators discussed in Lemma 9.6. Indeed, one can use a density argument to extend the conclusion to the whole space $H^\infty(\Sigma_\varphi)$. Operators with this property are said to admit a bounded H^∞-calculus, which is crucial for our stochastic regularity estimates.

Lemma 9.7. *Assume that (HTw) and (HS) hold and pick any $0 \le \theta \le \beta$. There exists $\varphi \in (\omega(B^Q_\theta), \frac{\pi}{2})$ together with a constant $K > 0$ so that for any $h \in H^1(\Sigma_\varphi) \cap H^\infty(\Sigma_\varphi)$ we have*

$$\left\| h(B^Q_\theta) \right\| \le K \, \|h\|_{H^\infty_\varphi} . \tag{9.25}$$

Proof. Since $\mathcal{L}_{\mathrm{tw}} - A_*$ is a first order differential operator with continuous coefficients, the perturbation theory described in [56, §8] can be applied to our setting. In particular, we can find constants $\Theta_0 \gg 1$ and $C_1 > 0$ together with an angle $\varphi_0 \in (\omega(B_{-\Theta_0}), \frac{\pi}{2})$ for which

$$\|h(B_{-\Theta_0})\|_{\mathcal{L}(L^2, L^2)} \le C_1 \, \|h\|_{H^\infty_{\varphi_0}} \tag{9.26}$$

holds for all $h \in H^1(\Sigma_{\varphi_0}) \cap H^\infty(\Sigma_{\varphi_0})$. By restriction, we hence also have

$$\left\| h(B^Q_{-\Theta_0}) \right\|_{\mathcal{L}(L^2_Q, L^2_Q)} \leq \| h(B_{-\Theta_0}) \|_{\mathcal{L}(L^2, L^2)} C_1 \| h \|_{H^\infty_{\varphi_0}} \tag{9.27}$$

for all such h. Fix two constants

$$\max\{\omega(B^Q_\theta), \varphi_0\} < \nu < \varphi < \frac{\pi}{2} \tag{9.28}$$

and pick $h \in H^1(\Sigma_\varphi) \cap H^\infty(\Sigma_\varphi)$. Using the resolvent identity, we may compute

$$\begin{array}{rcl} h(B^Q_\theta) - h(B^Q_{-\Theta_0}) & = & \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} h(z) \big[R(z, B^Q_\theta) - R(z, B^Q_{-\Theta_0}) \big] \, dz \\ & = & \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} h(z) \big[R(z, B^Q_\theta) - R(z - \theta - \Theta_0, B^Q_\theta) \big] \, dz \\ & = & (\theta + \Theta_0) \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} h(z) R(z, B^Q_\theta) R(z - \theta - \Theta_0, B^Q_\theta) \, dz. \end{array} \tag{9.29}$$

Since zero is contained in the resolvent set of B^Q_θ, there exists $C_2 > 0$ for which the estimate

$$\left\| R(z, B^Q_\theta) R(z - \theta - \Theta_0, B^Q_\theta) \right\|_{\mathcal{L}(L^2_Q, L^2_Q)} \leq \frac{C_2}{1 + |z|^2} \tag{9.30}$$

holds for all $z \in \partial \Sigma_\nu$. This decays sufficently fast to ensure that

$$\left\| h(B^Q_\theta) - h(B^Q_{-\Theta_0}) \right\|_{\mathcal{L}(L^2_Q, L^2_Q)} \leq C_3 \| h \|_{H^\infty_\varphi} \tag{9.31}$$

for some $C_3 > 0$ that does not depend on the choice of h. The desired bound now follows from the inequality

$$\| h \|_{H^\infty_{\varphi_0}} \leq \| h \|_{H^\infty_\varphi} . \tag{9.32}$$

□

Now that the formal framework has been set up, we are ready to return to the quantity $N_{\varepsilon,\alpha}(t)$ defined in (9.7), which is the main object of our interest. For convenience, we use the shorthand notation $\tau = \tau_{\varepsilon,\alpha}(T, \eta)$ ubiquitously throughout the remainder of this section. Writing $\nu = \alpha + \varepsilon$, we introduce the splitting

$$\begin{array}{rcl} N_{\varepsilon,\alpha;I}(t) & = & e^{\alpha t} \| V(t) \|^2_{L^2}, \\ N_{\varepsilon,\alpha;II}(t) & = & \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \| V(s) \|^2_{H^1} \, ds \\ & = & e^{-\varepsilon t} \int_0^t e^{\nu s} \| V(s) \|^2_{H^1} \, ds. \end{array} \tag{9.33}$$

In order to understand $N_{\varepsilon,\alpha;I}$, we introduce the expression

$$\mathcal{E}_0(t) = S(t) Q V(0), \tag{9.34}$$

together with the long-term integrals

$$\begin{array}{rcl} \mathcal{E}^{\rm lt}_{F;\rm lin}(t) & = & \int_0^{t-1} S(t-s) Q F_{\rm lin}\big(V(s)\big) \mathbf{1}_{s<\tau} \, ds, \\ \mathcal{E}^{\rm lt}_{F;\rm nl}(t) & = & \int_0^{t-1} S(t-s) Q F_{\rm nl}\big(V(s)\big) \mathbf{1}_{s<\tau} \, ds, \\ \mathcal{E}^{\rm lt}_{B;\rm lin}(t) & = & \int_0^{t-1} S(t-s) Q B_{\rm lin}\big(V(s)\big) \mathbf{1}_{s<\tau} \, d\beta_s, \\ \mathcal{E}^{\rm lt}_{B;\rm cn}(t) & = & \int_0^{t-1} S(t-s) Q B_{\rm cn} \mathbf{1}_{s<\tau} \, d\beta_s \end{array} \tag{9.35}$$

and their short-term counterparts

$$
\begin{array}{rcl}
\mathcal{E}^{\rm sh}_{F;\rm lin}(t) & = & \int_{t-1}^t S(t-s) Q F_{\rm lin}\big(V(s)\big) \mathbf{1}_{s < \tau} \, ds, \\
\mathcal{E}^{\rm sh}_{F;\rm nl}(t) & = & \int_{t-1}^t S(t-s) Q F_{\rm nl}\big(V(s)\big) \mathbf{1}_{s < \tau} \, ds, \\
\mathcal{E}^{\rm sh}_{B;\rm lin}(t) & = & \int_{t-1}^t S(t-s) Q B_{\rm lin}\big(V(s)\big) \mathbf{1}_{s < \tau} \, d\beta_s, \\
\mathcal{E}^{\rm sh}_{B;\rm cn}(t) & = & \int_{t-1}^t S(t-s) Q B_{\rm cn} \mathbf{1}_{s < \tau} \, d\beta_s.
\end{array}
\tag{9.36}
$$

Here we use the convention that integrands are set to zero for $s < 0$. For convenience, we also write

$$
\mathcal{E}_{F;\#}(t) = \mathcal{E}^{\rm lt}_{F;\#}(t) + \mathcal{E}^{\rm sh}_{F;\#}(t) \tag{9.37}
$$

for $\# \in \{\rm lin, nl\}$ and

$$
\mathcal{E}_{B;\#}(t) = \mathcal{E}^{\rm lt}_{B;\#}(t) + \mathcal{E}^{\rm sh}_{B;\#}(t) \tag{9.38}
$$

for $\# \in \{\rm lin, cn\}$.

Turning to the terms in (9.6) that are relevant for evaluating $N_{\varepsilon,\alpha;II}$, we introduce the expression

$$
\mathcal{I}_{\nu,\delta;0}(t) \quad = \quad \int_0^t e^{\nu s} \left\| S(\delta) \mathcal{E}_0(s) \right\|_{H^1}^2 \, ds, \tag{9.39}
$$

together with

$$
\begin{array}{rcl}
\mathcal{I}^{\#}_{\nu,\delta;F;\rm lin}(t) & = & \int_0^t e^{\nu s} \left\| S(\delta) \mathcal{E}^{\#}_{F;\rm lin}(s) \right\|_{H^1}^2 \, ds, \\
\mathcal{I}^{\#}_{\nu,\delta;F;\rm nl}(t) & = & \int_0^t e^{\nu s} \left\| S(\delta) \mathcal{E}^{\#}_{F;\rm nl}(s) \right\|_{H^1}^2 \, ds, \\
\mathcal{I}^{\#}_{\nu,\delta;B;\rm lin}(t) & = & \int_0^t e^{\nu s} \left\| S(\delta) \mathcal{E}^{\#}_{B;\rm lin}(s) \right\|_{H^1}^2 \, ds, \\
\mathcal{I}^{\#}_{\nu,\delta;B;\rm cn}(t) & = & \int_0^t e^{\nu s} \left\| S(\delta) \mathcal{E}^{\#}_{B;\rm cn}(s) \right\|_{H^1}^2 \, ds
\end{array}
\tag{9.40}
$$

for $\# \in \{\rm lt, sh\}$. The extra $S(\delta)$ factor will be used to ensure that all the integrals we encounter are well-defined. We emphasize that all our estimates are uniform in $0 < \delta < 1$, allowing us to take $\delta \downarrow 0$. The estimates concerning $\mathcal{I}^{\rm sh}_{\nu,\delta;F;\rm nl}$ and $\mathcal{I}^{\rm sh}_{\nu,\delta;B;\rm lin}$ in Lemma's 9.12 and 9.18 are particularly delicate in this respect, as a direct application of the bounds in Lemma 9.5 would result in expressions that diverge as $\delta \downarrow 0$.

9.2 Deterministic regularity estimates

In this part we set out to analyze the deterministic integrals in (9.6). The main complication is that we only have integrated control over the squared H^1-norm of V. This is particularly delicate for $\mathcal{I}^{\rm sh}_{\nu,\delta;F;\rm nl}$, where the nonlinearity itself is quadratic in V.

Lemma 9.8. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \frac{\alpha}{2} < \beta$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \leq \delta < 1$ and any $0 \leq t \leq T$, we have the bound*

$$
e^{\alpha t} \left\| \mathcal{E}_0(t) \right\|_{L^2}^2 \quad \leq \quad M^2 e^{-\varepsilon t} \left\| V(0) \right\|_{L^2}^2, \tag{9.41}
$$

together with

$$
e^{-\varepsilon t} \mathcal{I}_{\nu,\delta;0}(t) \quad \leq \quad \tfrac{M^2}{2\beta - \nu} e^{-\varepsilon t} \left\| V(0) \right\|_{H^1}^2, \tag{9.42}
$$

Proof. We compute

$$
\begin{array}{rcl}
e^{\alpha t} \| \mathcal{E}_0(t) \|_{L^2}^2 & \leq & M^2 e^{\alpha t} e^{-2\beta t} \| V(0) \|_{L^2}^2 \\
& \leq & M^2 e^{-\varepsilon t} \| V(0) \|_{L^2}^2,
\end{array}
\tag{9.43}
$$

together with

$$\begin{array}{lll} e^{-\varepsilon t}\mathcal{I}_{\nu,\delta;0}(t) & \leq & M^2 e^{-\varepsilon t}\int_0^t e^{\nu s}e^{-2\beta(s+\delta)}\left\|V(0)\right\|_{H^1}^2\,ds \\ & \leq & \frac{M^2}{2\beta-\nu}e^{-\varepsilon t}\|V(0)\|_{H^1}^2. \end{array} \tag{9.44}$$

□

Lemma 9.9. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \frac{\alpha}{2} < \beta$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \leq \delta < 1$ and any $0 \leq t \leq \tau$, we have the bound*

$$e^{\alpha t}\left\|\mathcal{E}_{F;\mathrm{lin}}(t)\right\|_{L^2}^2 \quad \leq \quad K_{F;\mathrm{lin}}^2\frac{M^2}{2\beta-\nu}N_{\varepsilon,\alpha;II}(t), \tag{9.45}$$

together with

$$e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{lin}}(t) \quad \leq \quad K_{F;\mathrm{lin}}^2\frac{M^2}{2(\beta+\frac{\alpha}{2}-\nu)\varepsilon}N_{\varepsilon,\alpha;II}(t). \tag{9.46}$$

Proof. We first observe that

$$\begin{array}{lll} \|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 & \leq & K_{F;\mathrm{lin}}^2 M^2\left(\int_0^t e^{-\beta(t-s)}\left\|V(s)\right\|_{H^1}ds\right)^2, \\ \left\|S(\delta)\mathcal{E}^{\mathrm{lt}}_{F;\mathrm{lin}}(t)\right\|_{H^1}^2 & \leq & K_{F;\mathrm{lin}}^2 M^2\left(\int_0^t e^{-\beta(t-s)}\left\|V(s)\right\|_{H^1}ds\right)^2. \end{array} \tag{9.47}$$

This allows us to compute

$$\begin{array}{lll} e^{\alpha t}\|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 & \leq & K_{F;\mathrm{lin}}^2 M^2 e^{\alpha t}\left(\int_0^t e^{-(\beta-\frac{\nu}{2})(t-s)}e^{-\frac{\nu}{2}(t-s)}\left\|V(s)\right\|_{H^1}ds\right)^2 \\ & \leq & K_{F;\mathrm{lin}}^2\frac{M^2}{2\beta-\nu}e^{\alpha t}\int_0^t e^{-\nu(t-s)}\left\|V(s)\right\|_{H^1}^2 ds \\ & = & K_{F;\mathrm{lin}}^2\frac{M^2}{2\beta-\nu}N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.48}$$

Exploiting the inequality $2\beta - \nu > \varepsilon$, we write

$$\gamma_2 = \frac{\varepsilon+\nu}{2\beta} < 1 \tag{9.49}$$

and observe that

$$2\gamma_2\beta - \nu = \varepsilon. \tag{9.50}$$

Upon fixing $\gamma_1 = 1 - \gamma_2$, we readily see that

$$2\gamma_1\beta = 2\beta - \varepsilon - \nu = 2(\beta + \frac{\alpha}{2} - \nu). \tag{9.51}$$

This allows us to compute

$$\begin{array}{lll} e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{lin}}(t) & \leq & K_{F;\mathrm{lin}}^2 M^2 e^{-\varepsilon t}\int_0^t e^{\nu s}\left(\int_0^s e^{-\beta(s-s')}\left\|V(s')\right\|_{H^1}ds'\right)^2 ds \\ & \leq & K_{F;\mathrm{lin}}^2 M^2 e^{-\varepsilon t}\int_0^t e^{\nu s}\left(\int_0^s e^{-2\gamma_1\beta(s-s')}ds'\right)\left(\int_0^s e^{-2\gamma_2\beta(s-s')}\left\|V(s')\right\|_{H^1}^2 ds'\right) ds \\ & \leq & K_{F;\mathrm{lin}}^2\frac{M^2}{2\gamma_1\beta}e^{-\varepsilon t}\int_0^t e^{\nu s}\int_0^s e^{-2\gamma_2\beta(s-s')}\left\|V(s')\right\|_{H^1}^2 ds'\,ds \\ & = & K_{F;\mathrm{lin}}^2\frac{M^2}{2\gamma_1\beta}e^{-\varepsilon t}\int_0^t\int_{s'}^t e^{\nu s}e^{-2\gamma_2\beta(s-s')}\left\|V(s')\right\|_{H^1}^2 ds\,ds' \\ & = & K_{F;\mathrm{lin}}^2\frac{M^2}{2\gamma_1\beta}e^{-\varepsilon t}\int_0^t\left[\int_{s'}^t e^{-(2\gamma_2\beta-\nu)s}ds\right]e^{2\gamma_2\beta s'}\left\|V(s')\right\|_{H^1}^2 ds' \\ & \leq & K_{F;\mathrm{lin}}^2\frac{M^2}{(2\gamma_1\beta)(2\gamma_2\beta-\nu)}e^{-\varepsilon t}\int_0^t e^{-(2\gamma_2\beta-\nu)s'}e^{2\gamma_2\beta s'}\left\|V(s')\right\|_{H^1}^2 ds' \\ & = & K_{F;\mathrm{lin}}^2\frac{M^2}{(2\gamma_1\beta)(2\gamma_2\beta-\nu)}e^{-\varepsilon t}\int_0^t e^{\nu s'}\left\|V(s')\right\|_{H^1}^2 ds' \\ & = & K_{F;\mathrm{lin}}^2\frac{M^2}{2(\beta+\frac{\alpha}{2}-\nu)\varepsilon}N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.52}$$

□

Lemma 9.10. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \leq \delta < 1$ and any $0 \leq t \leq \tau$, we have the bound*

$$e^{-\varepsilon t}\mathcal{I}^{\rm sh}_{\nu,\delta;F;{\rm lin}}(t) \quad \leq \quad 4e^{\nu}M^2K^2_{F;{\rm lin}}N_{\varepsilon,\alpha;II}(t). \tag{9.53}$$

Proof. Using Cauchy-Schwarz, we compute

$$\begin{array}{lcl} e^{-\varepsilon t}\mathcal{I}^{\rm sh}_{\nu,\delta;F;{\rm lin}}(t) & \leq & M^2K^2_{F;{\rm lin}}e^{-\varepsilon t}\int_0^t e^{\nu s}\left(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\|V(s')\|_{H^1}ds'\right)^2 ds \\ & \leq & M^2K^2_{F;{\rm lin}}e^{-\varepsilon t}\int_0^t e^{\nu s}\left(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\,ds'\right)\left(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\|V(s')\|^2_{H^1}ds'\right) ds \\ & \leq & 2M^2K^2_{F;{\rm lin}}e^{-\varepsilon t}\int_0^t e^{\nu s}\left(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\|V(s')\|^2_{H^1}ds'\right) ds \\ & = & 2M^2K^2_{F;{\rm lin}}e^{-\varepsilon t}\int_0^t \left[\int_{s'}^{\min\{t,s'+1\}} e^{\nu s}\frac{1}{\sqrt{s+\delta-s'}}\,ds\right]\|V(s')\|^2_{H^1}\,ds' \\ & \leq & 4e^{\nu}M^2K^2_{F;{\rm lin}}e^{-\varepsilon t}\int_0^t e^{\nu s'}\|V(s')\|^2_{H^1}\,ds' \\ & = & 4e^{\nu}M^2K^2_{F;{\rm lin}}N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.54}$$

□

Lemma 9.11. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \frac{\alpha}{2} < \beta$ and write $\nu = \alpha + \varepsilon$. Then for any $\eta > 0$, any $0 \leq \delta < 1$ and any $0 \leq t \leq \tau$, we have the bound*

$$e^{\alpha t}\left\|\mathcal{E}_{F;{\rm nl}}(t)\right\|^2_{L^2} \quad \leq \quad \eta K^2_{F;{\rm nl}}M^2(1+\eta^m)^2N_{\varepsilon,\alpha;II}(t), \tag{9.55}$$

together with

$$e^{-\varepsilon t}\mathcal{I}^{\rm lt}_{\nu,\delta;F;{\rm nl}}(t) \quad \leq \quad \eta K^2_{F;{\rm nl}}(1+\eta^m)^2\frac{M^2}{\beta+\frac{\alpha}{2}-\nu}N_{\varepsilon,\alpha;II}(t). \tag{9.56}$$

Proof. We first notice that

$$\begin{array}{lcl} \|\mathcal{E}_{F;{\rm nl}}(t)\|^2_{L^2} & \leq & K_{F;{\rm nl}}(1+\eta^m)^2M^2\left(\int_0^t e^{-\beta(t-s)}\left\|V(s)\right\|^2_{H^1}\,ds\right)^2, \\ \left\|S(\delta)\mathcal{E}^{\rm lt}_{F;{\rm nl}}(t)\right\|^2_{H^1} & \leq & K_{F;{\rm nl}}(1+\eta^m)^2M^2\left(\int_0^t e^{-\beta(t-s)}\left\|V(s)\right\|^2_{H^1}\,ds\right)^2. \end{array} \tag{9.57}$$

Using $\beta > \nu - \frac{1}{2}\alpha = \frac{1}{2}\alpha + \varepsilon$, we compute

$$\begin{array}{lcl} \int_0^t e^{-\beta(t-s)}\left\|V(s)\right\|^2_{H^1}\,ds & = & e^{\frac{\alpha}{2}t}\int_0^t e^{-\beta(t-s)}e^{-\frac{\alpha}{2}t}\left\|V(s)\right\|^2_{H^1}\,ds \\ & \leq & e^{\frac{\alpha}{2}t}\int_0^t e^{-\beta(t-s)}e^{-\frac{\alpha}{2}(t-s)}\left\|V(s)\right\|^2_{H^1}\,ds \\ & \leq & e^{\frac{\alpha}{2}t}\int_0^t e^{-\nu(t-s)}\left\|V(s)\right\|^2_{H^1}\,ds. \end{array} \tag{9.58}$$

This yields the desired bound

$$\begin{array}{lcl} e^{\alpha t}\|\mathcal{E}_{F;{\rm nl}}(t)\|^2_{L^2} & \leq & K^2_{F;{\rm nl}}(1+\eta^m)^2M^2e^{\alpha t}\left(\int_0^t e^{-\beta(t-s)}\left\|V(s)\right\|^2_{H^1}\,ds\right)^2 \\ & \leq & K^2_{F;{\rm nl}}(1+\eta^m)^2M^2e^{2\alpha t}\left(\int_0^t e^{-\nu(t-s)}\left\|V(s)\right\|^2_{H^1}\,ds\right)^2 \\ & \leq & K^2_{F;{\rm nl}}(1+\eta^m)^2M^2\eta N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.59}$$

In a similar spirit, we compute

$$\begin{array}{rcl}
e^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{nl}}(t) & \leq & K^2_{F;\mathrm{nl}}(1+\eta^m)^2 M^2 e^{-\varepsilon t}\int_0^t e^{\nu s}\Big(\int_0^s e^{-\beta(s-s')}\left\|V(s')\right\|^2_{H^1}\, ds'\Big)^2\, ds \\
& \leq & K^2_{F;\mathrm{nl}}(1+\eta^m)^2 M^2 e^{-\varepsilon t}\int_0^t e^{\nu s} e^{\frac{\alpha}{2}s}\Big(\int_0^s e^{-\nu(s-s')}\left\|V(s')\right\|^2_{H^1}\, ds'\Big) \\
& & \qquad \times\Big(\int_0^s e^{-\beta(s-s')}\left\|V(s')\right\|^2_{H^1}\, ds'\Big)\, ds \\
& \leq & \eta K^2_{F;\mathrm{nl}}(1+\eta^m)^2 M^2 e^{-\varepsilon t}\int_0^t e^{\nu s} e^{-\frac{\alpha}{2}s}\int_0^s e^{-\beta(s-s')}\left\|V(s')\right\|^2_{H^1}\, ds'\, ds \\
& = & \eta K^2_{F;\mathrm{nl}}(1+\eta^m)^2 M^2 e^{-\varepsilon t}\int_0^t\Big[\int_{s'}^t e^{-(\frac{\alpha}{2}-\nu+\beta)s}\, ds\Big] e^{\beta s'}\left\|V(s')\right\|^2_{H^1}\, ds' \\
& \leq & \eta K^2_{F;\mathrm{nl}}(1+\eta^m)^2 \frac{M^2}{\beta+\frac{\alpha}{2}-\nu} e^{-\varepsilon t}\int_0^t e^{-(\frac{\alpha}{2}-\nu+\beta)s'} e^{\beta s'}\left\|V(s')\right\|^2_{H^1}\, ds' \\
& = & \eta K^2_{F;\mathrm{nl}}(1+\eta^m)^2 \frac{M^2}{\beta+\frac{\alpha}{2}-\nu} e^{-\varepsilon t}\int_0^t e^{\nu s'} e^{-\frac{\alpha}{2}s'}\left\|V(s')\right\|^2_{H^1}\, ds' \\
& \leq & \eta K^2_{F;\mathrm{nl}}(1+\eta^m)^2 \frac{M^2}{\beta+\frac{\alpha}{2}-\nu} N_{\varepsilon,\alpha;II}(t).
\end{array} \tag{9.60}$$

□

Lemma 9.12. *Fix $T>0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon>0$, $\alpha\geq 0$ and write $\nu=\alpha+\varepsilon$. Then for any $\eta>0$, any $0\leq\delta<1$ and any $0\leq t\leq\tau_{\varepsilon,\alpha}(T,\eta)$, we have the bound*

$$e^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;F,\mathrm{nl}}(t) \quad\leq\quad \eta M^2 K^2_{F;\mathrm{nl}}(1+\eta^m)^2(1+\rho^{-1})e^{3\nu}(4+\nu)N_{\varepsilon,\alpha;II}(t). \tag{9.61}$$

Proof. We start by observing that

$$\left\|v\right\|^2_{H^1} = \left\|v\right\|^2_{L^2} + \rho^{-1}\left\|A_*^{1/2}v\right\|^2_{L^2}. \tag{9.62}$$

In addition, for any $w\in L^2$, $\vartheta>0$, $\vartheta_A\geq 0$ and $\vartheta_B\geq 0$ we have

$$\begin{array}{rcl}
\frac{d}{d\vartheta}\langle S(\vartheta+\vartheta_A)w, S(\vartheta+\vartheta_B)w\rangle_{L^2} & = & \langle\mathcal{L}_{\mathrm{tw}}S(\vartheta+\vartheta_A)w, S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& & \quad +\langle S(\vartheta+\vartheta_A)w, \mathcal{L}_{\mathrm{tw}}S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& = & \langle S(\vartheta+\vartheta_A)w, \mathcal{L}^{\mathrm{adj}}_{\mathrm{tw}}S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& & \quad +\langle S(\vartheta+\vartheta_A)w, \mathcal{L}_{\mathrm{tw}}S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& = & \langle S(\vartheta+\vartheta_A)w, \big[\mathcal{L}^{\mathrm{adj}}_{\mathrm{tw}}-A_*\big]S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& & \quad +\langle S(\vartheta+\vartheta_A)w,\ \big[\mathcal{L}_{\mathrm{tw}}-A_*\big]S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& & \quad -2\langle A_*^{1/2}S(\vartheta+\vartheta_A)w, A_*^{1/2}S(\vartheta+\vartheta_B)w\rangle_{L^2}.
\end{array} \tag{9.63}$$

Assume for the moment that $\delta>0$. For convenience, we introduce the expression

$$\mathcal{E}_{s,s',s'';\mathcal{H}} = \langle S(s+\delta-s')QF_{\mathrm{nl}}\big(V(s')\big), S(s+\delta-s'')QF_{\mathrm{nl}}\big(V(s'')\big)\rangle_{\mathcal{H}}, \tag{9.64}$$

where we allow $\mathcal{H}\in\{L^2,H^1\}$. Exploiting (9.63) and the fact that $\delta>0$, we obtain the bound

$$\begin{array}{rcl}
\mathcal{E}_{s,s',s'';H^1} & \leq & M^2K^2_{F;\mathrm{nl}}(1+\eta^m)^2\left\|V(s')\right\|^2_{H^1}\left\|V(s'')\right\|^2_{H^1} \\
& & \quad +M^2K^2_{F;\mathrm{nl}}(1+\eta^m)^2\rho^{-1}\frac{1}{\sqrt{s+\delta-s''}}\left\|V(s')\right\|^2_{H^1}\left\|V(s'')\right\|^2_{H^1} \\
& & \quad -\rho^{-1}\frac{1}{2}\frac{d}{ds}\mathcal{E}_{s,s',s'';L^2}
\end{array} \tag{9.65}$$

for the values of (s, s', s'') that are relevant below. Upon introducing the integrals

$$\begin{array}{lll} \mathcal{I}_I & = & e^{-\varepsilon t} \int_0^t e^{\nu s} \int_{s-1}^s \int_{s-1}^s \big[1 + \frac{1}{\sqrt{s+\delta-s''}}\big] \|V(s')\|_{H^1}^2 \, \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \, ds, \\ \mathcal{I}_{II} & = & e^{-\varepsilon t} \int_0^t e^{\nu s} \int_{s-1}^s \int_{s-1}^s \frac{d}{ds} \mathcal{E}_{s,s',s'';L^2} ds'' \, ds' \, ds, \end{array} \tag{9.66}$$

we hence readily obtain the estimate

$$e^{-\varepsilon t} \mathcal{I}^{\mathrm{sh}}_{\nu,\delta;F;\mathrm{nl}}(t) \quad \leq \quad (1+\rho^{-1}) M^2 K_{F;\mathrm{nl}}^2 (1+\eta^m)^2 \mathcal{I}_I - \tfrac{1}{2}\rho^{-1} \mathcal{I}_{II}. \tag{9.67}$$

Changing the order of the integrals, we find

$$\begin{array}{lll} \mathcal{I}_I & = & e^{-\varepsilon t} \int_0^t \int_{s'-1}^{\min\{t,s'+1\}} \Big[\int_{\max\{s',s''\}}^{\min\{t,s'+1,s''+1\}} e^{\nu s} \big[1 + \frac{1}{\sqrt{s+\delta-s''}}\big] \, ds \Big] \|V(s')\|_{H^1}^2 \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \\ & \leq & 3 e^{-\varepsilon t} \int_0^t e^{\nu s'} e^{\nu} \|V(s')\|_{H^1}^2 \int_{s'-1}^{\min\{t,s'+1\}} \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \\ & \leq & 3 e^{-\varepsilon t} \int_0^t e^{\nu s'} e^{\nu} \|V(s')\|_{H^1}^2 e^{2\nu} \int_{s'-1}^{\min\{t,s'+1\}} e^{-\nu(\min\{t,s'+1\} - s'')} \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \\ & \leq & 3 \eta e^{3\nu} e^{-\varepsilon t} \int_0^t e^{\nu s'} \|V(s')\|_{H^1}^2 e^{-\alpha \min\{t,s'+1\}} \, ds' \\ & \leq & 3 \eta e^{3\nu} N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.68}$$

In a similar fashion, we may use an integration by parts to write

$$\begin{array}{lll} \mathcal{I}_{II} & = & e^{-\varepsilon t} \int_0^t \int_{s'-1}^{\min\{t,s'+1\}} \Big[\int_{\max\{s',s''\}}^{\min\{t,s'+1,s''+1\}} e^{\nu s} \frac{d}{ds} \mathcal{E}_{s,s',s'';L^2} \, ds \Big] \, ds'' \, ds' \\ & = & \mathcal{I}_{II;A} + \mathcal{I}_{II;B} + \mathcal{I}_{II;C}, \end{array} \tag{9.69}$$

in which we have introduced

$$\begin{array}{lll} \mathcal{I}_{II;A} & = & e^{-\varepsilon t} \int_0^t \int_{s'-1}^{\min\{t,s'+1\}} e^{\nu \min\{t,s'+1,s''+1\}} \mathcal{E}_{\min\{t,s'+1,s''+1\},s',s'';L^2} \, ds'' \, ds', \\ \mathcal{I}_{II;B} & = & -e^{-\varepsilon t} \int_0^t \int_{s'-1}^{\min\{t,s'+1\}} e^{\nu \max\{s',s''\}} \mathcal{E}_{\max\{s',s''\},s',s'';L^2} \, ds'' \, ds', \\ \mathcal{I}_{II;C} & = & -e^{-\varepsilon t} \int_0^t \int_{s'-1}^{\min\{t,s'+1\}} \Big[\int_{\max\{s',s''\}}^{\min\{t,s'+1,s''+1\}} \nu e^{\nu s} \mathcal{E}_{s,s',s'';L^2} \, ds \Big] \, ds'' \, ds'. \end{array} \tag{9.70}$$

Note here that $\mathcal{I}_{II;B}$ is well defined because $\delta > 0$. A direct inspection of these terms yields the bound

$$\begin{array}{lll} |\mathcal{I}_{II}| & \leq & e^{\nu}(2+\nu) M^2 K_{F;\mathrm{nl}}^2 (1+\eta^m)^2 e^{-\varepsilon t} \int_0^t e^{\nu s'} \, \|V(s')\|_{H^1}^2 \int_{s'-1}^{\min\{t,s'+1\}} \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \\ & \leq & e^{\nu}(2+\nu) M^2 K_{F;\mathrm{nl}}^2 (1+\eta^m)^2 e^{-\varepsilon t} \int_0^t e^{\nu s'} \, \|V(s')\|_{H^1}^2 \, e^{2\nu} \\ & & \qquad \times \int_{s'-1}^{\min\{t,s'+1\}} e^{-\nu(\min\{t,s'+1\} - s'')} \, \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \\ & \leq & \eta e^{3\nu}(2+\nu) M^2 K_{F;\mathrm{nl}}^2 (1+\eta^m)^2 e^{-\varepsilon t} \int_0^t e^{\nu s'} \, \|V(s')\|_{H^1}^2 \, e^{-\alpha \min\{t,s'+1\}} \, ds' \\ & \leq & \eta e^{3\nu}(2+\nu) M^2 K_{F;\mathrm{nl}}^2 (1+\eta^m)^2 N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.71}$$

It hence remains to consider the case $\delta = 0$. We may apply Fatou's lemma to conclude

$$\begin{array}{lll} \mathcal{I}^{\mathrm{sh}}_{\nu,0;F;\mathrm{nl}}(t) & = & \int_0^t e^{\nu s} (\lim_{\delta \to 0} \|S(\delta) \mathcal{E}^{\mathrm{sh}}_{B;\mathrm{lin}}(s)\|_{H^1})^2 \mathbf{1}_{s<\tau} \, ds \\ & \leq & \liminf_{\delta \to 0} \mathcal{I}^{\mathrm{sh}}_{\nu,\delta;F;\mathrm{nl}}(t). \end{array} \tag{9.72}$$

The result now follows from the fact that the bounds obtained above do not depend on δ. □

9.3 Stochastic regularity estimates

We are now ready to discuss the stochastic integrals in (9.6). These require special care because they cannot be bounded in a pathwise fashion, unlike the deterministic integrals above. Expectations of suprema are particularly delicate in this respect. Indeed, the powerful Burkholder-Davis-Gundy inequalities cannot be directly applied to the stochastic convolutions that arise in our mild formulation. However, the H^∞-calculus obtained in Lemma 9.7 allows us to use the following mild version, which is the source of the extra T factors that appear in our estimates.

Lemma 9.13. *Fix $T > 0$ and assume that (HA), (HTw), (HS) and (Hβ) all hold. There exists a constant $K_{\mathrm{cnv}} > 0$ so that for any $W \in \mathcal{N}^2([0,T];(\mathcal{F})_t;L^2)$ and any $0 \le \alpha \le 2\beta$ we have*

$$E \sup_{0 \le t \le T} e^{\alpha t} \left\| \int_0^t S(t-s) Q W(s)\, d\beta_s \right\|_{L^2}^2 \le K_{\mathrm{cnv}} E \int_0^T e^{\alpha s} \|W(s)\|_{L^2}^2 \, ds. \tag{9.73}$$

Proof. Lemma 9.7 implies that the generator $B^Q_\alpha = \mathcal{L}_{\mathrm{tw}} + \frac{1}{2}\alpha$ of the semigroup $e^{\frac{1}{2}\alpha t} S(t)$ on L^2_Q satisfies assumption (H) in [53]. On account of the identity

$$e^{\alpha t} \left\| \int_0^t S(t-s) Q W(s)\, d\beta_s \right\|_{L^2}^2 = \left\| \int_0^t e^{\frac{1}{2}\alpha(t-s)} S(t-s) Q e^{\frac{1}{2}\alpha s} W(s)\, d\beta_s \right\|_{L^2}^2, \tag{9.74}$$

the desired estimate is now an immediate consequence of [53, Thm. 1.1]. □

Lemma 9.14. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Then for any pair of constants $\varepsilon > 0$ and $0 \le \alpha \le 2\beta$ we have the bound*

$$E \sup_{0 \le t \le \tau} e^{\alpha t} \|\mathcal{E}_{B;\mathrm{lin}}(t)\|_{L^2}^2 \quad \le \quad (T+1) K_{\mathrm{cnv}} K_{B;\mathrm{lin}}^2 e^{\varepsilon} E \sup_{0 \le t \le \tau} N_{\varepsilon,\alpha;II}(t). \tag{9.75}$$

Proof. Using Lemma 9.13 we compute

$$\begin{array}{lcl} E \sup_{0 \le t \le \tau} e^{\alpha t} \|\mathcal{E}_{B;\mathrm{lin}}(t)\|_{L^2}^2 & \le & E \sup_{0 \le t \le T} e^{\alpha t} \|\mathcal{E}_{B;\mathrm{lin}}(t)\|_{L^2}^2 \\ & = & E \sup_{0 \le t \le T} e^{\alpha t} \left\| \int_0^t S(t-s) Q B_{\mathrm{lin}}\big(V(s)\big) \mathbf{1}_{s<\tau}\, d\beta_s \right\|_{L^2}^2 \\ & \le & K_{\mathrm{cnv}} E \int_0^T e^{\alpha s} \left\| B_{\mathrm{lin}}\big(V(s)\big) \mathbf{1}_{s<\tau} \right\|_{L^2}^2 ds \\ & \le & K_{\mathrm{cnv}} K_{B;\mathrm{lin}}^2 E \int_0^\tau e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds. \end{array} \tag{9.76}$$

By dividing up the integral, we obtain

$$\begin{array}{lcl} \int_0^\tau e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds & \le & e^{\varepsilon} \int_0^1 e^{-\varepsilon(1-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<\tau}\, ds \\ & & \quad + e^{\varepsilon} \int_1^2 e^{-\varepsilon(2-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<\tau}\, ds \\ & & \quad + \ldots + e^{\varepsilon} \int_{\lfloor T \rfloor}^{\lfloor T \rfloor + 1} e^{-\varepsilon(\lfloor T \rfloor + 1 - s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<\tau}\, ds \\ & \le & (T+1) e^{\varepsilon} \sup_{0 \le t \le T+1} \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \mathbf{1}_{s<\tau}\, ds \\ & \le & (T+1) e^{\varepsilon} \sup_{0 \le t \le \tau} \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds \\ & = & (T+1) e^{\varepsilon} \sup_{0 \le t \le \tau} N_{\varepsilon,\alpha;II}(t), \end{array} \tag{9.77}$$

which yields the desired bound upon taking expectations. □

Lemma 9.15. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Then we have the bound*

$$E \sup_{0 \le t \le \tau} \|\mathcal{E}_{B;\mathrm{cn}}(t)\|_{L^2}^2 \quad \le \quad T K_{\mathrm{cnv}} K_{B;\mathrm{cn}}^2. \tag{9.78}$$

Proof. This bound follows directly from (9.76) by picking $\alpha = 0$ and making the substitutions

$$K_{B;\mathrm{lin}} \mapsto K_{B;\mathrm{cn}}, \qquad \|V(s)\|_{H^1}^2 \mapsto 1. \tag{9.79}$$

□

We now set out to bound the expectation of the supremum of the remaining double integrals $\mathcal{I}^{\#}_{\nu,\delta;B;\mathrm{lin}}(t)$ and $\mathcal{I}^{\#}_{\nu,\delta;B;\mathrm{cn}}(t)$ with $\# \in \{\mathrm{lt}, \mathrm{sh}\}$. This is performed in Lemma's 9.20 and 9.21, but we first compute several time independent bounds for the expectation of the integrals themselves.

Lemma 9.16. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick constants $\varepsilon > 0$, $\alpha \geq 0$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \leq \delta < 1$ and $0 \leq t \leq T$, we have the identities*

$$\begin{array}{lcl} E\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;B;\mathrm{lin}}(t) & = & E\int_0^t e^{\nu s} \int_0^{s-1} \big\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\big\|_{L^2}^2 \mathbf{1}_{s'<\tau}\, ds'\, ds, \\ E\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;B;\mathrm{cn}}(t) & = & E\int_0^t e^{\nu s} \int_0^{s-1} \|S(s+\delta-s')QB_{\mathrm{cn}}\|_{L^2}^2 \mathbf{1}_{s'<\tau}\, ds'\, ds, \end{array} \tag{9.80}$$

together with their short-time counterparts

$$\begin{array}{lcl} E\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{lin}}(t) & = & E\int_0^t e^{\nu s} \int_{s-1}^{s} \big\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\big\|_{L^2}^2 \mathbf{1}_{s'<\tau}\, ds'\, ds, \\ E\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{cn}}(t) & = & E\int_0^t e^{\nu s} \int_{s-1}^{s} \|S(s+\delta-s')QB_{\mathrm{cn}}\|_{L^2}^2 \mathbf{1}_{s'<\tau}\, ds'\, ds. \end{array} \tag{9.81}$$

Proof. The linearity of the expectation operator, the Itô Isometry (see e.g. [42, §2.3]) and the integrability of the integrands imply that

$$\begin{array}{lcl} E\mathcal{I}^{\mathrm{lt}}_{B;\mathrm{lin}}(t) & = & E\int_0^t e^{\nu s} \Big\|\int_0^{s-1} S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\mathbf{1}_{s'<\tau} d\beta_{s'}\Big\|_{H_1}^2 ds \\ & = & E\int_0^t e^{\nu s} \int_0^{s-1} \big\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\big\|_{L^2}^2 \mathbf{1}_{s'<\tau}\, ds'\, ds. \end{array} \tag{9.82}$$

The remaining expressions follow in a similar fashion. □

Lemma 9.17. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \alpha < 2\beta$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \leq \delta < 1$ and any $0 \leq t \leq T$, we have the bound*

$$Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;B;\mathrm{lin}}(t) \quad \leq \quad \tfrac{M^2}{2\beta-\nu} K_{B;\mathrm{lin}}^2 EN_{\varepsilon;\alpha;II}(t\wedge\tau). \tag{9.83}$$

Proof. Using (9.80) and switching the integration order, we obtain

$$\begin{array}{lcl} Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;B;\mathrm{lin}}(t) & \leq & M^2 K_{B;\mathrm{lin}}^2 Ee^{-\varepsilon t}\int_0^t e^{\nu s}\int_0^{s\wedge\tau} e^{-2\beta(s-s')}\|V(s')\|_{H^1}^2\, ds'\, ds \\ & = & M^2 K_{B;\mathrm{lin}}^2 Ee^{-\varepsilon t}\int_0^{t\wedge\tau}\Big[\int_{s'}^t e^{-(2\beta-\nu)s}\, ds\Big] e^{2\beta s'}\|V(s')\|_{H^1}^2\, ds' \\ & \leq & \tfrac{M^2}{2\beta-\nu} K_{B;\mathrm{lin}}^2 Ee^{-\varepsilon t}\int_0^{t\wedge\tau} e^{-(2\beta-\nu)s'} e^{2\beta s'}\|V(s')\|_{H^1}^2\, ds' \\ & = & \tfrac{M^2}{2\beta-\nu} K_{B;\mathrm{lin}}^2 Ee^{-\varepsilon t}\int_0^{t\wedge\tau} e^{\nu s'}\|V(s')\|_{H^1}^2\, ds' \\ & \leq & \tfrac{M^2}{2\beta-\nu} K_{B;\mathrm{lin}}^2 Ee^{-\varepsilon t\wedge\tau}\int_0^{t\wedge\tau} e^{\nu s'}\|V(s')\|_{H^1}^2\, ds' \\ & = & \tfrac{M^2}{2\beta-\nu} K_{B;\mathrm{lin}}^2 EN_{\varepsilon,\alpha;II}(t\wedge\tau). \end{array} \tag{9.84}$$

□

Lemma 9.18. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ and write $\nu = \alpha + \varepsilon$. Then for any $0 \leq \delta < 1$ and any $0 \leq t \leq T$, we have the bound*

$$Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{lin}}(t) \quad \leq \quad K_{B;\mathrm{lin}}^2 M^2(1+\rho^{-1})e^{\nu}(4+\nu)EN_{\varepsilon,\alpha;II}(t\wedge\tau). \tag{9.85}$$

Proof. We only consider the case $\delta > 0$ here, noting that the limit $\delta \downarrow 0$ can be handled as in the proof of Lemma 9.12. Applying the identity (9.63) with $\vartheta_A = \vartheta_B$, we obtain the bound

$$\begin{array}{rcl} \big\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\big\|_{H^1}^2 & \leq & M^2 K_{B;\mathrm{lin}}^2 \left\|V(s')\right\|_{H^1}^2 \\ & & +M^2 K_{B;\mathrm{lin}}^2 \rho^{-1} \frac{1}{\sqrt{s+\delta-s'}} \left\|V(s')\right\|_{H^1}^2 \\ & & -\rho^{-1}\frac{1}{2}\frac{d}{ds}\big\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\big\|_{L^2}^2 \end{array} \tag{9.86}$$

for the values of (s, s') that are relevant below. Upon writing

$$\begin{array}{rcl} \mathcal{I}_I & = & Ee^{-\varepsilon t}\int_0^t e^{\nu s}\int_{s-1}^s \big[1+\frac{1}{\sqrt{s+\delta-s'}}\big] \left\|V(s')\right\|_{H^1}^2 \mathbf{1}_{s'<\tau}\, ds'\, ds, \\ \mathcal{I}_{II} & = & Ee^{-\varepsilon t}\int_0^t e^{\nu s}\int_{s-1}^s \frac{d}{ds}\big\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\big\|_{L^2}^2 \mathbf{1}_{s'<\tau} ds'\, ds, \end{array} \tag{9.87}$$

we obtain the estimate

$$Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{lin}}(t) \quad \leq \quad (1+\rho^{-1})M^2 K_{B;\mathrm{lin}}^2 \mathcal{I}_I - \tfrac{1}{2}\rho^{-1}\mathcal{I}_{II}. \tag{9.88}$$

Changing the integration order, we obtain

$$\begin{array}{rcl} \mathcal{I}_I & = & Ee^{-\varepsilon t}\int_0^t \Big[\int_{s'}^{\min\{t,s'+1\}} e^{\nu s}\big[1+\frac{1}{\sqrt{s+\delta-s'}}\big]\, ds\Big] \|V(s')\|_{H^1}^2 \mathbf{1}_{s'<\tau}\, ds' \\ & \leq & 3e^{\nu} Ee^{-\varepsilon t}\int_0^t e^{\nu s'}\|V(s')\|_{H^1}^2 \mathbf{1}_{s'<\tau}\, ds' \\ & \leq & 3e^{\nu} Ee^{-\varepsilon t\wedge\tau}\int_0^{t\wedge\tau} e^{\nu s'}\|V(s')\|_{H^1}^2\, ds' \\ & = & 3e^{\nu} EN_{\varepsilon,\alpha;II}(t\wedge\tau). \end{array} \tag{9.89}$$

Integrating by parts, we arrive at the identity

$$\begin{array}{rcl} \mathcal{I}_{II} & = & Ee^{-\varepsilon t}\int_0^t \Big[\int_{s'}^{\min\{t,s'+1\}} e^{\nu s}\frac{d}{ds}\big\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\big\|_{L^2}^2\, ds\Big]\mathbf{1}_{s'<\tau}\, ds' \\ & = & \mathcal{I}_{II;A} + \mathcal{I}_{II;B} + \mathcal{I}_{II;C}, \end{array} \tag{9.90}$$

in which we have introduced the expressions

$$\begin{array}{rcl} \mathcal{I}_{II;A} & = & Ee^{-\varepsilon t}\int_0^t e^{\nu\min\{t,s'+1\}}\big\|S(\min\{t,s'+1\}+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\big\|_{L^2}^2 \mathbf{1}_{s'<\tau}\, ds', \\ \mathcal{I}_{II;B} & = & -Ee^{-\varepsilon t}\int_0^t e^{\nu s'}\big\|S(\delta)QB_{\mathrm{lin}}\big(V(s')\big)\big\|_{L^2}^2 \mathbf{1}_{s'<\tau}\, ds', \\ \mathcal{I}_{II;C} & = & -Ee^{-\varepsilon t}\int_0^t \Big[\int_{s'}^{\min\{t,s'+1\}} \nu e^{\nu s}\big\|S(s+\delta-s')QB_{\mathrm{lin}}\big(V(s')\big)\big\|_{L^2}^2\, ds\Big]\mathbf{1}_{s'<\tau}\, ds'. \end{array} \tag{9.91}$$

Inspecting these expressions, we readily obtain the bound

$$\begin{array}{rcl} |\mathcal{I}_{II}| & \leq & e^{\nu}(2+\nu)M^2 K_{B;\mathrm{lin}}^2 Ee^{-\varepsilon t}\int_0^t e^{\nu s'}\left\|V(s')\right\|_{H_1}^2 \mathbf{1}_{s'<\tau}\, ds' \\ & \leq & e^{\nu}(2+\nu)M^2 K_{B;\mathrm{lin}}^2 EN_{\varepsilon,\alpha;II}(t\wedge\tau). \end{array} \tag{9.92}$$

□

Lemma 9.19. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick a constant $0 < \varepsilon < 2\beta$. Then for any $0 \leq \delta < 1$, and any $0 \leq t \leq T$, we have the bounds*

$$\begin{array}{rcl} Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;B;\mathrm{cn}}(t) & \leq & \frac{M^2}{(2\beta-\varepsilon)\varepsilon}K_{B;\mathrm{cn}}^2, \\ Ee^{-\varepsilon t}\mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{cn}}(t) & \leq & K_{B;\mathrm{cn}}^2\frac{M^2}{\varepsilon}(1+\rho^{-1})e^{\varepsilon}(4+\varepsilon). \end{array} \tag{9.93}$$

Proof. Using the fact that

$$e^{-\varepsilon t} \int_0^t e^{\varepsilon s}\, ds \leq \frac{1}{\varepsilon}, \tag{9.94}$$

these bounds can be obtained from Lemma's 9.17 and 9.18 by picking $\alpha = 0$ and making the substitutions

$$K_{B;\mathrm{lin}} \mapsto K_{B;\mathrm{cn}}, \qquad EN_{\varepsilon,0;II}(t \wedge \tau) \mapsto \frac{1}{\varepsilon}. \tag{9.95}$$

□

Lemma 9.20. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \alpha < 2\beta$ and write $\nu = \alpha + \varepsilon$. Then we have the bounds*

$$\begin{array}{lcl} E \sup_{0 \leq t \leq \tau} e^{-\varepsilon t} \mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(t) & \leq & e^{\varepsilon}(T+1)\frac{M^2}{2\beta - \nu} K^2_{B;\mathrm{lin}} E \sup_{0 \leq t \leq \tau} N_{\varepsilon;\alpha;II}(t), \\ E \sup_{0 \leq t \leq \tau} e^{-\varepsilon t} \mathcal{I}^{\mathrm{sh}}_{\nu,0;B;\mathrm{lin}}(t) & \leq & e^{\varepsilon}(T+1) K^2_{B;\mathrm{lin}} M^2 (1 + \rho^{-1}) e^{\nu}(4 + \nu) E \sup_{0 \leq t \leq \tau} N_{\varepsilon;\alpha;II}(t). \end{array} \tag{9.96}$$

Proof. By splitting the integration interval we obtain

$$\begin{array}{lcl} \sup_{0 \leq t \leq \tau} e^{-\varepsilon t} \mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(t) & \leq & \sup_{0 \leq t \leq T} e^{-\varepsilon t} \mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(t) \\ & = & \sup_{0 \leq t \leq T} e^{-\varepsilon t} \int_0^t e^{\nu s} \left\| \mathcal{E}^{\mathrm{lt}}_{B;\mathrm{lin}}(s) \right\|^2_{H^1} ds \\ & \leq & e^{\varepsilon} e^{-\varepsilon} \int_0^1 e^{\nu s} \left\| \mathcal{E}^{\mathrm{lt}}_{B;\mathrm{lin}}(s) \right\|^2_{H^1} ds \\ & & \quad + e^{\varepsilon} e^{-2\varepsilon} \int_1^2 e^{\nu s} \left\| \mathcal{E}^{\mathrm{lt}}_{B;\mathrm{lin}}(s) \right\|^2_{H^1} ds \\ & & \quad + e^{\varepsilon} e^{-(\lfloor T \rfloor + 1)\varepsilon} \int_{\lfloor T \rfloor}^{\lfloor T \rfloor + 1} e^{\nu s} \left\| \mathcal{E}^{\mathrm{lt}}_{B;\mathrm{lin}}(s) \right\|^2_{H^1} ds \\ & = & e^{\varepsilon} \big[e^{-\varepsilon} \mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(1) + e^{-2\varepsilon} \mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(2) \\ & & \quad + \ldots + e^{-\varepsilon(\lfloor T \rfloor + 1)} \mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(\lfloor T \rfloor + 1) \big]. \end{array} \tag{9.97}$$

Applying Lemma 9.17, we hence see

$$\begin{array}{lcl} E \sup_{0 \leq t \leq \tau} e^{-\varepsilon t} \mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{lin}}(t) & \leq & e^{\varepsilon} \frac{M^2}{2\beta - \nu} K^2_{B;\mathrm{lin}} E \big[N_{\varepsilon;\alpha;II}(1 \wedge \tau) + \ldots + N_{\varepsilon;\alpha;II}((\lfloor T \rfloor + 1) \wedge \tau) \big] \\ & \leq & (T+1) e^{\varepsilon} \frac{M^2}{2\beta - \nu} K^2_{B;\mathrm{lin}} \sup_{0 \leq t \leq \tau} N_{\varepsilon;\alpha;II}(t). \end{array} \tag{9.98}$$

The same procedure works for the second estimate. □

Lemma 9.21. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick a constant $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon < 2\beta$. Then we have the bounds*

$$\begin{array}{lcl} E \sup_{0 \leq t \leq \tau} e^{-\varepsilon t} \mathcal{I}^{\mathrm{lt}}_{\nu,0;B;\mathrm{cn}}(t) & \leq & e^{\varepsilon}(T+1) \frac{M^2}{(2\beta - \varepsilon)\varepsilon} K^2_{B;\mathrm{cn}}, \\ E \sup_{0 \leq t \leq \tau} e^{-\varepsilon t} \mathcal{I}^{\mathrm{sh}}_{\nu,0;B;\mathrm{cn}}(t) & \leq & e^{\varepsilon}(T+1) K^2_{B;\mathrm{cn}} \frac{M^2}{\varepsilon}(1 + \rho^{-1}) e^{\varepsilon}(4 + \varepsilon). \end{array} \tag{9.99}$$

Proof. Following the procedure in the proof of Lemma 9.20, these bounds can be obtained from the estimates in Lemma 9.19 . □

Proof of Proposition 9.1. Pick $T > 0$ and $0 < \eta < \eta_0$ and write $\tau = \tau_{\varepsilon,\alpha}(T, \eta)$. Since the identities (9.3) with $v = V(t \wedge \tau)$ hold for all $0 \leq t \leq T$, we may compute

$$\begin{array}{lcl} E \sup_{0 \leq t \leq \tau} N_{\varepsilon,0;I}(t) & \leq & 5 E \sup_{0 \leq t \leq \tau} \Big[\left\| \mathcal{E}_0(t) \right\|^2_{L^2} + \sigma^4 \left\| \mathcal{E}_{F;\mathrm{lin}}(t) \right\|^2_{L^2} + \left\| \mathcal{E}_{F;\mathrm{nl}}(t) \right\|^2_{L^2} \\ & & \quad + \sigma^2 \left\| \mathcal{E}_{B;\mathrm{lin}}(t) \right\|^2_{L^2} + \sigma^2 \left\| \mathcal{E}_{B;\mathrm{cn}}(t) \right\|^2_{L^2} \Big] \end{array} \tag{9.100}$$

by applying Young's inequality. The inequalities in Lemma's 9.8-9.21 now imply that

$$E \sup_{0 \le t \le \tau} N_{\varepsilon,0;I}(t) \quad \le \quad C_1 \big[\, \|V(0)\|_{H^1}^2 + \sigma^2 T + \Big(\eta + \sigma^2 T + \sigma^4 \Big) E \sup_{0 \le t \le \tau} N_{\varepsilon,0;II}(t) \big]. \tag{9.101}$$

In addition, we note that

$$\begin{aligned} E \sup_{0 \le t \le \tau} N_{\varepsilon,0;II}(t) \quad \le \quad & 9E \sup_{0 \le t \le \tau} e^{-\varepsilon t} \Big[\mathcal{I}_{\nu,0;0}(t) + \sigma^4 \mathcal{I}^{\rm lt}_{\nu,0;F;{\rm lin}}(t) + \sigma^4 \mathcal{I}^{\rm sh}_{\nu,0;F;{\rm lin}}(t) \\ & + \mathcal{I}^{\rm lt}_{\nu,0;F;{\rm nl}}(t) + \mathcal{I}^{\rm sh}_{\nu,0;F;{\rm nl}}(t) \\ & + \sigma^2 \mathcal{I}^{\rm lt}_{\nu,0;B;{\rm lin}}(t) + \sigma^2 \mathcal{I}^{\rm sh}_{\nu,0;B;{\rm lin}}(t) \\ & + \sigma^2 \mathcal{I}^{\rm lt}_{\nu,0;B;{\rm cn}}(t) + \sigma^2 \mathcal{I}^{\rm sh}_{\nu,0;B;{\rm cn}}(t) \Big]. \end{aligned} \tag{9.102}$$

The inequalities in Lemma's 9.8-9.21 now imply that

$$E \sup_{0 \le t \le \tau} N_{\varepsilon,0;II}(t) \quad \le \quad C_2 \big[\, \|V(0)\|_{H^1}^2 + \sigma^2 T + \Big(\eta + \sigma^2 T + \sigma^4 \Big) \sup_{0 \le t \le \tau} N_{\varepsilon,0;II}(t) \big]. \tag{9.103}$$

In particular, we see that

$$E \sup_{0 \le t \le \tau} N_{\varepsilon,0}(t) \le C_3 \big[\, \|V(0)\|_{H^1}^2 + \sigma^2 T + (\eta + \sigma^2 T + \sigma^4) E \sup_{0 \le t \le \tau} N_{\varepsilon,0}(t) \big]. \tag{9.104}$$

The desired bound hence follows by appropriately restricting the size of $\eta + \sigma^2 T + \sigma^4$. □

Proof of Proposition 9.2. Ignoring the contributions arising from $B_{\rm cn}$, we can follow the proof of Proposition 9.1 to obtain the bound

$$E \sup_{0 \le t \le \tau} N_{\varepsilon,\alpha}(t) \le C_4 \big[\, \|V(0)\|_{H^1}^2 + (\eta + \sigma^2 T + \sigma^4) E \sup_{0 \le t \le \tau} N_{\varepsilon,\alpha}(t) \big]. \tag{9.105}$$

The desired estimate hence follows by appropriately restricting the size of $\eta + \sigma^2 T + \sigma^4$. □

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